# Estimation procedures for a flexible extension of Maxwell distribution with data modeling

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**Abstract**

*In this paper, we introduce a flexible extension of the Maxwell distribution for modeling various practical data with non-monotone failure rate. Some main properties of this distribution are obtained, and then the estimation of the parameters for the proposed distribution has been addressed by maximum likelihood estimation method and Bayes estimation method. The Bayes estimators have been obtained under gamma prior using squared error loss function. Also, a simulation study is gained to assess the estimates performance. A real-life applications for the proposed distribution have been illustrated through different lifetime data.*

**Keywords:** Family of Maxwell distributions, Entropy, Classical and Bayes estimation, Interval estimation, Asymptotic confidence length.

## 1. Introduction

The Maxwell distribution has broad application in statistical physics, physical chemistry, and their related areas. Besides Physics and Chemistry it has also a good number of applications in reliability theory. At first, the Maxwell distribution was used as lifetime distribution by [1]. The inferences based on generalized Maxwell distribution have been discussed by [2]. [3] considered the estimation of reliability characteristics for Maxwell distribution under Bayes paradigm. [4] discussed the prior selection procedure in case of Maxwell distribution. [5] studied the distributions of the product $|XY|$ and ratio $|X/Y|$, where X and Y are independent random variables having the Maxwell and Rayleigh distributions, respectively. [6] proposed the Bayesian estimation of the Maxwell parameters. [7] discussed the estimation procedure for the Maxwell parameters under progressive type-I hybrid censored data. Furthermore, several generalizations based on Maxwell distribution are advocated and statistically justified. Recently, two more extensions of Maxwell distribution has been introduced by [8], [9] and discussed the classical as well as Bayesian estimation of the parameter along with real-life applications.

A random variable $Z$ follows the Maxwell distribution (MaD) with scale parameter $\alpha$, denoted as $Z \sim MaD(\alpha)$, if its probability density function (PDF) and cumulative distribution function (CDF) are given by

$$f(z,\alpha) = \frac{4}{\sqrt{\pi}}\alpha^{\frac{3}{2}} z^2 e^{-\alpha z^2} \quad z \geq 0, \alpha > 0 \tag{1}$$

and

$$F(z, \alpha) = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}, \alpha z^2\right), \tag{2}$$

respectively, where $\Gamma(a, z) = \int_0^z p^{a-1} e^{-p} dp$ is the incomplete gamma function.

In this article, we propose a flexible extension of the Maxwell distribution. The objective of this article is to get some main properties of this distribution for showing its merit in modeling various practical data, and then estimate the unknown parameters using classical and Bayes estimation methods. Other motivations regarding the advantages of the distribution comes from its flexibility to model the data with non-monotone failure rates. The former aim is justified, where the proposed distribution provides better fit to the reliability/survival data comparing to the some known and recent versions of the Maxwell distribution. Further, the distribution is that having the nature of platykurtic, mesokurtic and leptokurtic, hence it can be used to model skewed and symmetric data as well. Also, the Bayes procedure under informative prior provides the more efficient estimates as compared to the maximum likelihood estimates (MLEs) concerning the estimation point of view. Another motivation for the confidence interval of the distribution parameters is that increasing the sample size decreases the width of confidence intervals, because it decreases the standard error, and this justified by simulation study and using sizes of four practical data sets.

The reminder of the considered work has been structured in the following manner. Section 2 provides some statistical properties related to the proposed model for purpose of data modeling. In Section 3, some types of entropy are investigated. The maximum likelihood (ML) and Bayes estimation procedures have been discussed in Section 4. Also, a simulation study is carried out to compare the performance of Bayes estimates with MLEs. In Section 5, we illustrate the application and usefulness of the proposed model by applying it to four practical data sets. Section 5 offers some concluding remarks.

## 2. The model and some of its properties

This section provides another generalization of the MaD using power transformation of Maxwell random variates for estimations issues of the distribution parameters and modeling practical data. For this purpose, consider the transformation $X = Z^{\frac{1}{\beta}}$, where $Z \sim MaD(\alpha)$, hence the resulting distribution of $X$ is called as power Maxwell distribution (for short PMaD) and denoted by $X \sim PMaD(\alpha, \beta)$, where, $\alpha$ and $\beta$ are the scale and shape parameters, respectively. The PDF and CDF of the PMaD are given by

$$f(x, \alpha, \beta) = \frac{4}{\sqrt{\pi}} \alpha^{\frac{3}{2}} \beta x^{3\beta-1} e^{-\alpha x^{2\beta}}, \quad x \geq 0, \alpha, \beta > 0, \tag{3}$$

$$F(x, \alpha, \beta) = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}, \alpha x^{2\beta}\right), \tag{4}$$

respectively. Plots of the PDF are given by Figure 1 for different choices of $\alpha$ and $\beta$. The plots show different kurtosis, positive skewness and symmetric shapes.

Some main mathematical and statistical properties of PMaD have been obtained in the following.

### 2.1. Behaviour with some reliability functions

This subsection, described the asymptotic nature of density and survival functions for the proposed distribution. To illustrate asymptotic behaviour, at first, we will show that $\lim_{x \to 0} f(x, \alpha, \beta) = 0$ and $\lim_{x \to \infty} f(x, \alpha, \beta) = 0$ . Therefore, using (2.1)

$$\lim_{x \to 0} f(x, \alpha, \beta) = \frac{4}{\sqrt{\pi}} \alpha^{\frac{3}{2}} \beta \lim_{x \to 0} x^{3\beta-1} e^{-\alpha x^{2\beta}} = 0,$$

