MULTI-SERVER MARKOVIAN QUEUE WITH SUCCESSIVE OPTIONAL SERVICES

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Abstract

In this study, we analyze a multi-server queueing model with two successive optional services. Each server provides FES as well as two optional services to each arriving customer, for a total of c servers. Every new customer requires the first essential service (FES). The customer may quit the system with probability $(1 - r_0)$ or choose optional services supplied by the same server after finishing the FES. With probability r_0, customer chooses the first optional service (OS - 1). Following that, the customer has the option of joining the second optional service (OS - 2) with probability r_1 or leave the system with probability $(1 - r_1)$. We obtain the steady-state probability distributions by applying matrix-geometric method. We also derive a number of performance measures of the queueing model. Sensitivity analysis is used to investigate the impact of various parameters on performance of the queueing model.

Keywords: queue, multi-server, first essential service, optional services, matrix-geometric method

I. Introduction

A common goal of service systems is reducing customer waiting times, which is usually achieved by using faster services or hiring more servers. Various fields like call centers, hospitals, supermarkets, and other situations that occur every day make use of multi-server queues. In classic works like Medhi [15] and Gross, Shortle, Thompson, and Harris [5], numerous results have been obtained in all aspects of the $M/M/c$ queue. The steady-state distribution of a truncated multi-channel queueing system with customers' impatience and general balk function has been considered by Abou-El-Ata and Hariri [1]. For more research topics regarding $M/M/c$ queues, refer to Kumar [9], Levy and Yechiali [13], Li and Stanford [11], Mora [16], Bouchentouf et al. [3] and the references therein.

Real-time service systems have instances where everyone needs the first essential service (FES) and only a few others need optional services provided by the same server. Madan [14] was the first to suggest an optional second service in an $M/G/1$ queueing system using the supplementary variable approach. Similarly, Ke [6] analyzed a queueing model using startup time, in which all arriving customers need FES, while some may require additional J optional services. Jain et al. [10] investigated multiple types of optional services and vacations for an unreliable server in an $M/G/1$ queue, in which the customer may prefer to select an optional service with probability r_1 or depart from the system with probability $(1 - r_1)$. Further, the customer may also join for any one of i $(2 \leq i \leq l)$ optional services. In a study by Ke et al. [7], they examined an $M/M/c$ retrial queue with an additional optional service. In Yang et al. [19], they discussed an $M/M/R$ queueing model with a second optional channel and obtained steady-state probabilities and various system performance measures by using a matrix-geometric method. Later, Ke et al. [8] extended this model to unlimited capacity. Research on a variety of queueing models dealing with optional services is available in Li and Wang [12], Yang and Chen [18], Anitha et al. [2], Chandrika and Kalaiselvi [4], etc.

There has been no research on a multi-server queueing model with two successive optional services despite the vast body of literature. The combination of multiple servers and successive optional services gives the queueing model more realism and versatility. In practice, there are several instances in which services are provided in stages, for example, once a customer enters multi-channel service facilities, they may proceed to the next stage in turn after finishing the previous stage. This applies to many different fields, including manufacturing systems, transportation systems, telecommunications, and many daily operations. The main objective of this study is to explore the steady-state behavior of an $M/M/c$ queue with one essential service and two successive optional services.

The remainder of this paper is organized as follows. Section 2 presents a model description. Section 3 contains the mathematical formulation of the proposed queueing model. In Section 4, we apply a matrix geometric approach to find the steady-state solution. The system characteristics are described in Section 5. Section 6 is devoted to present numerical illustrations of the queueing model through practical application. Finally, we wind up our study with conclusions in Section 7.

II. Description of the Model

Consider a multi-channel queueing model with infinite capacity, FES, and two successive optional services. The pictorial representation of the model is shown in Figure 1.

The following is a description of system's fundamental operation.

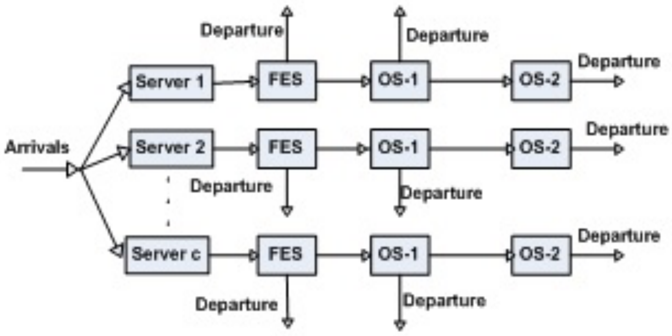

Figure 1: Model diagram

1. Arrival pattern follows Poisson process with parameter λ. There are c number of servers and each server provides FES as well as two optional services to each arriving customer.
2. After completing the FES, customer may leave the system with probability $(1 - r_0)$ or choose optional services provided by the same server. Customer opts for first optional service (OS - 1) with probability r_0 $(0 \le r_0 \le 1)$. After this, customer may join for second optional service (OS - 2) with probability r_1 $(0 \le r_1 \le 1)$ or may quit the system with probability $(1 - r_1)$. During FES, OS - 1 and OS - 2, the service times are exponentially distributed with rates of μ_0, μ_1, and μ_2, respectively.
3. The customer quits the system as soon as OS - 2 is completed, and the next consumer in the queue is allocated to FES. Each server can only serve one customer at a time and can only deliver one of three services (FES, OS - 1, OS - 2) at any given instant.
4. Upon arrival, the customer finds that all the servers are busy and must wait in the queue until one becomes available.

Practical Application

This model has real-time applications in motor vehicle service centers. The general services of vehicles include checking spark plugs, brake fluid, brake discs, checking for the normal functioning of lights, etc., which are mainly required by all vehicles. Engine oil replacement service is based on the distance traveled by the vehicle. Vehicles that have reached the certain miles enter the engine oil replacement service facility. After changing the oil, the mechanic performs the task of checking the oil filter. If it is misaligned or loose, it can be replaced. In this scenario, vehicles, mechanics, general service, engine oil replacement, and oil filter replacement, respectively, correspond to arrivals, servers, FES, OS - 1, OS - 2 in the queueing terminology.

