Reliability analysis for a class of exponential distribution based on progressive first-failure censoring

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Abstract

Based on progressively first-failure censored data, the problem of estimating parameters as well as reliability and hazard rate functions for a class of an exponential distribution is considered. The classic and Bayes approaches are used to estimate the parameters. The maximum likelihood estimates and exact confidence interval as well as exact confidence region for parameters are developed based on this censoring scheme. Also, when the parameters have discrete and continuous priors, several Bayes estimators with respect to different symmetric and asymmetric loss functions such as squared error, linear-exponential (LINEX) and general entropy are derived. Finally two numerical examples are presented to illustrate the methods of inference developed in this paper.

Keywords: Bayes estimator, Confidence region, Exponential distribution, Maximum likelihood estimator, Loss function, Progressive first-failure censoring scheme

1. Introduction

In many life test studies, it is common that the lifetimes of the test units may not be able to record exactly. Censoring is very common in reliability data analysis, in the past several decades. It usually applies when the exact lifetimes are known for only a portion of the products and the remainder of the lifetimes has only partial information. There are several types of censoring schemes in survival analysis and the type-II censoring scheme is one of the most common for consideration. In type-II censoring, the test terminates after a predetermined number of failure occurs in order to save time or cost, but the conventional type-II censoring scheme does not has the flexibility of allowing removal of units at points other than the terminal point of the experiment. For this reason, a more general censoring scheme called progressive type-II right censoring is proposed. Although, progressive censoring scheme was introduced long ago in the statistical literature, in recent years the progressive censoring scheme has received considerable attention in the statistical literature, see for instance [1], [2] and [3]. For an exhaustive list of references and further details on progressive censoring, readers are referred to [4]. In some cases, the lifetime of products is quite long and so the experimental time of the progressive type-II censoring scheme can still be too long. In order to give an efficient experiment, the other test methods are proposed by statisticians where one of them is the progressive first-failure censoring scheme. It can be described as follows.

Suppose that n independent groups with k items within each group are put on a life test and experimenter decides beforehand the quantity m, the number of units to be failed. At the time of the first failure, $X^{\mathbf{r}}_{1;m,n,k}$, r_1 groups and the group in which the first failure is observed are randomly removed. r_2 groups and the group with observed failure are randomly removed as soon as the second failure, $X^{\mathbf{r}}_{2;m,n,k}$, has occurred. The procedure is continued until all r_m groups and the group with observed failure are removed at the time of the m-th failure, $X^{\mathbf{r}}_{m;m:k:n}$. Then $X^{\mathbf{r}}_{1;m,n,k} < X^{\mathbf{r}}_{2;m,n,k} < \ldots < X^{\mathbf{r}}_{m;m,n,k}$ are called progressively first-failure censored order statistics with the censoring scheme $\mathbf{r} = (r_1, r_2, \ldots, r_m)$. We notice that if $k = 1$, progressively first-failure censored reduces to the progressive type-II censoring. Also, if $k = 1$ and $r_1 = r_2 = \cdots = r_{m-1} = 0, r_m = n - m$, it reduce to the type-II censoring. Wu et al. [5] and Wu and Yu [6] obtained the maximum likelihood estimates (MLEs), exact confidence intervals and exact confidence regions for the parameters of the Gompertz and Burr type XII distributions based on first-failure censored sampling, respectively. Wu and Kuş [7] studied the Weibull distribution under progressive first-failure censoring to make some classical inference on the parameters of a Weibull distribution and they proved that the progressive first-failure censoring scheme had shorter expected test times than the progressive type-II censoring scheme. Abou-Elheggag [8] studied the Rayleigh distribution under progressive first-failure censoring. He derived the maximum likelihood estimates and Bayes estimates of scale parameter, survival and hazard rate functions. Also, one can refer to [9], [10], [11],[12] and [13].

To simplify the notation, we will use X_i in place of $X^{\mathbf{r}}_{i;m,n,k}$. Let $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ be a progressive first-failure censored sample from a continuous population with the cumulative distribution function (CDF), $F(x)$, the probability density function (PDF), $f(x)$, and $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ is an observed value of $\mathbf{X}$. The joint pdf of $\mathbf{X}$ is given by [7] as follows

$$f_{1,2,\ldots,m}(\mathbf{x}) = Ak^m \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{k(r_i+1)-1}, \quad 0 < x_1 < x_2 < \ldots < x_m < \infty, \tag{1}$$

where $A = n(n - r_1 - 1)(n - r_1 - r_2 - 2) \cdots (n - r_1 - r_2 - \cdots - r_{m-1} - m + 1)$.

In this paper, our main object is to study the classical and Bayes estimation procedures for the parameter(s) of a general class of exponential- type distribution based on a progressively first-failure censored sample.