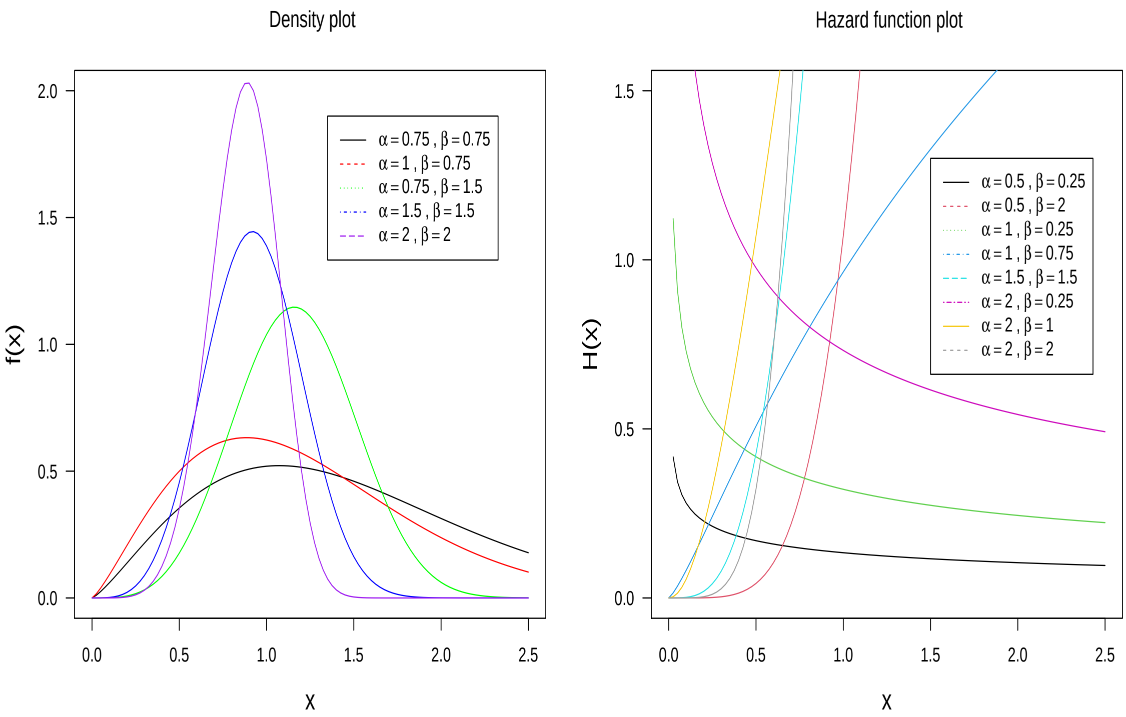


**Figure 1:** *Density function and hazard function plot for different choices of α and β.*

and

$$\lim_{x\to\infty} f(x,\alpha,\beta) = \frac{4}{\sqrt{\pi}}\alpha^{\frac{3}{2}}\beta \lim_{x\to\infty} x^{3\beta-1} \lim_{x\to\infty} e^{-\alpha x^{2\beta}} = 0$$

The characteristics based on reliability function and hazard function are very useful to study the pattern of any lifetime phenomenon. Let $X$ be a random variable with PDF (2.1) and CDF (2.2), different reliability measures for the proposed distribution are obtained by following equations.

The reliability function $R(x)$ is given by

$$R(x) = P(X > x) = 1 - \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}, \alpha x^{2\beta}\right) \tag{5}$$

The mean time to system failure $M(x)$ is

$$M(x) = E(x) = \frac{2}{\sqrt{\pi}}\left(\frac{1}{\alpha}\right)^{\frac{1}{2\beta}}\Gamma\left(\frac{3\beta+1}{2\beta}\right) \tag{6}$$

The hazard function $H(x)$ is given as

$$H(x) = \frac{f(x,\alpha,\beta)}{1 - F(x,\alpha,\beta)} = \frac{4\alpha^{\frac{3}{2}}\beta x^{3\beta-1}e^{-\alpha x^{2\beta}}}{\sqrt{\pi} - 2\Gamma\left(\frac{3}{2}, \alpha x^{2\beta}\right)} \tag{7}$$

The plots, in Figure 1, show that the proposed density is unimodel and positively skewed with monotone failure rate function for the different combination of the model parameters. The comparative behavior of the random variables can be measured by stochastic ordering concept that is summarized in the next proposition.

**Proposition:** Let $X \sim PMaD(\alpha_1, \beta_1)$ and $Y \sim PMaD(\alpha_2, \beta_2)$, then the likelihood ratio is

$$\Phi = \frac{f_X(x)}{f_Y(x)} = \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{3}{2}} \left(\frac{\beta_1}{\beta_2}\right) x^{3(\beta_1-\beta_2)} e^{-(\alpha_1 x^{2\beta_1} + \alpha_2 x^{2\beta_2})}.$$

Therefore,

$$\Phi^{'} = \log\left(\frac{f_X(x)}{f_Y(x)}\right) = \frac{1}{x}\left[3(\beta_1 - \beta_2) - (\alpha_1 x^{2\beta_1} + \alpha_2 x^{2\beta_2})\right]$$

If $\beta_1 = \beta_2 = \beta$, then $\Phi^{'} < 0$, which implies that the random variable $X$ is a likelihood ratio order than $Y$, that is $X \leq_{lr} Y$. Also, if $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 < \beta_2$, then again $\Phi^{'} < 0$, which shows that $X \leq_{lr} Y$. Other stochastic orderings behaviour follow using $X \leq_{lr} Y$, such as hazard rate order $(X \leq_{hr} Y)$, mean residual life order $(X \leq_{mrl} Y)$ and scholastically greater $(X \leq_{st} Y)$.

## 2.2. Moments and some conditional ones

Let $x_1, x_2, \cdots x_n$ be random observations from the PMaD$(\alpha, \beta)$. The $r^{th}$ moment, $\mu_r^{'}$, about origin is

$$\mu_r^{'} = \int_{x=0}^{\infty} x^r f(x, \alpha, \beta) \quad dx = \frac{2}{\sqrt{\pi}}\left(\frac{1}{\alpha}\right)^{\frac{r}{2\beta}} \Gamma\left(\frac{3\beta + r}{2\beta}\right), \quad r \geqslant 1.$$

The coefficient of skewness and kurtosis measure the convexity of the curve and its shape. Using the moments above, the two earlier measures are obtained by moments based relations suggested by Pearson and given by

$$\beta_1 = \frac{\left[\mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\left(\mu_1^{'}\right)^3\right]^2}{\left[\mu_2^{'} - (\mu_1^{'})^2\right]^3}$$

and

$$\beta_2 = \frac{\mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\left(\mu_1^{'}\right)^2 - 3\left(\mu_1^{'}\right)^4}{\left[\mu_2^{'} - (\mu_1^{'})^2\right]^2}.$$

Numerical values of some measures above are calculated in Table 1 for different combination of the model parameters, and it is observed that the shape of the PMaD is right skewed and almost symmetrical for some choices of $\alpha, \beta$. Also, it can has the nature of platykurtic, mesokurtic and leptokurtic, thus PMaD may be used to model skewed and symmetric data as well.

The mode $(M_0)$ for PMaD $(\alpha, \beta)$ is obtained by solving the following expression $\frac{d}{dx} f(x, \alpha, \beta)|_{M_0} = 0$, which yields

$$M_0 = \left(\frac{3\beta - 1}{2\alpha\beta}\right)^{\frac{1}{2\beta}}.$$

Moreover, the median $(M_d)$ of the proposed distribution can be calculated by using the empirical relation among the mean, median and mode. Thus, the median is,

$$M_d = \frac{1}{3}M_0 + \frac{2}{3}\mu_1^{'} = \frac{1}{3}\left[\left(\frac{3\beta - 1}{2\alpha\beta}\right)^{\frac{1}{2\beta}} + \frac{4}{\sqrt{\pi}}\left(\frac{1}{\alpha}\right)^{\frac{1}{2\beta}} \Gamma\left(\frac{3\beta + 1}{2\beta}\right)\right].$$

The moment generating function (mgf) $M_X(t)$ for a PMaD random variable $X$ is obtained as

$$M_X(t) = E(e^{tx}) = \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{1}{j!}\left(\frac{t}{\alpha^{2\beta}}\right)^r \Gamma\left(\frac{3\beta + r}{2\beta}\right).$$