III. Mathematical Formulation of the Model

Let $L(t)$ be the number of customers in the FES, $J_1(t)$ be the number of customers in OS - 1, and $J_2(t)$ be the number of customers in OS - 2 at time t. The process $\{(L(t), J_1(t), J_2(t)), t \geq 0\}$ defines a continuous-time Markov process with state space

$\chi = \{(i, j_1, j_2): i \geq 0, j_1 = 0,1,2,\ldots,c, j_2 = 0,1,2,\ldots,c.\}$.

It is noted that if $i + j_1 + j_2 \leq c$, the customer will receive the service immediately, if $i + j_1 + j_2 > c$, the newly arrived customer must wait in the queue.

We define the following steady-state probabilities for mathematical formulation.

P_{i,j_1,j_2} = Probability that i number of customers in the FES, j_1 number of customers in OS - 1, and j_2 number of customers in OS - 2, $i \geq 0,\ 0 \leq j_1,\ j_2 \leq c$.

Steady-State Equations:

Here, we develop the steady-state probability equations using the Markov process, which controls the dynamics of the queueing system as below.

Case I: When $j_1 = 0$ and $j_2 = 0$.

$$\lambda P_{0,0,0} = (1-r_0)\mu_0 P_{1,0,0} + (1-r_1)\mu_1 P_{0,1,0} + \mu_2 P_{0,0,1}, \tag{1}$$

$$(\lambda + i\mu_0)P_{i,0,0} = (i+1)(1-r_0)\mu_0 P_{i+1,0,0} + (1-r_1)\mu_1 P_{i,1,0} + \mu_2 P_{i,0,1} + \lambda P_{i-1,0,0}, 1 \leq i \leq c-1 \tag{2}$$

$$(\lambda + c\mu_0)P_{i,0,0} = c(1-r_0)\mu_0 P_{i+1,0,0} + (1-r_1)\mu_1 P_{i,1,0} + \mu_2 P_{i,0,1} + \lambda P_{i-1,0,0}, i \geq c. \tag{3}$$

Case II: When $1 \leq j_1 \leq c-1$ and $j_2 = 0$.

$$(\lambda + j_1\mu_1)P_{0,j_1,0} = (1-r_0)\mu_0 P_{1,j_1,0} + r_0\mu_0 P_{1,j_1-1,0} + (j_1+1)(1-r_1)\mu_1 P_{0,j_1+1,0} + \mu_2 P_{0,j_1,1}, \tag{4}$$

$$(\lambda + i\mu_0 + j_1\mu_1)P_{i,j_1,0} = \lambda P_{i-1,j_1,0} + (i+1)(1-r_0)\mu_0 P_{i+1,j_1,0} + \mu_2 P_{i,j_1,1} + (i+1)r_0\mu_0 P_{i+1,j_1-1,0} + (j_1+1)(1-r_1)\mu_1 P_{i,j_1+1,0}, 1 \leq i \leq c-j_1-1, \tag{5}$$

$$(\lambda + (c-j_1)\mu_0 + j_1\mu_1)P_{i,j_1,0} = \lambda P_{i-1,j_1,0} + (c-j_1)(1-r_0)\mu_0 P_{i+1,j_1,0} + (c-j_1+1)r_0\mu_0 P_{i+1,j_1-1,0} + (j_1+1)(1-r_1)\mu_1 P_{i,j_1+1,0} + \mu_2 P_{i,j_1,1}, i \geq c-j_1. \tag{6}$$

Case III: When $j_1 = 0$ and $1 \leq j_2 \leq c-1$.

$$(\lambda + j_2\mu_2)P_{0,0,j_2} = (1-r_0)\mu_0 P_{1,0,j_2} + (1-r_1)\mu_1 P_{0,1,j_2} + r_1\mu_1 P_{0,1,j_2-1} + (j_2+1)\mu_2 P_{0,0,j_2+1}, \tag{7}$$

$$(\lambda + i\mu_0 + j_2\mu_2)P_{i,0,j_2} = \lambda P_{i-1,0,j_2} + (i+1)(1-r_0)\mu_0 P_{i+1,0,j_2} + (1-r_1)\mu_1 P_{i,1,j_2} + r_1\mu_1 P_{i,1,j_2-1}, 1 \leq i \leq c-j_2-1, \tag{8}$$

$$(\lambda + (c-j_2)\mu_0 + j_2\mu_2)P_{i,0,j_2} = \lambda P_{i-1,0,j_2} + (c-j_2)(1-r_0)\mu_0 P_{i+1,0,j_2} + (1-r_1)\mu_1 P_{i,1,j_2} + r_1\mu_1 P_{i,1,j_2-1} + (j_2+1)\mu_2 P_{i,0,j_2+1}, i \geq c-j_2. \tag{9}$$

Case IV: When $1 \leq j_1 \leq c-1$, $1 \leq j_2 \leq c-1$, and $j_1 + j_2 \leq c$.