The rest of this paper is organized as follows. In Section 2, the model is described. Some classical estimation, such as maximum likelihood estimation and interval estimation are presented in Section 3. Section 4 develops the Bayes estimators for different loss functions such as squared error, LINEX and general entropy. One illustrative example and a simulation study via a Monte Carlo method are conducted in Section 5. Finally, we conclude the paper in Section 6.

2. Model description

Suppose the lifetime random variable T has a continuous distribution with two parameters as α and λ, and with the PDF and CDF as

$$f(t; \alpha, \lambda) = \alpha\psi(t; \lambda) \exp\{-\alpha\Psi(t; \lambda)\}, \qquad 0 < t < \infty, \tag{2}$$

$$F(t; \alpha, \lambda) = 1 - \exp\{-\alpha\Psi(t; \lambda)\}, \tag{3}$$

where $\psi(t; \lambda) = \frac{\partial\Psi(t;\lambda)}{\partial t}$, $\Psi(t; \lambda)$ is increasing in t with $\Psi(0; \lambda) = 0$ and $\Psi(\infty; \lambda) = \infty$. The corresponding reliability and hazard rate functions becomes:

$$R(t) = \exp\{-\alpha\Psi(t; \lambda)\}, \tag{4}$$

$$h(t) = \alpha\psi(t; \lambda), \tag{5}$$

respectively. This general form for lifetime model including some well-known and useful models such as Burr XII distribution with $\Psi(t; \lambda) = \ln(1 + t^\lambda)$, Gompertz distribution with $\Psi(t; \lambda) = \frac{e^{\lambda t}-1}{\lambda}$, Weibull distribution with $\Psi(t; \lambda) = t^\lambda$, two parameters bathtub-shaped lifetime distribution (see [14]) with $\Psi(t; \lambda) = e^{t^\lambda} - 1$ and so on. For more details, we refer the reader to [15].

3. Classical estimation

In this section, we consider the maximum likelihood estimation and interval estimation for the unknown parameters when the data are progressively first-failure censored.

3.1. Point estimation

Let $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ be a progressive first-failure censored sample from (2), with censoring scheme $(r_1, r_2, \ldots, r_m)$. From (1) the likelihood function is given by

$$L(\alpha, \lambda; \mathbf{x}) = Ak^m \alpha^m \exp\{-\alpha k \sum_{i=1}^{m} (r_i + 1)\Psi(x_i; \lambda)\} \prod_{i=1}^{m} \psi(x_i; \lambda). \quad (6)$$

By setting the derivatives of the log-likelihood function with respective to α or λ to zero, the MLE of λ, say $\hat{\lambda}$, is the solution to the following likelihood equation

$$\sum_{i=1}^{m} \frac{(\partial/\partial\lambda)\psi(x_i; \lambda)}{\psi(x_i; \lambda)} = \frac{m \sum_{i=1}^{m} (r_i + 1)(\partial/\partial\lambda)\Psi(x_i; \lambda)}{\sum_{i=1}^{m} (r_i + 1)\Psi(x_i; \lambda)}, \quad (7)$$

and the MLE of α, say $\hat{\alpha}$, can be obtained as

$$\hat{\alpha} = \frac{m}{k \sum_{i=1}^{m} (r_i + 1)\Psi(x_i; \hat{\lambda})}. \quad (8)$$

It is not easy to solve the equation(7) analytically in order to achieve the MLE of λ. Some numerical methods can be employed such as the Newton-Raphson method. Finally, using the invariance property, the MLEs of $R(t)$ and $h(t)$ are obtained as

$$\hat{R}(t) = \exp\{-\hat{\alpha}\Psi(t; \hat{\lambda})\}, \;\; t > 0,$$

and

$$\hat{h}(t) = \hat{\alpha}\psi(t; \hat{\lambda}), \;\; t > 0,$$

respectively.

3.2. Interval estimation

Let $Y_i = k\alpha\Psi(X_i; \lambda)$ for $i = 1, 2, \ldots, m$. It can be seen that $Y_1 < Y_2 < \ldots < Y_m$, are the progressive first-failure censored order statistics from an exponential distribution with mean 1. Consider the following transformation:

$$Z_1 = nY_1,$$
$$Z_i = (n - r_1 - r_2 - \cdots - r_{i-1} - i + 1)(Y_i - Y_{i-1}), \quad i = 2, 3, \ldots, m.$$