**Table 1:** *Values of mean, variance, skewness, kurtosis, mode and coefficient of variation for different α, β*

| $\alpha,\ \beta$ | $\mu_1'$ | $\mu_2$ | $\beta_1$ | $\beta_2$ | $x_0$ | CV |
|---|---|---|---|---|---|---|
| | when $\alpha$ fixed and $\beta$ varying | | | | | |
| 0.5, 0.5 | 3.0008 | 5.9992 | 2.6675 | 7.0010 | 1.0000 | 0.8162 |
| 0.5, 1.0 | 1.5962 | 0.4530 | 0.2384 | 3.1071 | 1.4142 | 0.4217 |
| 0.5, 1.5 | 1.3376 | 0.1499 | 0.0102 | 2.7882 | 1.3264 | 0.2894 |
| 0.5, 2.5 | 1.1780 | 0.0445 | 0.0481 | 2.7890 | 1.2106 | 0.1792 |
| 0.5, 3.5 | 1.1204 | 0.0211 | 0.1037 | 2.4351 | 1.1533 | 0.1298 |
| when $\beta$ fixed $\alpha$ varying | | | | | | |
| 0.5, 0.75 | 1.9392 | 1.1443 | 0.7425 | 3.8789 | 1.4057 | 0.5516 |
| 1.0, 0.75 | 1.2216 | 0.4541 | 0.7425 | 3.8789 | 0.8855 | 0.5516 |
| 1.5, 0.75 | 0.9323 | 0.2645 | 0.7425 | 3.8789 | 0.6758 | 0.5516 |
| 2.5, 0.75 | 0.6632 | 0.1338 | 0.7425 | 3.8789 | 0.4807 | 0.5516 |
| 3.5, 0.75 | 0.5299 | 0.0855 | 0.7425 | 3.8789 | 0.3841 | 0.5516 |
| when both varying | | | | | | |
| 1, 1 | 1.1287 | 0.2265 | 0.2384 | 3.1071 | 1.0000 | 0.4217 |
| 2, 2 | 0.8723 | 0.0372 | 0.0102 | 2.7895 | 0.8891 | 0.2212 |
| 3, 3 | 0.8484 | 0.0163 | 0.0831 | 2.6907 | 0.8736 | 0.1506 |
| 4, 4 | 0.8509 | 0.0094 | 0.1069 | 1.9643 | 0.8750 | 0.1140 |
| 5, 5 | 0.8586 | 0.0062 | 0.0677 | 0.1072 | 0.8805 | 0.0915 |

For lifetime distributions, the conditional moments are of interest in prediction. Another application of conditional moments is the mean residual life (MRL). For this purpose, let $X$ observed from PDF(2.1), the conditional moments, $E(X^r|X>k)$ and the conditional mgf $E(e^{tx}|X>k)$ are obtained as follows;

$$E(X^r|X>k)=\frac{\int_{x>k}x^r f(x,\alpha,\beta)dx}{\int_{x>k}f(x,\alpha,\beta)dx}=\frac{2\left(\frac{1}{\alpha}\right)^{\frac{r}{2\beta}}\Gamma\left(\frac{3\beta+r}{2\beta},\alpha k^{2\beta}\right)}{\sqrt{\pi}-2\Gamma\left(\frac{3}{2},\alpha k^{2\beta}\right)}$$

and

$$\begin{aligned}E(e^{tx}|X>k)&=\frac{\int_{x>k}e^{tx}f(x,\alpha,\beta)dx}{\int_{x>k}f(x,\alpha,\beta)dx}\\&=\frac{2\sum_{i=0}^{\infty}\frac{t^i}{i!}\left(\frac{1}{\alpha}\right)^{\frac{r}{2\beta}}\Gamma\left(\frac{3\beta+r}{2\beta},\alpha k^{2\beta}\right)}{\sqrt{\pi}-2\Gamma\left(\frac{3}{2},\alpha k^{2\beta}\right)},\end{aligned}$$

respectively. The MRL is the expected remaining life $X-x$, given that the equipment has survived to time $k$. The MRL function in terms of the first conditional moments is given as

$$m(x)=E[X-x|X>k]=\frac{2\left(\frac{1}{\alpha}\right)^{\frac{1}{2\beta}}\Gamma\left(\frac{3\beta+1}{2\beta},\alpha k^{2\beta}\right)}{\sqrt{\pi}-2\Gamma\left(\frac{3}{2},\alpha k^{2\beta}\right)}-x$$

## 3. Entropy measurements

In information theory, entropy measurement plays a vital role to study the uncertainty associated with the random variable. In this section, we discuss the different entropy measures for $PMaD$. For more detail about entropy measurement, see [10].

### 3.1. Renyi entropy

Renyi entropy of a r.v. $X$ with PDF (2.1) is given as

$$R_E = \frac{1}{(1-\in)} \ln \left[ \int_{x=0}^{\infty} \left\{ \frac{4}{\sqrt{\pi}} \alpha^{\frac{3}{2}} \beta x^{3\beta-1} e^{-\alpha x^{2\beta}} \right\}^{\in} dx \right]$$

Hence, after some algebra, we get

$$R_E = \frac{1}{(1-\in)} \left[ \lambda \ln 4 - \frac{\lambda}{2} \ln \pi + \lambda \ln \beta - \frac{1-\lambda-2\beta}{2\beta} \ln \alpha - \frac{3\lambda\beta-\lambda+1}{2\beta} \ln \lambda + \ln \left( \frac{3\beta\lambda-\lambda+1}{2\beta} \right) \right].$$

### 3.2. Δ-entropy

The $\Delta$ entropy is also known as $\beta$ entropy. The $\Delta$ entropy for a random variable $X$ having PDF (2.1) is defined as

$$\Delta_E = \frac{1}{\Delta-1} \left[ 1 - \int_{x=0}^{\infty} f^{\Delta}(x, \alpha, \beta) dx \right].$$

Using PDF (2.1) and after simplification, the expression for $\beta$-entropy is given by;

$$\Delta_E = \frac{1}{\Delta-1} \left[ 1 - \left( \frac{4}{\sqrt{\pi}} \right)^{\Delta} \beta^{\Delta} \left( \frac{1}{\alpha} \right)^{\frac{1-\Delta-2\beta}{2\beta}} \left( \frac{\Gamma\left( \frac{3\Delta\beta-\Delta+1}{2\beta} \right)}{\Delta^{\frac{3\Delta\beta-\Delta+1}{2\beta}}} \right) \right]. \quad (8)$$

### 3.3. Generalized entropy

The generalized entropy is defined by

$$G_E = \frac{\nu_\lambda \mu^{-\lambda} - 1}{\lambda(\lambda-1)} \quad ; \lambda \neq 0, 1,$$

where, $\nu_\lambda = \int_{x=0}^{\infty} x^{\lambda} f(x, \alpha, \theta) dx$ and $\mu = E(X)$. After some algebra, we get

$$G_E = \left( \frac{4}{\pi} \right)^{\frac{1-\lambda}{2}} \left[ \frac{\Gamma\left( \frac{3\beta+\lambda}{2\beta} \right) \left\{ \Gamma\left( \frac{3\beta+1}{2\beta} \right) \right\}^{-\lambda}}{\lambda(\lambda-1)} \right], \quad \lambda \neq 0, 1. \quad (9)$$

## 4. Parameter estimation with a simulation study

Here, we describe the maximum likelihood estimation method and Bayes estimation method for estimating the unknown parameters $\alpha, \beta$ of the PMaD. The estimators obtained under these methods are not in nice closed form; thus, numerical approximation techniques are used to get the solution. Further, the performances of these estimators are studied through a Monte Carlo simulation.