$$(\lambda + j_1\mu_1 + j_2\mu_2)P_{0,j_1,j_2} = (1-r_0)\mu_0 P_{1,j_1,j_2} + r_0\mu_0 P_{1,j_1-1,j_2} + (j_1+1)(1-r_1)\mu_1 P_{0,j_1+1,j_2} + (j_1+1)r_1\mu_1 P_{0,j_1+1,j_2-1} + (j_2+1)\mu_2 P_{0,j_1,j_2+1},$$

$$j_1 + j_2 \le c-1, \tag{10}$$

$$(\lambda + j_1\mu_1 + j_2\mu_2)P_{0,j_1,j_2} = r_0\mu_0 P_{1,j_1-1,j_2} + (j_1+1)r_1\mu_1 P_{0,j_1+1,j_2-1},\, j_1+j_2 = c. \tag{11}$$

$$\begin{aligned}(\lambda + i\mu_0 + j_1\mu_1 + j_2\mu_2)P_{i,j_1,j_2} &= \lambda P_{i-1,j_1,j_2} + (i+1)(1-r_0)\mu_0 P_{i+1,j_1,j_2} + \\ &(i+1)r_0\mu_0 P_{i+1,j_1-1,j_2} + (j_1+1)(1-r_1)\mu_1 P_{i,j_1+1,j_2} + (j_1+1)r_1\mu_1 P_{i,j_1+1,j_2-1} + \\ &(j_2+1)\mu_2 P_{i,j_1,j_2+1},\, j_1+j_2 \le c-1,\, 1 \le i \le c-j_1-j_2-1,\end{aligned} \tag{12}$$

$$\begin{aligned}(\lambda + \beta\mu_0 + j_1\mu_1 + j_2\mu_2)P_{i,j_1,j_2} &= \lambda P_{i-1,j_1,j_2} + \beta(1-r_0)\mu_0 P_{i+1,j_1,j_2} + (\beta+1) \\ &r_0\mu_0 P_{i+1,j_1-1,j_2} + (j_1+1)\mu_1 P_{i,j_1+1,j_2} + (j_2+1)\mu_2 P_{i,j_1,j_2+1},\, j_1+j_2 \le c-1, \\ &i \ge c-j_1-j_2.\end{aligned} \tag{13}$$

$$\begin{aligned}(\lambda + j_1\mu_1 + j_2\mu_2)P_{i,j_1,j_2} &= \lambda P_{i-1,j_1,j_2} + (\beta+1)r_0\mu_0 P_{i+1,j_1-1,j_2} + \\ &(j_1+1)r_1\mu_1 P_{i,j_1+1,j_2-1},\, i \ge \beta+1, \\ &j_1+j_2 = c,\, \beta = c-j_1-j_2.\end{aligned} \tag{14}$$

Case V: When $j_1 = c$ and $j_2 = 0$.

$$(\lambda + c\mu_1)P_{0,c,0} = r_0\mu_0 P_{1,c-1,0}, \tag{15}$$

$$(\lambda + c\mu_1)P_{i,c,0} = \lambda P_{i-1,c,0} + r_0\mu_0 P_{i+1,c-1,0},\, i \ge 1. \tag{16}$$

Case VI: When $j_1 = 0$ and $j_2 = c$.

$$(\lambda + c\mu_2)P_{0,0,c} = r_1\mu_1 P_{0,1,c-1}, \tag{17}$$

$$(\lambda + c\mu_1)P_{i,0,c} = \lambda P_{i-1,0,c} + r_1\mu_1 P_{i,1,c-1},\, i \ge 1. \tag{18}$$

IV. Matrix-Geometric Method

For the QBD process introduced in Section 3, obtaining a closed form solution is very difficult. We employ the matrix-geometric method to analyze the probabilities of the Markov chain in order to develop an effective and numerically stable solution. We can simply obtain the stationary probability vector using the matrix-geometric method because the transition rate matrix contains repeated block sub-matrices.

Applying the concept of Neuts [17], the infinitesimal generator $\mathbf{Q}$ for the process could be given as follows.

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A}_0 & \mathbf{C} & & & & & & \\ \mathbf{B}_0 & \mathbf{A}_1 & \mathbf{C} & & & & & \\ & \mathbf{B}_1 & \mathbf{A}_1 & \mathbf{C} & & & & \\ & & \mathbf{B}_2 & \mathbf{A}_2 & \mathbf{C} & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & \mathbf{B}_c & \mathbf{A}_c & \mathbf{C} & \\ & & & & & \mathbf{B}_c & \mathbf{A}_c & \mathbf{C} \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

We denote the transition probability from state (i, j_1, j_2) to the state $(\hat{\imath}, \hat{\jmath}_1, \hat{\jmath}_2)$ by $P_{(i,j_1,j_2),(\hat{\imath},\hat{\jmath}_1,\hat{\jmath}_2)}$. The elements of the sub-matrix $\mathbf{A}_i$, $i \ge 0$ are given as:

- $P_{(i,j_1,j_2),(\hat{\imath},\hat{\jmath}_1,\hat{\jmath}_2)} = -(\lambda + i\mu_0)$, for $0 \le i \le c-1$, $j_1 = j_2 = 0$, $\hat{\imath} = i$, $\hat{\jmath}_1 = j_1$, $\hat{\jmath}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{\jmath}_1,\hat{\jmath}_2)} = -(\lambda + c\mu_0)$, for $i \ge c$, $j_1 = j_2 = 0$, $\hat{\imath} = i$, $\hat{\jmath}_1 = j_1$, $\hat{\jmath}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{\jmath}_1,\hat{\jmath}_2)} = -(\lambda + i\mu_0 + j_2\mu_2)$, for $i \ge 0$, $1 \le j_2 \le c$, $i + j_2 \le c$, $j_1 = 0$, $\hat{\imath} = i$, $\hat{\jmath}_1 = j_1$, $\hat{\jmath}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{\jmath}_1,\hat{\jmath}_2)} = -(\lambda + (c-j_2)\mu_0 + j_2\mu_2)$, for $i \ge 0$, $1 \le j_2 \le c$, $i + j_2 > c$, $j_1 = 0$, $\hat{\jmath}_2 = j_2$, $\hat{\imath} = i$, $\hat{\jmath}_1 = j_1$, $\hat{\jmath}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{\jmath}_1,\hat{\jmath}_2)} = -(\lambda + i\mu_0 + j_1\mu_1)$, for $i \ge 0$, $0 \le j_1 \le c$ $i + j_1 \le c$, $j_2 = 0$, $\hat{\jmath}_1 = j_1$, $\hat{\imath} = i$, $\hat{\jmath}_1 = j_1$, $\hat{\jmath}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{\jmath}_1,\hat{\jmath}_2)} = -(\lambda + (c-j_1)\mu_0 + j_1\mu_2)$, for $i \ge 0$, $i = \hat{\imath}$, $0 \le j_1 \le c$, $i + j_1 > c$, $j_2 = 0$, $\hat{\jmath}_1 = j_1$, $\hat{\jmath}_2 = j_2$.

- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = -(\lambda + (c - j_1 - j_2)\mu_0 + j_1\mu_1 + j_2\mu_2)$, for $i \geq 0$, $\hat{\imath} = i$, $j_1 + j_2 \leq c$, $1 \leq j_1 \leq c-1$, $1 \leq j_2 \leq c-1$, $\hat{j}_1 = j_1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = j_1(1 - r_1)\mu_1$, for $i = 0$, $\hat{\imath} = i$, $1 \leq j_1 \leq c$, $0 \leq j_2 \leq c-1$, $j_1 + j_2 \leq c$, $\hat{j}_1 = j_1 - 1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = j_2\mu_2$, for $i = 0$, $\hat{\imath} = i$, $1 \leq j_2 \leq c, 0 \leq j_1 \leq c-1$, $\hat{j}_2 = j_2 - 1$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = j_1 r_1 \mu_1$, for $i = 0$, $\hat{\imath} = i$, $1 \leq j_1 \leq c, 0 \leq j_2 \leq c-1$, $\hat{j}_1 = j_1 - 1$ $\hat{j}_2 = j_2 + 1$.

The elements of the sub-matrix $\mathbf{B}_i$, $i \geq 0$ are taken as:

- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = i(1 - r_0)\mu_0$, for $1 \leq i \leq c-1$, $\hat{\imath} = i-1$, $j_1 = j_2 = 0$, $\hat{j}_1 = j_1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = c(1 - r_0)\mu_0$, for $i \geq c$, $\hat{\imath} = i-1$, $j_1 = j_2 = 0$, $\hat{j}_1 = j_1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = i(1 - r_0)\mu_0$, for $i \geq 0$, $\hat{\imath} = i-1$, $0 \leq j_2 \leq c$, $i + j_2 \leq c$, $j_1 = 0$, $\hat{j}_1 = j_1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = (c - j_2)(1 - r_0)\mu_0$, for $i \geq 0$, $\hat{\imath} = i-1$, $0 \leq j_2 \leq c$, $i + j_2 > c$, $j_1 = 0$, $\hat{j}_1 = j_1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = i(1 - r_0)\mu_0$, for $i \geq 0$, $\hat{\imath} = i-1$, $0 \leq j_1 \leq c$, $i + j_1 \leq c$, $j_2 = 0$, $\hat{j}_1 = j_1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = (c - j_1)(1 - r_0)\mu_0$, for $i \geq 0$, $\hat{\imath} = i-1$, $0 \leq j_1 \leq c$, $i + j_1 > c$, $j_2 = 0$, $\hat{j}_1 = j_1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = (c - j_1 - j_2)(1 - r_0)\mu_0$, for $i \geq 0$, $\hat{\imath} = i-1$, $1 \leq j_1 \leq c-1$, $1 \leq j_2 \leq c-1$, $\hat{j}_1 = j_1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = i r_0 \mu_0$, for $i \geq 0$, $\hat{\imath} = i-1$, $0 \leq j_2 \leq c$, $i + j_2 \leq c$, $j_1 = 0$, $\hat{j}_1 = j_1 + 1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = (c - j_2) r_0 \mu_0$, for $i \geq 0$, $\hat{\imath} = i-1$, $0 \leq j_2 \leq c$, $i + j_2 > c$, $j_1 = 0$, $\hat{j}_1 = j_1 + 1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = i r_0 \mu_0$, for $i \geq 0$, $\hat{\imath} = i-1$, $0 \leq j_1 \leq c$, $i + j_1 \leq c$, $j_2 = 0$, $\hat{j}_1 = j_1 + 1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = (c - j_1) r_0 \mu_0$, for $i \geq 0$, $\hat{\imath} = i-1$, $0 \leq j_1 \leq c$, $i + j_1 > c$, $j_2 = 0$, $\hat{j}_1 = j_1 + 1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = (c - j_1 - j_2) r_0 \mu_0$, for $i \geq 0$, $\hat{\imath} = i-1$, $1 \leq j_1 \leq c-1$, $1 \leq j_2 \leq c-1$, $j_1 + j_2 \leq c$, $\hat{j}_1 = j_1 + 1$, $\hat{j}_2 = j_2$.
- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = 0$, when $j_1 + j_2 = c$.

The elements of sub-matrix $\mathbf{C}$ are given as follows:

- $P_{(i,j_1,j_2),(\hat{\imath},\hat{j}_1,\hat{j}_2)} = \lambda$, for $i \geq 0$, $\hat{\imath} = i + 1$, $0 \leq j_1 \leq c$, $0 \leq j_2 \leq c$, $\hat{j}_1 = j_1$, and $\hat{j}_2 = j_2$.

Here, the sub-matrices $\mathbf{C}$, $\mathbf{A}_i$, $\mathbf{B}_i$, $i \geq 0$ are of order $1 + \sum_{n=1}^{c-1} n + 2c$.

4.1 Steady-state solution

Based on $\mathbf{Q}$ matrix structure, one can easily notice that the process $\{L(t), J_1(t), J_2(t), t \geq 0\}$ is a QBD process. As per the block structure of $\mathbf{Q}$, the stationary distribution of the process can be composed as segmented vectors, denoted as,

$$P_{i,j_1,j_2} = \lim_{t\to\infty} P\{L(t) = i, J_1(t) = j_1, J_2(t) = j_2\}, (i, j_1, j_2) \in \chi$$

According to Neuts (1981), the system is stable and the steady state probability vector exists if and only if $\tilde{\mathbf{Y}}\mathbf{Ce} < \tilde{\mathbf{Y}}\mathbf{B_c e}$ where $\tilde{\mathbf{Y}}$ is an invariant probability of the matrix $\boldsymbol{\psi} = \mathbf{B_c} + \mathbf{A_c} + \mathbf{C}$. The equations $\tilde{\mathbf{Y}}\boldsymbol{\psi} = \mathbf{0}$ and $\tilde{\mathbf{Y}}\mathbf{e} = \mathbf{1}$ can be satisfied by $\tilde{\mathbf{Y}}$.