The generalized spacings $Z_1, Z_2, \ldots, Z_m$ are independent and identically distributed as an exponential distribution with mean 1 (see [1]). Hence, for $j = 1, 2, \ldots, m-1$,

$$\tau_j = 2\sum_{i=1}^{j} Z_i = 2k\alpha[\sum_{i=1}^{j} (r_i + 1)\Psi(X_i; \lambda) + \sum_{i=j+1}^{m} (r_i + 1)\Psi(X_j, \lambda)] \quad (9)$$

and

$$\gamma_j = 2\sum_{i=j+1}^{m} Z_i = 2k\alpha \sum_{i=j+1}^{m} (r_i + 1)[\Psi(X_i; \lambda) - \Psi(X_j; \lambda)] \quad (10)$$

are independently Chi-squared distributed with $2j$ and $2(m-j)$ degrees of freedom, respectively. We consider the following pivotal quantities:

$$\eta_j = \frac{j}{m-j} . \frac{\sum_{i=j+1}^{m} (r_i + 1)(\Psi(X_i; \lambda) - \Psi(X_j; \lambda))}{\sum_{i=1}^{j} (r_i + 1)\Psi(X_i; \lambda) + \sum_{i=j+1}^{m} (r_i + 1)\Psi(X_j; \lambda)}, \;\; j = 1, 2, \ldots, m-1, \quad (11)$$

$$\xi = 2k\alpha \sum_{i=1}^{m} (r_i + 1)\Psi(X_i; \lambda). \quad (12)$$

It is clearly that η_j has a F distribution with $2(m-j)$ and $2j$ degrees of freedom and ξ has a Chi-squared distribution with $2m$ degree of freedom. Meanwhile, η_j and ξ are independent. To construct an exact confidence interval for λ and the joint confidence region for the parameters α and λ, we need the following lemma.

Lemma 1. Suppose that for $x_1 < x_2 < \cdots < x_m$,

$$w_j(\lambda) = \frac{\sum_{i=j+1}^{m}(r_i+1)(\Psi(x_i;\lambda) - \Psi(x_j;\lambda))}{\sum_{i=1}^{j}(r_i+1)\Psi(x_i;\lambda) + \sum_{i=j+1}^{m}(r_i+1)\Psi(x_j;\lambda)}, \quad j = 1, 2, \ldots, m-1. \tag{13}$$

Then $w_j(\lambda)$ is strictly increasing in λ, if function $\frac{\Psi'(t;\lambda)}{\Psi(t;\lambda)}$ is strictly increasing in t, where $\Psi'(t;\lambda)$ is $(\partial/\partial\lambda)\Psi(t;\lambda)$.

Proof. Let $w_j(\lambda) = w_{1j}(\lambda)/w_{2j}(\lambda)$, where

$$w_{1j}(\lambda) = \sum_{i=j+1}^{m}(r_i+1)\frac{\Psi(x_i;\lambda)}{\Psi(x_j;\lambda)} - \sum_{i=j+1}^{m}(r_i+1), \tag{14}$$

$$w_{2j}(\lambda) = \sum_{i=1}^{j}(r_i+1)\frac{\Psi(x_i;\lambda)}{\Psi(x_j;\lambda)} + \sum_{i=j+1}^{m}(r_i+1). \tag{15}$$

Since $w_{1j}(\lambda)$ and $w_{2j}(\lambda)$ are positive, the proof is obtained if we can show that $w_{1j}(\lambda)$ and $w_{2j}(\lambda)$ are strictly increasing and decreasing functions in λ, respectively. It is observed that

$$w'_{1j}(\lambda) = \frac{1}{\Psi^2(x_j;\lambda)}\sum_{i=j+1}^{m}(r_i+1)[\Psi'(x_i,\lambda)\Psi(x_j;\lambda)) - \Psi(x_i;\lambda)\Psi'(x_j;\lambda)] > 0, \tag{16}$$

$$w'_{2j}(\lambda) = \frac{1}{\Psi^2(x_j;\lambda)}\sum_{i=1}^{j}(r_i+1)[\Psi'(x_i;\lambda)\Psi(x_j;\lambda)) - \Psi(x_i;\lambda)\Psi'(x_j;\lambda)] \le 0, \tag{17}$$

Since, when $\frac{\Psi'(t;\lambda)}{\Psi(t;\lambda)}$ is strictly increasing in t, then $\frac{\Psi'(x_i;\lambda)}{\Psi(x_i;\lambda)} < \frac{\Psi'(x_j;\lambda)}{\Psi(x_j;\lambda)}$ for $i = 1, 2, \ldots, j-1$, and $\frac{\Psi'(x_i;\lambda)}{\Psi(x_i;\lambda)} > \frac{\Psi'(x_j;\lambda)}{\Psi(x_j;\lambda)}$ for $i = j+1, j+2, \ldots, m$. ■

Remark 1. For all of well-known lifetime distributions mentioned in Section 2, it can be shown that $\frac{\Psi'(t;\lambda)}{\Psi(t;\lambda)}$ is strictly increasing in t. For instance, when $\Psi(t;\lambda) = \ln(1+t^{\lambda})$, it turns out to be Burr XII distribution and see [5].