### 4.1. Maximum likelihood estimation

The most popular and efficient method of classical estimation of the parameter(s) is maximum likelihood estimation. The estimators obtained by this method passes several desirable properties such as consistency, efficiency etc. Let $X_1, X_2, \cdots, X_n$ be an iid random sample of size $n$ taken from PMaD $(\alpha, \beta)$, then the likelihood function is

$$L(\alpha,\theta) = \prod_{i=1}^{n} \frac{4}{\sqrt{\pi}} \alpha^{\frac{3}{2}} \beta x_i^{3\beta-1} e^{-\alpha x_i^{2\beta}} = \frac{4^n}{\pi^{n/2}} \alpha^{\frac{3n}{2}} \beta^n e^{-\alpha \sum_{i=1}^{n} x_i^{2\beta}} \left( \prod_{i=1}^{n} x_i^{3\beta-1} \right),$$

hence the corresponding log-likelihood function is written as

$$\ln L(\alpha,\theta) = l = n \ln 4 - \frac{n}{2} \ln \pi + \frac{3n}{2} \ln \alpha + n \ln \beta - \alpha \sum_{i=1}^{n} x_i^{2\beta} + (3\beta - 1) \sum_{i=1}^{n} \ln x_i. \tag{10}$$

The MLEs of $\alpha$ and $\beta$ are the solution of $\frac{\partial l}{\partial \alpha} = 0$ and $\frac{\partial l}{\partial \beta} = 0$, hence

$$\frac{3n}{2\alpha} - \sum_{i=1}^{n} x_i^{2\beta} = 0 \tag{11}$$

$$\frac{n}{\beta} - 2\alpha \sum_{i=1}^{n} x_i^{2\beta} \ln x_i + 3 \sum_{i=1}^{n} \ln x_i = 0. \tag{12}$$

The MLEs of the parameters are obtained by solving the two equations above simultaneously, and non-linear maximization techniques is used to get the solution.

#### 4.1.1 Uniqueness of MLEs

The uniqueness of the MLEs discussed in the previous section can be checked by using following propositions.

**Proposition 1:** If $\beta$ is fixed, then $\hat{\alpha}$ exists and is unique.

**Proof:** Let $L_\alpha = \frac{3n}{2\alpha} - \sum_{i=1}^{n} x_i^{2\beta}$, since $L_\alpha$ is continuous and it has been verified that $\lim_{\alpha \to 0} L_\alpha = \infty$ and $\lim_{\alpha \to \infty} L_\alpha = -\sum_{i=1}^{n} x_i^{2\beta} < 0$. This implies that $L_\alpha$ will have at least one root in interval $(0, \infty)$ and hence $L_\alpha$ is a decreasing function in $\alpha$. Thus, $L_\alpha = 0$ has a unique solution in $(0, \infty)$.

**Proposition 2:** If $\alpha$ is fixed, then $\hat{\beta}$ exists and is unique.

**Proof:** Let $L_\beta = \frac{n}{\beta} - \alpha \sum_{i=1}^{n} x_i^{2\beta} \ln x_i + 3 \sum_{i=1}^{n} \ln x_i$, since $L_\beta$ is continuous and it has been verified that $\lim_{\beta \to 0} L_\beta = \infty$ and $\lim_{\beta \to \infty} L_\beta = -2 \sum_{i=1}^{n} \ln x_i < 0$. This implies, as above, $\hat{\beta}$ exists and it is unique.

#### 4.1.2 Fisher Information Matrix

Here, we derive the Fisher information matrix for constructing $100(1 - \Psi)\%$ asymptotic confidence interval for the parameters using large sample theory. The Fisher information matrix can be obtained, by using equations (4.2) and (4.3), as

$$I(\hat{\alpha}, \hat{\beta}) = -E \begin{pmatrix} l_{\alpha\alpha} & l_{\alpha\beta} \\ l_{\beta\alpha} & l_{\beta\beta} \end{pmatrix}_{(\hat{\alpha}, \hat{\beta})} \tag{5.2.1}$$

where,

$$l_{\alpha\alpha} = -\frac{3n}{2\alpha^2}, \quad l_{\alpha\beta} = -2 \sum_{i=1}^{n} x_i^{2\beta} \ln x_i, \quad l_{\beta\beta} = -\frac{n}{\beta^2} - 4\alpha \sum_{i=1}^{n} x_i^{2\beta} (\ln x_i)^2.$$

The above matrix can be inverted and the diagonal elements of $I^{-1}(\hat{\alpha}, \hat{\beta})$ provide the asymptotic variance of $\alpha$ and $\beta$, respectively. Now, two sided $100(1-\Psi)\%$ asymptotic confidence interval for $\alpha$, $\beta$ can be obtained as

$$\alpha \in [\hat{\alpha} - Z_{1-\frac{\Psi}{2}}\sqrt{var(\hat{\alpha})}, \hat{\alpha} + Z_{1-\frac{\Psi}{2}}\sqrt{var(\hat{\alpha})}],$$

$$\beta \in [\hat{\beta} - Z_{1-\frac{\Psi}{2}}\sqrt{var(\hat{\beta})}, \hat{\beta} + Z_{1-\frac{\Psi}{2}}\sqrt{var(\hat{\beta})}],$$

respectively.

## 4.2. Bayes estimation

In this subsection, the Bayes estimation procedure for the PMaD parameters has been developed. Here, we consider two independent gamma priors for both shape and scale parameter. The considered prior is very flexible due to its flexibility of assuming different shape. Thus, the joint prior $g(\alpha, \beta)$ is given by;

$$g(\alpha, \beta) \propto \alpha^{a-1}\beta^{c-1}e^{-b\alpha - d\beta} \quad ; \quad \alpha, \beta > 0, \tag{13}$$

where $a$, $b$, $c$ and $d$ are the hyper-parameters of the considered priors. Using likelihood function of PMaD and equation above, the joint posterior density function $\pi(\alpha, \beta|x)$ is derived as

$$\begin{aligned} \pi(\alpha, \beta|x) &= \frac{L(x|\alpha, \beta)g(\alpha, \beta)}{\int_\alpha \int_\beta L(x|\alpha, \beta)g(\alpha, \beta)d\alpha\, d\beta} \\ &= \frac{\alpha^{\frac{3n}{2}+a-1}\beta^{n+c-1}e^{-\alpha\left(b+\sum_{i=1}^n x_i^{2\beta}\right)}e^{-d\beta}\left(\prod_{i=1}^n x_i^{3\beta-1}\right)}{\int_\alpha \int_\beta \alpha^{\frac{3n}{2}+a-1}\beta^{n+c-1}e^{-\alpha\left(b+\sum_{i=1}^n x_i^{2\beta}\right)}e^{-d\beta}\left(\prod_{i=1}^n x_i^{3\beta-1}\right) \quad d\alpha \quad d\beta}. \end{aligned} \tag{14}$$

In the Bayesian analysis, the specification of proper loss function plays an important role. We talk most frequently used the square error loss function (SELF) to obtain the estimators of the parameters, which defined as

$$L(\phi, \hat{\phi}) \propto \left(\phi - \hat{\phi}\right)^2, \tag{15}$$

where $\hat{\phi}$ is estimate of $\phi$. Bayes estimators under SELF is the posterior mean and evaluated by

$$\hat{\phi}_{SELF} = [E(\phi|x)], \tag{16}$$

provided the expectation exist and finite. Thus, the Bayes estimators based on equation no. (4.5) under SELF are given by

$$\hat{\alpha}_{bs} = E_{\alpha,\beta|x}(\alpha|\beta, x) = \eta^{-1}\int_\alpha \int_\beta \alpha^{\frac{3n}{2}+a}\beta^{n+c-1}e^{-\alpha\left(b+\sum_{i=1}^n x_i^{2\beta}\right)}e^{-d\beta}\left(\prod_{i=1}^n x_i^{3\beta-1}\right) d\alpha\, d\beta, \tag{17}$$

and

$$\hat{\beta}_{bs} = E_{\alpha,\beta|x}(\beta|\alpha, x) = \eta^{-1}\int_\alpha \int_\beta \alpha^{\frac{3n}{2}+a-1}\beta^{n+c}e^{-\alpha\left(b+\sum_{i=1}^n x_i^{2\beta}\right)}e^{-d\beta}\left(\prod_{i=1}^n x_i^{3\beta-1}\right) d\alpha\, d\beta, \tag{18}$$

where $\eta^{-1} = \int_\alpha \int_\beta \alpha^{\frac{3n}{2}+a-1}\beta^{n+c-1}e^{-\alpha\left(b+\sum_{i=1}^n x_i^{2\beta}\right)}e^{-d\beta}\left(\prod_{i=1}^n x_i^{3\beta-1}\right) d\alpha\, d\beta.$