Under the stability condition, let $\mathbf{P}$ be the stationary probability vector of the generator $\mathbf{Q}$ satisfying the balance equation $\mathbf{PQ} = \mathbf{0}$ and $\mathbf{Pe}_i = 1$, where $\mathbf{0}$ is the row vector with all elements as zero and $\mathbf{e}_i$ is the column vector of appropriate dimension i with every element as one. The vector $\mathbf{P}$ partitioned as $\mathbf{P} = [\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \dots]$, where

$$\mathbf{P}_i = [P_{i,0,0}, P_{i,0,1}, P_{i,0,2}, \dots, P_{i,0,c}, P_{i,1,0}, P_{i,2,0}, \dots, P_{i,c,0}, P_{i,1,1}, P_{i,1,2}, \dots, P_{i,1,c-1}, P_{i,2,1}, P_{i,2,2}, \dots, P_{i,2,c-2}, \dots, P_{i,c-2,1}, P_{i,c-2,2}, P_{i,c-1,1}],\ i \geq 0.$$

When the stability criterion is met, the sub-vectors of $\mathbf{P}$ pertaining to various levels appear to satisfy

$$\mathbf{P}_i = \mathbf{P}_c\mathbf{R}^{i-c}, i \geq c, \tag{19}$$

where the matrix $\mathbf{R}$ is the minimal non-negative solution of the matrix quadratic equation

$$\mathbf{C} + \mathbf{RA}_c + \mathbf{R}^2\mathbf{B}_c = \mathbf{0}. \tag{20}$$

The QBD process is positive recurrent if and only if the spectral radius $Sp(\mathbf{R}) < 1$. Further, it is rather complex to determine the explicit expression of the matrix $\mathbf{R}$ by solving equation (20). Neuts [17] has devised an iterative algorithm for calculating $\mathbf{R}$ numerically. We starting with initial iteration $\mathbf{R}_0 = 0$, and calculate the successive approximations using

$$\mathbf{R}_{i+1} = -(\mathbf{C} + \mathbf{R}_i^2\mathbf{B}_c)(\mathbf{A}_c)^{-1}, i \geq 0.$$

Now, $\mathbf{R}$ can be determined iteratively until it converges, i.e., $\lim_{i\to\infty}\mathbf{R}_i = \mathbf{R}$.

Using the equation $\mathbf{PQ} = \mathbf{0}$, the governing system of difference equations are expressed as follows

$$\mathbf{P}_0\mathbf{A}_0 + \mathbf{P}_1\mathbf{B}_1 = 0, \tag{21}$$

$$\mathbf{P}_{i-1}\mathbf{C} + \mathbf{P}_i\mathbf{A}_i + \mathbf{P}_{i+1}\mathbf{B}_{i+1} = 0, 1 \leq i \leq c-1, \tag{22}$$

$$\mathbf{P}_{i-1}\mathbf{C} + \mathbf{P}_i\mathbf{A}_c + \mathbf{P}_{i+1}\mathbf{B}_c = 0, i \geq c, \tag{23}$$

and the normalizing condition

$$\sum_{i=0}^{\infty} \mathbf{P}_i\mathbf{e}_i = 1. \tag{24}$$

After applying some mathematical manipulations to equations (21) to (23), we get

$$\mathbf{P}_{i-1} = \mathbf{P}_i\boldsymbol{\phi}_i, 1 \leq i \leq c, \tag{25}$$

$$\mathbf{P}_c[\boldsymbol{\phi}_c\mathbf{C} + \mathbf{A}_c + \mathbf{RB}_c] = \mathbf{0}, \tag{26}$$

where

$$\boldsymbol{\phi}_1 = -\mathbf{B}_0(\mathbf{A}_0^{-1}), \boldsymbol{\phi}_i = -\mathbf{B}_i(\mathbf{A}_{i-1} + \boldsymbol{\phi}_{i-1}\mathbf{C})^{-1}, 1 \leq i \leq c.$$

Using equations (24) and (25), we obtain

$$\mathbf{P}_c[\sum_{j=1}^{c} \prod_{n=c}^{l} \boldsymbol{\phi}_n + (\mathbf{I} - \mathbf{R})^{-1}]\mathbf{e}_i = 1. \tag{27}$$

Solving equations (26) and (27) yields $\mathbf{P}_c$. We use equations (19) and (25) to get $\mathbf{P}_i$ for $i \geq 0$.

V. Performance Measures

An infinite capacity multi-server queueing system with two successive optional services has several system characteristics, such as the expected length of the system in FES, OS - 1, and OS - 2, the expected number of customers in the system, the expected number of idle servers, the expected number of busy servers, probability that the system is empty, can be obtained by using steady-state probabilities. The expressions of above are given as follows:

- Expected number of customers in FES

$$E[L_f] = \sum_{i=1}^{\infty} iP_{i,0,0} + \sum_{i=1}^{\infty} i\sum_{j_1=1}^{c-1} P_{i,j_1,0} + \sum_{i=1}^{\infty} i\sum_{j_2=1}^{c-1} P_{i,0,j_2} + \sum_{i=1}^{\infty} i\sum_{j_1=1}^{c-1}\sum_{j_2=1}^{c-j_1} P_{i,j_1,j_2} + \sum_{i=1}^{\infty} iP_{i,c,0} + \sum_{i=1}^{\infty} iP_{i,0,c}.$$