From equations (4.8) and (4.9), it is easy to observe that the posterior expectations are appearing in the form of the ratio of two integrals. Thus, the analytical solution of these expectations are not presumable. Therefore, any numerical approximation techniques may be implemented to secure the solutions. Here, we used one of the most popular and quite effective approximation technique suggested by [11]. The detailed description is as follows.

$$(\hat{\alpha},\hat{\beta})_{Bayes} = \frac{\int_\alpha \int_\beta u(\alpha,\beta) e^{\rho(\alpha,\beta)+l} \quad d\alpha d\beta}{\int_\alpha \int_\beta e^{\rho(\alpha,\beta)+l} \quad d\alpha d\beta} \tag{19}$$

$$\begin{aligned} &= (\hat{\alpha},\hat{\beta})_{ml} + \frac{1}{2}[(u_{\alpha\alpha}+2u_\alpha\rho_\alpha)\tau_{\alpha\alpha} + (u_{\alpha\beta}+2u_\alpha\rho_\beta)\tau_{\alpha\beta} + (u_{\beta\alpha}+2u_\beta\rho_\alpha)\tau_{\beta\alpha} \\ &+ (u_{\beta\beta}+2u_\beta\rho_\beta)\tau_{\beta\beta}] + \frac{\alpha}{\beta}[(u_\alpha\tau_{\alpha\alpha}+u_\beta\tau_{\alpha\beta})(l_{111}\tau_{\alpha\alpha}+2l_{21}\tau_{\alpha\beta}+l_{12}\tau_{\beta\beta}) \\ &+ (u_\alpha\tau_{\beta\alpha}+u_\beta\tau_{\beta\beta})(l_{21}\tau_{\alpha\alpha}+2l_{12}\tau_{\beta\alpha}+l_{222}\tau_{\beta\beta})], \end{aligned} \tag{20}$$

where $u(\alpha,\beta)=(\alpha,\beta)$, $\rho(\alpha,\beta)=\ln g(\alpha,\beta)$ and $l=\ln L(\alpha,\beta|\underline{x})$,

$$l_{ab} = \frac{\partial^3 l}{\partial\alpha^a\partial\beta^b}, \quad a,b=0,1,2,3 \quad a+b=3, \quad \rho_\alpha=\frac{\partial\rho}{\partial\alpha}, \quad \rho_\beta=\frac{\partial\rho}{\partial\beta}$$

$$u_\alpha = \frac{\partial u}{\partial\alpha}, \quad u_\beta=\frac{\partial u}{\partial\beta}, \quad u_{\alpha\alpha}=\frac{\partial^2 u}{\partial\alpha^2}, \quad u_{\beta\beta}=\frac{\partial^2 u}{\partial\beta^2}, \quad u_{\alpha\beta}=\frac{\partial^2 u}{\partial\alpha\partial\beta},$$

$$\tau_{\alpha\alpha}=\frac{1}{l_{20}},\ \tau_{\alpha\beta}=\frac{1}{l_{11}}=\tau_{\beta\alpha},\ \tau_{\beta\beta}=\frac{1}{l_{02}}.$$

Since $u(\alpha,\beta)$ is the function of $\alpha,\beta$,

- If $u(\alpha,\beta)=\alpha$ in (4.11), then

$$u_\alpha=1, \quad u_\beta=0, \quad u_{\alpha\alpha}=u_{\beta\beta}=0, \quad u_{\alpha\beta}=u_{\beta\alpha}=0.$$

- If $u(\alpha,\beta)=\beta$ in (4.11), then

$$u_\beta=1, \quad u_\alpha=0, \quad u_{\alpha\alpha}=u_{\beta\beta}=0, \quad u_{\alpha\beta}=u_{\beta\alpha}=0,$$

and the rest derivatives based on likelihood function are obtained as

$$l_{30}=\frac{3n}{\alpha^3},\ \ l_{11}=-2\sum_{i=1}^{n} x_i^{2\beta}\ln x_i,\ \ l_{03}=\frac{2n}{\beta^3}-8\alpha\sum_{i=1}^{n} x_i^{2\beta}(\ln x_i)^3$$

$$l_{12}=-4\sum_{i=1}^{n} x_i^{2\beta}(\ln x_i)^2=l_{21}.$$

Using these derivatives the Bayes estimators of $(\alpha,\beta)$ are obtained by expressions

$$\begin{aligned} \hat{\alpha}_{bl} =&\hat{\alpha}_{ml} + \frac{1}{2}[(2u_\alpha\rho_\alpha)\tau_{\alpha\alpha} + (2u_\alpha\rho_\beta)\tau_{\alpha\beta}] + \frac{1}{2}[(u_\alpha\tau_{\alpha\alpha})(l_{30}\tau_{\alpha\alpha}+2l_{21}\tau_{\alpha\beta}+l_{12}\tau_{\beta\beta}) \\ &+ (u_\alpha\tau_{\beta\alpha})(l_{21}\tau_{\alpha\alpha}+2l_{12}\tau_{\beta\alpha}+l_{03}\tau_{\beta\beta})], \end{aligned} \tag{21}$$

$$\begin{aligned} \hat{\beta}_{bl} = &\hat{\beta}_{ml} + \frac{1}{2}[(2u_\beta\rho_\alpha)\tau_{\beta\alpha} + (2u_\beta\rho_\beta)\tau_{\beta\beta}] + \frac{1}{2}[(u_\beta\tau_{\alpha\beta})(l_{30}\tau_{\alpha\alpha}+2l_{21}\tau_{\alpha\beta}+l_{12}\tau_{\beta\beta}) \\ &+ (u_\beta\tau_{\beta\beta})(l_{21}\tau_{\alpha\alpha}+2l_{12}\tau_{\beta\alpha}+l_{03}\tau_{\beta\beta})]. \end{aligned} \tag{22}$$

## 4.3. Simulation study

In this section, a Monte Carlo simulation study has been performed to assess the performance of the obtained estimators in terms of their mean square errors (MSEs). The MLEs of the parameters are evaluated by using $nlm()$ function, and also the MLEs of reliability characteristics are obtained by using invariance properties. The Bayes estimates of the parameters are evaluated by Lindley's approximation technique. The hyper-parameters values are chosen in such a way that the prior mean is equal to the true value, and prior variance is taken as very small, say 0.5. All the computations are done by $R3.4.1$ software. At first, we generated 5000 random samples from the PMaD $(\alpha, \beta)$ using the Newton-Raphson algorithm for different variation of sample sizes as $n = 10$ (small), $n = 20, 30$ (moderate), $n = 50$ (large) for fixed $(\alpha = 0.75, \beta = 0.75)$ and secondly for different variation of $(\alpha, \beta)$ when sample size is fixed $(n = 20)$, respectively. Average estimates and mean square error (MSE) of the parameters are calculated for the above mentioned choices, and the corresponding results are reported in Table 2. The asymptotic confidence interval (ACI) and asymptotic confidence length (ACL) are also obtained and presented in Table 3. From this simulation study, it has been observed that the precision of MLEs and Bayes estimator are increasing when the sample size is increasing while average ACL is decreasing. The Bayes estimates under informative prior is more precise as compared to the MLEs especially for small sample sizes while for large sample the precision of the estimators is almost same for all the considered parametric choices.

**Table 2:** *Average estimates and mean square errors (in each second row) of the parameters and reliability characteristics based on simulated data.*

| n | $\alpha, \beta$ | $\alpha_{ml}$ | $\beta_{ml}$ | $M(t)_{ml}$ | $R(t)_{ml}$ | $H(t)_{ml}$ | $\alpha_{bl}$ | $\beta_{bl}$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 0.75,0.75 | 0.5070 | 1.1598 | 1.5119 | 0.9691 | 0.1663 | 0.5063 | 1.1028 |
| | | 0.0631 | 0.2588 | 0.0164 | 0.0049 | 0.0947 | 0.0631 | 0.2027 |
| 20 | | 0.6560 | 0.8848 | 1.4922 | 0.9343 | 0.2965 | 0.6521 | 0.8647 |
| | | 0.0098 | 0.0326 | 0.0093 | 0.0014 | 0.0703 | 0.0105 | 0.0263 |
| 30 | | 0.7096 | 0.8064 | 1.4883 | 0.9163 | 0.3504 | 0.7058 | 0.7951 |
| | | 0.0022 | 0.0103 | 0.0071 | 0.0004 | 0.0010 | 0.0025 | 0.0087 |
| 50 | | 0.7542 | 0.7453 | 1.4869 | 0.8988 | 0.3968 | 0.7514 | 0.7397 |
| | | 0.0003 | 0.0031 | 0.0046 | 0.0001 | 0.0003 | 0.0003 | 0.0031 |
| for fixed $n$ and different $\alpha, \beta$ | | | | | | | | |
| 20 | 0.5,0.75 | 0.6603 | 0.6832 | 1.7380 | 0.9044 | 0.3400 | 0.6574 | 0.6716 |
| | | 0.0261 | 0.0125 | 0.0585 | 0.0017 | 0.0099 | 0.0252 | 0.0117 |
| | 0.5, 1.5 | 0.7290 | 0.3033 | 4.6222 | 0.7871 | 0.3556 | 0.7258 | 0.3229 |
| | | 0.0528 | 1.4330 | 11.9171 | 0.0402 | 0.1139 | 0.0513 | 1.3866 |
| | 1.5, 0.5 | 0.5090 | 2.9297 | 1.1531 | 0.9983 | 0.0207 | 0.5517 | 2.8634 |
| | | 0.9907 | 6.6465 | 0.0242 | 0.1274 | 26.0695 | 0.9087 | 6.3006 |
| | 2.5,2.5 | 1.0448 | 0.5958 | 1.4084 | 0.7953 | 0.6393 | 1.2825 | 0.6727 |
| | | 2.1402 | 3.6573 | 0.3860 | 0.0373 | 0.3553 | 1.5058 | 3.3715 |

**Table 3:** *Interval estimates and asymptotic confidence length (ACL) of the parameters.*

| n | $\alpha, \beta$ | $\alpha_L$ | $\alpha_U$ | $ACL_\alpha$ | $\beta_L$ | $\beta_U$ | $ACl_\beta$ |
|---|---|---|---|---|---|---|---|
| 10 | 0.75,0.75 | 0.0874 | 0.9266 | 0.8393 | 0.5711 | 1.7485 | 1.1775 |
| 20 | 0.75,0.75 | 0.3209 | 0.9911 | 0.6703 | 0.5525 | 1.2171 | 0.6646 |
| 30 | 0.75,0.75 | 0.4263 | 0.9928 | 0.5665 | 0.5555 | 1.0574 | 0.5019 |
| 50 | 0.75,0.75 | 0.5290 | 0.9794 | 0.4505 | 0.5631 | 0.9275 | 0.3644 |
| for fixed n and different $\alpha, \beta$ | | | | | | | |
| 20 | 0.5, 0.75 | 0.3255 | 0.9951 | 0.6696 | 0.4142 | 0.9523 | 0.5381 |
| | 0.5, 1.5 | 0.3794 | 1.0785 | 0.6991 | 0.4819 | 1.7425 | 1.2429 |
| | 1.5, 0.5 | 0.4206 | 1.7812 | 0.76058 | 0.2260 | 1.8334 | 1.3807 |
| | 2.5, 2.5 | 0.5804 | 2.9509 | 0.9788 | 0.54133 | 2.7783 | 1.1365 |

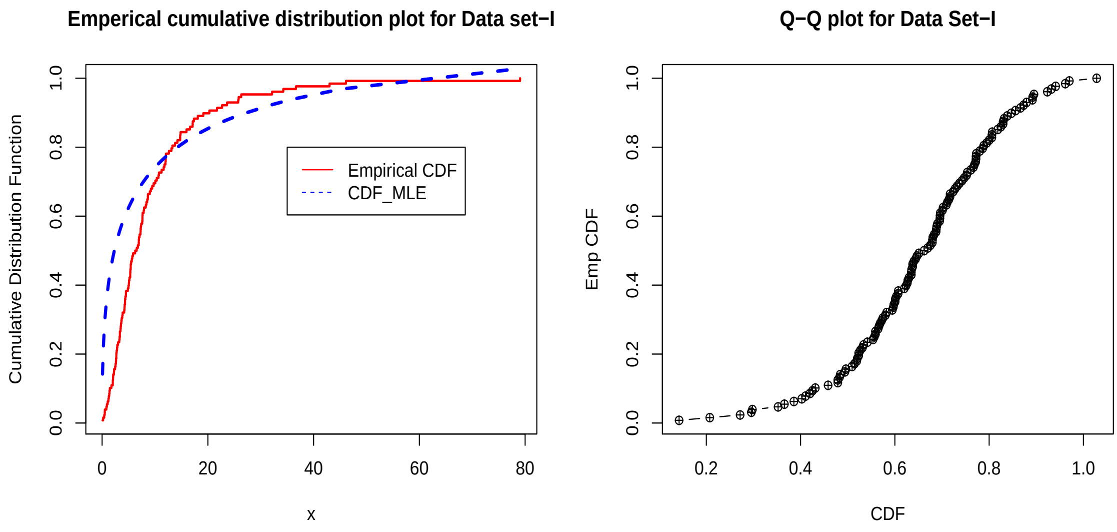


**Figure 2:** *Empirical cumulative distribution function and QQ plot for the data set-I.*

## 5. Practical data modeling

This section demonstrates the practical applicability of the proposed model in real-life scenario, especially for the survival/reliability data taken from different sources. The proposed distribution is compared with Maxwell distribution (MaD) and its different generalizations, such as, length biased Maxwell distribution (LBMaD), see [9], area biased Maxwell distribution (ABMaD), see [9], extended Maxwell distribution (EMaD), see [8] and generalized Maxwell distribution (GMaD), see [2]. For these models the estimates of the parameter(s) are obtained by method of maximum likelihood and the compatibility of PMaD has been discussed using model selection tools (which depend on the MLE) such as log-likelihood (-log L), Akaike information criterion (AIC), corrected Akaike information criterion (AICC), Bayesian information criterion (BIC) and Kolmogorov Smirnov (K-S) test. In general, the smaller values of these statistics indicate the better fit to the data.