- Expected number of customers in OS - 1

$$E[L_{s_1}] = \sum_{j_1=1}^{c-1} j_1\sum_{i=0}^{\infty} P_{i,j_1,0} + \sum_{j_1=1}^{c-1} j_1\sum_{i=0}^{\infty}\sum_{j_2=1}^{c-j_1} P_{i,j_1,j_2} + c\sum_{i=0}^{\infty} P_{i,c,0}.$$

- Expected number of customers in OS - 2

$$E[L_{s_2}] = \sum_{j_2=1}^{c-1} j_2 \sum_{i=0}^{\infty} P_{i,0,j_2} + \sum_{j_2=1}^{c-1} j_2 \sum_{i=0}^{\infty} \sum_{j_1=1}^{c-j_2} P_{i,j_1,j_2} + c\sum_{i=0}^{\infty} P_{i,0,c}.$$

- Expected number of customers in the system

$$E[L] = E[L_f] + E[L_{s_1}] + E[L_{s_2}].$$

- Expected number of idle servers

$$E[I] = \sum_{i+j_1+j_2=0}^{c-1} [c - max(i, j_1, j_2)]P_{i,j_1,j_2}.$$

- Expected number of busy servers

$$E[B] = c - E[I].$$

- Probability that the system is empty is $P_{0,0,0}$.

VI. Numerical Investigations

To understand the system long run behaviour change with the parameters, we have conducted some numerical studies on the system characteristics by changing the parameter values. Considering the practical application given in Section 2, we perform the sensitivity analysis using arbitrarily selected parameters $\lambda = 1.0$, $\mu_0 = 5.0$, $\mu_1 = 4.5$, $\mu_2 = 3.0$, $r_0 = 0.6$, $r_1 = 0.5$, $c = 4$, where

λ = The rate at which vehicles arrive at the service center,
μ_0 = Service rate for general services, including spark plugs check, brake fluid, brake discs, etc. (FES),
μ_1 = Service rate of engine oil replacement service (OS - 1),
μ_2 = Service rate of oil filters replacement service (OS - 2),
r_0 = Probability that vehicles taken for engine oil replacement,
r_1 = Probability that vehicles taken for oil filter replacement,
c = Amount of mechanics in the vehicle service center.

Table 1: Effect of r_0 on $E[L]$

	$E[L]$			
r_0	$\mu_2 = 3.0$	$\mu_2 = 3.2$	$\mu_2 = 3.4$	$\mu_2 = 3.6$
0.1	0.12143	0.11988	0.11848	0.11721
0.3	0.32241	0.31685	0.31179	0.30718
0.5	0.70647	0.69319	0.68101	0.66976
0.7	1.45122	1.4277	1.40562	1.38481
0.9	2.17964	2.16400	2.14999	2.13729

Table 2: Effect of r_1 on $E[L]$

	$E[L]$			
r_1	$\mu_2 = 3.0$	$\mu_2 = 3.2$	$\mu_2 = 3.4$	$\mu_2 = 3.6$
0.2	0.81844	0.81319	0.80852	0.80434
0.4	0.92491	0.91515	0.90642	0.89855
0.6	1.07338	1.06013	1.04821	1.03741
0.8	1.27478	1.25986	1.24638	1.23412
1.0	1.55389	1.54122	1.52988	1.51967

Tables 1 and 2 show the impact of the probabilities r_0 and r_1, on the expected length of the system $E[L]$ for different values of the service rates in OS - 1 (μ_1) and OS - 2 (μ_2). We observe that

- As r_0 (r_1) increases, the number of vehicles opting for engine oil (oil filter) replacement facility increases, which tends to increase the waiting time of vehicles at the service center. Hence $E[L]$ increases.
- Also, an increase in the service rate μ_1 (μ_2) reduces $E[L]$, which agrees with our intuition.

Table 3: Effect of λ on performance measures

	$r_0 = 0.0$ and $r_1 = 0.0$			$r_0 = 1.0$ and $r_1 = 1.0$		
λ	$E[L]$	$E[I]$	$E[B]$	$E[L]$	$E[I]$	$E[B]$
0.3	0.01522	3.98478	0.01522	2.19881	1.49894	2.50106
0.6	0.03085	3.96915	0.03085	2.33150	1.37036	2.62964
0.9	0.04685	3.95315	0.04685	2.47134	1.25348	2.74652
1.2	0.06318	3.93681	0.06319	2.61849	1.14669	2.85331
1.5	0.07982	3.92015	0.07985	2.77359	1.04824	2.95176
1.8	0.09671	3.90322	0.09677	2.93725	0.95641	3.04360

Table 3 shows the impact of arrival rate λ on expected number of vehicles at the service center $E[L]$, expected amount of idle mechanics $E[I]$, and expected amount of busy mechanics $E[B]$ in two situations as follows:
Case a: When no vehicle is opting for optional services ($r_0 = 0$, $r_1 = 0$)
Case b: When all arriving vehicles are opting both optional services ($r_0 = 1$, $r_1 = 1$)
It is observed that

- An increase in λ results in increase of $E[L]$, $E[B]$ and decrease of $E[I]$ for a fixed r_0 and r_1, as expected.
- Further, for a fixed λ, $E[L]$ and $E[B]$ are seen smaller when no vehicles adopting any optional service provided by a service center. On the other hand, $E[I]$ is smaller when all arriving vehicles choose both optional services, as anticipated.

Table 4: Effect of r_0 and r_1 on performance measures

	$r_1 = 0.3$		$r_1 = 0.5$		$r_1 = 0.7$	
r_0	$E[L]$	$E[I]$	$E[L]$	$E[I]$	$E[L]$	$E[I]$
0.3	0.23906	3.76450	0.31179	3.68605	0.40389	3.58498
0.5	0.54262	3.45912	0.68101	3.28845	0.85592	3.07158
0.7	1.38111	2.62083	1.40562	2.50599	1.51586	2.31483

The effect of the probability of opting OS - 1 and OS - 2 (r_0 and r_1) on $E[L]$ and $E[I]$ is shown in Table 4.