The data sets description is as follows.

**Data Set-I (Bladder cancer data):** This data set represents the remission times (in months) of a 128 bladder cancer patients, and it was initially used by [12]. The same data set is used to show the superiority of extended Maxwell distribution by [8].

**Data Set-II (Item failure data)**: This data set is taken from [13]. It shows 50 items put into use at initial time $t = 0$ and failure items recorded in weeks.

**Data Set-III (Airborne communication transceiver):** The data set was initially considered by [14]. It represent the 46 repair times (in hours) for an airborne communication transceiver.

**Data Set-IV (Flood data)**. The data are the exceedances of flood peaks (in $m^3/s$) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958-1984, rounded to one decimal place. This data set was analyzed by [16].

Summary of the considered data sets is given in Table 5 and it can be seen that skewness is positive for all data sets which indicates that they have positive skewness which appropriately suited to the proposed model. This table also shows platykurtic, mesokurtic and leptokurtic nature of the data, which proves again the suitability of the proposed model to the data.

**Table 4:** *Goodness of fit values for different model.*

| Bladder cancer data N=128 | | | | | | | |
|---|---|---|---|---|---|---|---|
| Model | $\hat{\alpha}$ | $\hat{\beta}$ | -logL | AIC | AICC | BIC | K-S |
| **PMaD** | **0.7978** | **0.1637** | **366.3820** | **736.7639** | **732.8599** | **742.4680** | **0.3675** |
| MaD | 0.0076 | – | 1014.4440 | 2030.8870 | 2028.9190 | 2033.7400 | 0.4144 |
| LBMaD | 98.6386 | – | 669.3668 | 1340.7340 | 1338.7650 | 1343.5860 | 0.4906 |
| ABMaD | 78.9109 | – | 767.8122 | 1537.6240 | 1535.6560 | 1540.4770 | 0.5608 |
| ExMaD | 0.8447 | 1.4431 | 412.1232 | 828.2464 | 824.3424 | 833.9504 | 0.8265 |
| GMaD | 0.7484 | 527.2314 | 426.6019 | 857.2037 | 853.2997 | 862.9078 | 0.7086 |
| Item failure data N=50 | | | | | | | |
| Model | $\hat{\alpha}$ | $\hat{\beta}$ | -logL | AIC | AICC | BIC | K-S |
| **PMaD** | **0.8339** | **0.1820** | **135.8204** | **275.6407** | **271.8961** | **279.4648** | **0.2625** |
| MaD | 0.0104 | – | 367.8528 | 737.7056 | 735.7890 | 739.6177 | 0.4268 |
| LBMaD | 72.1146 | – | 315.1624 | 632.3248 | 630.4081 | 634.2368 | 0.5112 |
| ABMaD | 57.6917 | – | 374.1247 | 750.2494 | 748.3328 | 752.1615 | 0.5825 |
| ExMaD | 0.6186 | 1.0139 | 151.2998 | 306.5996 | 302.8550 | 310.4237 | 0.7327 |
| GMaD | 0.5400 | 534.1569 | 151.2643 | 306.5287 | 302.7840 | 310.3527 | 0.3920 |
| Airborne communication transceiver N=46 | | | | | | | |
| Model | $\hat{\alpha}$ | $\hat{\beta}$ | - logL | AIC | AICC | BIC | K-S |
| **PMaD** | **0.8735** | **0.2709** | **101.9125** | **207.8249** | **204.1040** | **211.4822** | **0.2136** |
| MaD | 0.0406 | – | 245.1383 | 492.2766 | 490.3675 | 494.1052 | 0.5027 |
| LBMaD | 18.4603 | – | 237.4945 | 476.9890 | 475.0799 | 478.8176 | 0.5771 |
| ABMaD | 14.7683 | – | 284.7017 | 571.4034 | 569.4943 | 573.2320 | 0.6324 |
| ExMaD | 0.7290 | 0.8672 | 103.3052 | 210.6104 | 206.8895 | 214.2677 | 0.2989 |
| GMaD | 0.6015 | 122.7666 | 110.8521 | 225.7042 | 221.9833 | 229.3615 | 0.4392 |
| River data N=72 | | | | | | | |
| Model | $\hat{\alpha}$ | $\hat{\beta}$ | - logL | AIC | AICC | BIC | K-S |
| **PMaD** | 0.805185 | 0.1504145 | 212.8942 | 429.7884 | 425.9623 | 434.3418 | 0.2760 |
| MaD | 0.005032 | – | 610.9235 | 1223.847 | 1221.904 | 1226.124 | 0.3821 |
| LBMaD | 149.0315 | – | 426.3076 | 854.6153 | 852.6724 | 856.8919 | 0.4113 |
| ABMaD | 119.2252 | – | 493.3271 | 988.6543 | 986.7114 | 990.9309 | 0.4529 |
| ExMaD | 0.697471 | 1.306933 | 251.9244 | 507.8487 | 504.0226 | 512.4021 | 0.7487 |
| GMaD | 0.648149 | 919.7356 | 251.2767 | 506.5534 | 502.7273 | 511.1068 | 0.4998 |

**Table 5:** *Summary of the data sets.*

| Data | Min | Q1 | Q2 | Mean | Q3 | Max | Kurtosis | Skewness |
|---|---|---|---|---|---|---|---|---|
| I | 0.080 | 3.348 | 6.395 | 9.366 | 11.838 | 79.050 | 18.483 | 3.287 |
| II | 0.013 | 1.390 | 5.320 | 7.821 | 10.043 | 48.105 | 9.408 | 2.306 |
| III | 0.200 | 0.800 | 1.750 | 3.607 | 4.375 | 24.500 | 11.803 | 2.888 |
| IV | 0.100 | 2.125 | 9.500 | 12.204 | 20.125 | 64.000 | 5.890 | 1.473 |

**Table 6:** *ML and Bayes estimates of the four data sets.*

| Data | $\alpha_{ml}$ | $\beta_{ml}$ | $\alpha_{bl}$ | $\beta_{bl}$ |
|---|---|---|---|---|
| I | 0.7978 | 0.1637 | 0.7962 | 0.1639 |
| II | 0.8339 | 0.1820 | 0.8292 | 0.1821 |
| III | 0.8735 | 0.2709 | 0.8675 | 0.2703 |
| IV | 0.8052 | 0.1504 | 0.8023 | 0.1506 |

**Table 7:** *Interval estimates based on the four data sets.*

| Data | $\alpha_L$ | $\alpha_U$ | $ACL_\alpha$ | $\beta_L$ | $\beta_U$ | $ACL_\beta$ |
|---|---|---|---|---|---|---|
| I | 0.6545 | 0.9411 | 0.2866 | 0.1373 | 0.1902 | 0.0529 |
| II | 0.5962 | 1.0717 | 0.4754 | 0.1376 | 0.2263 | 0.0888 |
| III | 0.6202 | 1.1269 | 0.5067 | 0.2081 | 0.3337 | 0.1256 |
| IV | 0.6126 | 0.9978 | 0.3852 | 0.1186 | 0.1822 | 0.0636 |