- For a fixed r_1, as r_0 increases, the number of vehicles opting for engine oil service grows, resulting in an increase in the number of vehicles waiting for service at the service center $E[L]$. Moreover, for a fixed r_0, the same trend is observed for $E[L]$ with increase in r_1.
- However, it can be seen that increase in r_0 (r_1) yields the lower $E[I]$. This is because an increase in these probabilities increases the vehicle service time.
- Also, considering the cases $r_0 < r_1 (= 0.7)$ and $r_0 > r_1 (= 0.3)$, we notice that $E[L]$ is higher when $r_0 < r_1$ for the chosen parameter values.

Table 5: Effect of λ on performance measures for different μ_0, μ_1, and μ_2

	$\mu_0 = 5.2, \mu_1 = 4.6, \mu_2 = 3.5$		$\mu_0 = 3.5, \mu_1 = 4.0, \mu_2 = 5.0$	
λ	$E[L]$	$E[I]$	$E[L]$	$E[I]$
0.4	0.45349	3.53054	0.47686	3.52990
0.8	0.77994	3.18595	0.82567	3.18415
1.2	1.03378	2.91530	1.09905	2.91338
1.6	1.24318	2.69213	1.32574	2.69058

The impact of λ on $E[L]$ and $E[I]$ for different μ_0, μ_1, and μ_2 is shown in Table 5. Here we depicted the comparison of cases $\mu_0 > \mu_1 > \mu_2$ and $\mu_0 < \mu_1 < \mu_2$. As shown in Table 3, increase of λ increases $E[L]$ and decreases $E[I]$. Evidently, from the table, expected size of vehicles at the service center can be reduced by taking $\mu_0 > \mu_1 > \mu_2$. This helps the service center managers to run the system effectively when the arrival rate of vehicles is high.

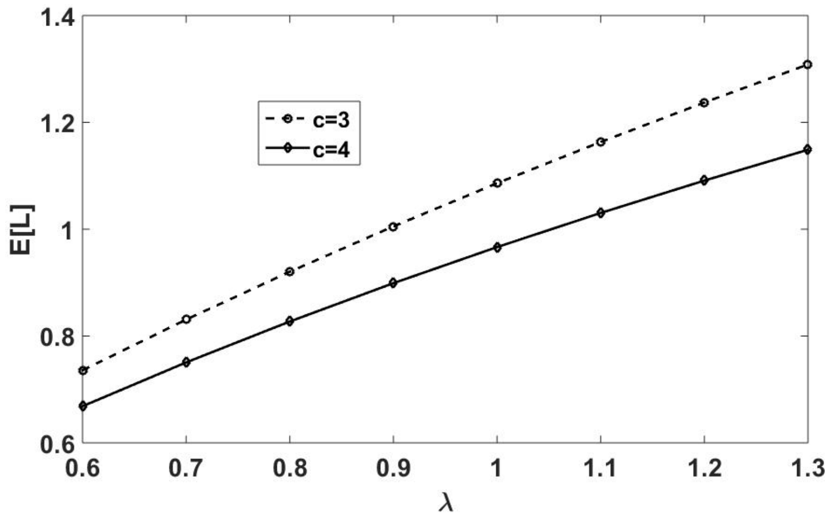

Figure 2: Effect of λ on $E[L]$ for different c

Figure 2 illustrates the impact of λ on $E[L]$ for various c values. As we seen in the tables, an increase in λ increases $E[L]$ for a fixed amount of mechanics c. Furthermore, an opposite effect is observed with the increase in c, this is due to the fact that increase of mechanics decreases vehicles waiting time. From this figure, we conclude that when the arrival rate of vehicles at service center is high, one can reduce the system size by increasing the number of mechanics, even though the service rates are kept constant.

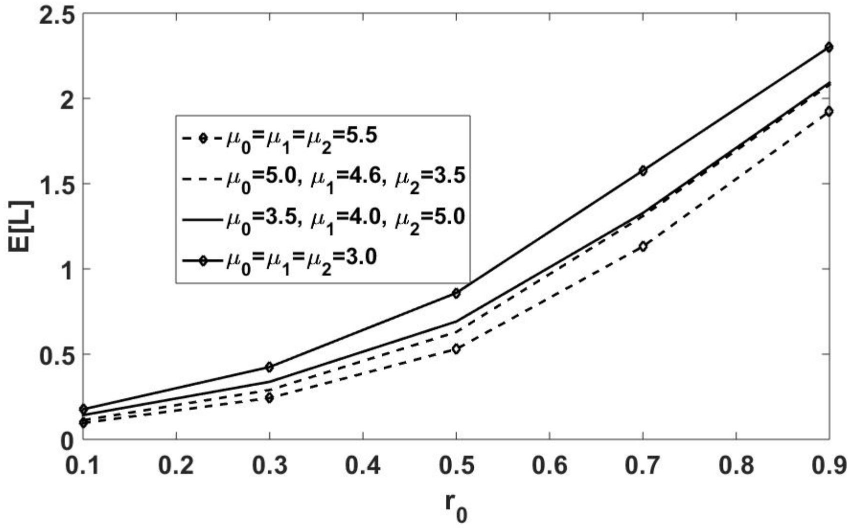

Figure 3: Effect of r_0 on $E[L]$

Figure 3 explores the impact of r_0 on $E[L]$ for different values of service rates. It is clear from the figure that as the number of vehicles opting for the first optional service facility increases, $E[L]$ increases. Subsequently, system performance can be ranked, with $\mu_0 = \mu_1 = \mu_2 = 5.5$ being best, followed by $\mu_0 = 5.0$, $\mu_1 = 4.6$, $\mu_2 = 3.5$ and then $\mu_0 = 3.5$, $\mu_1 = 4.0$, $\mu_2 = 5.0$, and lastly $\mu_0 = \mu_1 = \mu_2 = 3.0$.

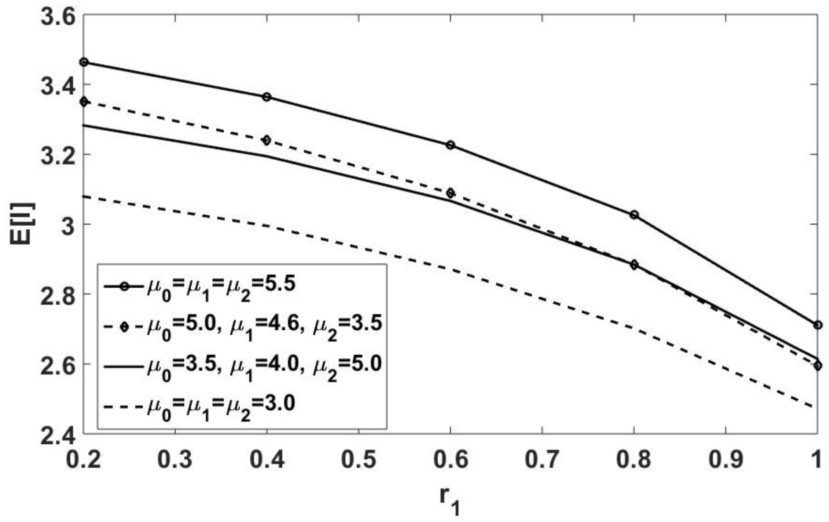

Figure 4: Effect of r_1 on $E[I]$

The impact of r_1 on expected number of idle servers $E[I]$ for different values of service rates in depicted in Figure 4. It is obvious that an increase in probability of vehicles choosing for oil filter service after getting engine oil service facility r_1 decreases the idle time of the mechanics at service center. Hence, $E[I]$ decreases. On the other hand, $E[I]$ is smaller when $\mu_0 = \mu_1 = \mu_2 = 3.0$ and higher when $\mu_0 = \mu_1 = \mu_2 = 5.5$. Also, it is quite interesting to note that while comparing the cases $\mu_0 = 5.0$, $\mu_1 = 4.6$, $\mu_2 = 3.5$ and $\mu_0 = 3.5$, $\mu_1 = 4.0$, $\mu_2 = 5.0$, $E[I]$ is observed higher for $\mu_0 = 5.0$, $\mu_1 = 4.6$, $\mu_2 = 3.5$ when $r_1 \leq 0.6$. At $r_1 = 0.7$ two curves are almost coincide and at $r_1 = 0.8$ two curves intersect each other. Further, for $r_1 > 0.8$, the trend is reversed and $E[I]$ is seen higher for $\mu_0 = 3.5$, $\mu_1 = 4.0$, $\mu_2 = 5.0$. This reveals the fact that as more and more vehicles are opting OS - 2 ($r_1 > 0.6$), by taking μ_2 as bigger than μ_0 and μ_1, $E[I]$ will be smaller (here r_0 value is chosen as 0.6).

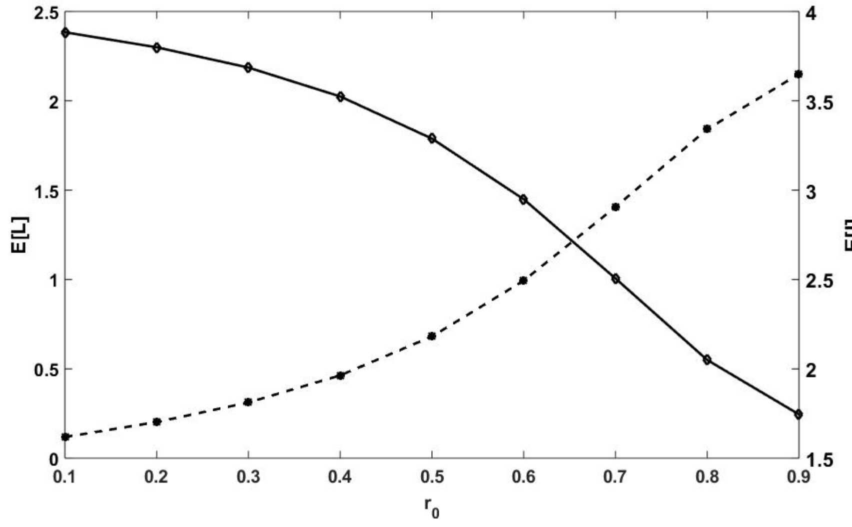

Figure 5: Effect of r_0 on $E[L]$ and $E[I]$

Figure 5 exhibits the effect of r_0 on $E[L]$ and $E[I]$. It demonstrates that $E[L]$ and $E[I]$ increases and decreases, respectively, with the increase in r_0. The point of intersection of two curves determines the value of r_0 at which $E[L]$ and $E[I]$ are the maximum and minimum, respectively. As a result, service center managers can optimize $E[L]$ or $E[I]$ by taking the appropriate measures using r_0 knowledge.

VII. Conclusion

In this study, we have carried out the analysis of $M/M/c$ queueing model with two successive optional services. Using QBD process and matrix geometric method, we have obtained the stationary probability distribution of the model. Further we have derived some performance measures of the model such as expected length of the system, expected number of idle servers and expected number of busy servers. Sensitivity analysis has been carried out by considering the practical application of the model. Through our numerical and graphical studies, it is observed that

- Expected number of vehicles at the service center increases with the increase of arrival rate and probability of opting optional services.
- System size can be reduced by increasing the amount of mechanics when the arrival rate is high.
- System size is smaller when $\mu_0 > \mu_1 > \mu_2$ for a constant arrival rate and optional service probabilities.
- When more number of vehicles opt for optional services, $E[L]$ can be decreased by taking equal higher service rates in FES, OS -1,OS -2.

This research work may extended further by incorporating the concepts of working vacations, customers' impatience, server breakdowns, etc.

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