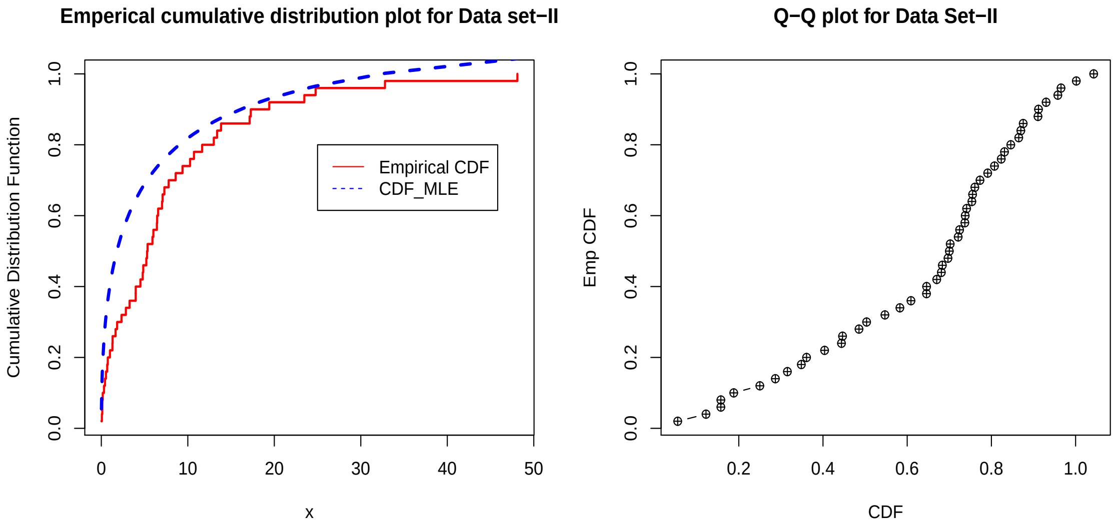


**Figure 3:** *Empirical cumulative distribution function and QQ plot for the data set-II.*

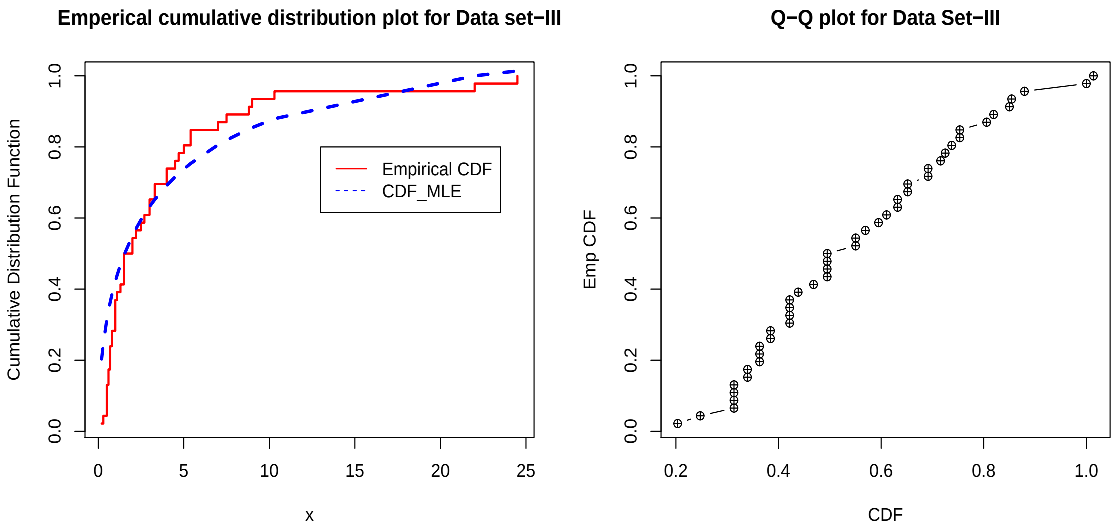


**Figure 4:** *Empirical cumulative distribution function and QQ plot for the data set-III.*

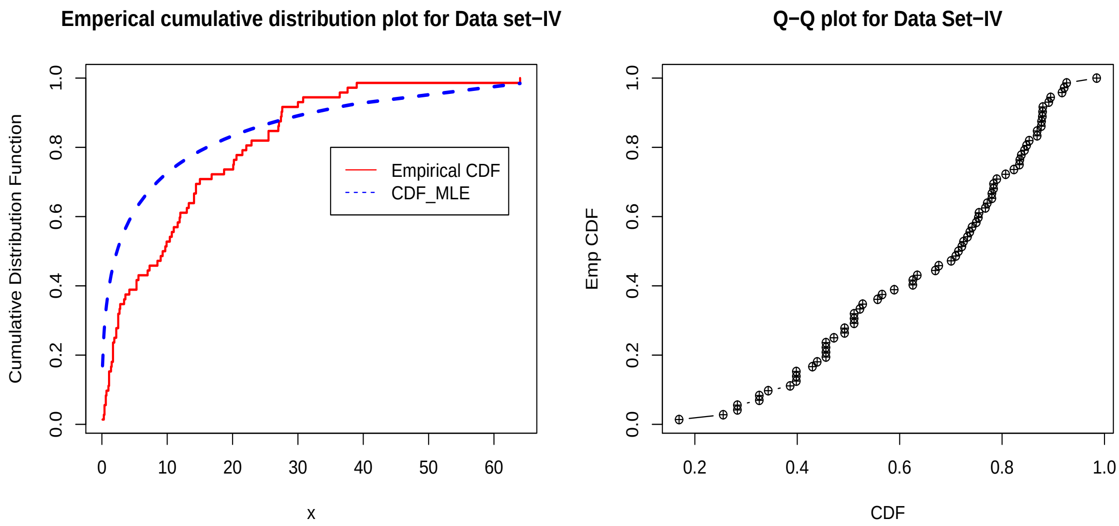


**Figure 5:** *Empirical cumulative distribution function and QQ plot for the data set-IV.*

From Table 4, it is clear that the proposed model (PMaD) has least value of the model selection tools, which reflects the merit of PMaD for modeling such four practical data sets than the the existing versions of the Maxwell distributions. The empirical cumulative distribution function (ECDF) plots and corresponding QQ plots for all the considered data set are plotted for $PMaD$, see Figures 2-5. From ECDF and QQ plots, it is clear that the considered data sets are adequately fitted to the proposed model. The point (ML and Bayes) estimates of the parameters for each data set are reported in Table 6. The Bayes estimates are calculated under non-informative prior, and it is observed that the obtained estimates (ML and Bayes) are almost same. The interval estimate of the parameter and corresponding asymptotic confidence length are also evaluated and presented in Table 7. This table shows that as the size of the data increases, the length of the interval is decreases, because it decreases the standard error, which support to our simulation part.

## 6. Conclusion

This article proposed the power Maxwell distribution (PMaD) as a flexible extension of the Maxwell distribution and studied some of its main properties for data modeling. We also study the skewness and kurtosis of the PMaD and found that it is capable of modeling the positively skewed as well as symmetric data. The unknown parameters of the PMaD are estimated by the maximum likelihood estimation (MLE) and Bayes estimation methods. The MLEs of the reliability function and hazard function are also obtained by using the invariance property. The 95% asymptotic confidence interval for the parameters are constructed using Fisher information matrix. The MLEs and Bayes estimators are compared through the Monte Carlo simulation and observed that Bayes estimators are more precise under informative prior. Finally, medical/reliability data have been used to show practical utility of the PMaD, and it is observed that it provides the better fit comparing to other versions of the Maxwell distributions. Thus, it can be recommended as an alternative model for the non-monotone failure rate models.

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