Let $F_{v_1,v_2}(p)$ is the percentile of F distribution with v_1 and v_2 degrees of freedom with the right-tail probability p. An exact confidence interval for the parameter λ, and the joint confidence region for the parameters α and λ are given in the following theorems, respectively.

Theorem 1. Suppose that $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ be a progressive first-failure censored sample from (2), with censoring scheme $(r_1, r_2, \ldots, r_m)$, $\frac{\Psi'(t;\lambda)}{\Psi(t;\lambda)}$ is strictly increasing in t, and

$$W_j(\lambda) = \frac{j}{m-j}w_j(\lambda), \quad j = 1, 2, \ldots, m-1, \tag{18}$$

where $w_j(\lambda)$ is defined in (13). Then, for any $0 < \nu < 1$ and $j = 1, 2, \ldots, m-1$, when $F_{2(m-j),2j}(\frac{\nu}{2})$ and $F_{2(m-j),2j}(1-\frac{\nu}{2})$ are in the range of the function $W_j(\lambda)$

$$\varphi_j\big(\mathbf{X}, F_{2(m-j),2j}(1-\frac{\nu}{2})\big) < \lambda < \varphi_j\big(\mathbf{X}, F_{2(m-j),2j}(\frac{\nu}{2})\big) \tag{19}$$

is a $100(1-\nu)\%$ confidence interval for λ, where $\varphi_j(\mathbf{X}, u)$ is the solution of λ for equation $W_j(\lambda) = u$.

Proof. By Lemma 1, $W_j(\lambda)$ is a strictly increasing function in λ and since $F_{2(m-j),2j}(1-\frac{\nu}{2})$ and $F_{2(m-j),2j}(\frac{\nu}{2})$ are in the range of function $W_j(\lambda)$, then equations $W_j(\lambda) = F_{2(m-j),2j}(\frac{\nu}{2})$ and $W_j(\lambda) = F_{2(m-j),2j}(1-\frac{\nu}{2})$ have unique solutions with respect to λ. Also we know that η_j has an F distribution with $2(m-j)$ and $2j$ degrees of freedom. Thus for $0 < \nu < 1$,

$$P\big(F_{2(m-j),2j}(1-\frac{\nu}{2}) < \eta_j < F_{2(m-j),2j}(\frac{\nu}{2})\big) = 1-\nu$$

is equivalent to

$$P\Big(\varphi_j\big(\mathbf{X}, F_{2(m-j),2j}(1-\frac{\nu}{2})\big) < \lambda < \varphi_j\big(\mathbf{X}, F_{2(m-j),2j}(\frac{\nu}{2})\big)\Big) = 1-\nu.$$

■

Theorem 2. Suppose that $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ be a progressive first-failure censored sample from (2), with censoring scheme $(r_1, r_2, \ldots, r_m)$, $\frac{\Psi'(t;\lambda)}{\Psi(t;\lambda)}$ is strictly increasing in t. Then, for any $0 < \nu < 1$ and $j = 1, 2, \ldots, m-1$, when $F_{2(m-j),2j}(\frac{1+\sqrt{1-\nu}}{2})$ and $F_{2(m-j),2j}(\frac{1-\sqrt{1-\nu}}{2})$are in the range of function $W_j(\lambda)$, a $100(1-\nu)\%$ confidence region for (α, λ) is given by

$$\begin{cases} \varphi_j\big(\mathbf{X}, F_{2(m-j),2j}(\frac{1+\sqrt{1-\nu}}{2})\big) < \lambda < \varphi_j\big(\mathbf{X}, F_{2(m-j),2j}(\frac{1-\sqrt{1-\nu}}{2})\big) \\ \frac{\chi^2_{2m}(\frac{1+\sqrt{1-\nu}}{2})}{2k\sum_{i=1}^m (r_i+1)\Psi(x_i;\lambda)} < \alpha < \frac{\chi^2_{2m}(\frac{1-\sqrt{1-\nu}}{2})}{2k\sum_{i=1}^m (r_i+1)\Psi(x_i;\lambda)} \end{cases}, \tag{20}$$

where $\chi^2_{v_1}(p)$ is the percentile of Chi-squared distribution with v_1 degree of freedom with the right-tail probability p and $\varphi_j(\mathbf{X}, u)$ is defined in Theorem 1 .

Proof. For $0 < \nu < 1$,

$$\begin{aligned} P\Big(\varphi_j\big(\mathbf{X}, F_{2(m-j),2j}(\frac{1+\sqrt{1-\nu}}{2})\big) < \lambda < \varphi_j\big(\mathbf{X}, F_{2(m-j),2j}(\frac{1-\sqrt{1-\nu}}{2})\big), \\ \frac{\chi^2_{2m}(\frac{1+\sqrt{1-\nu}}{2})}{2k\sum_{i=1}^m (r_i+1)\Psi(x_i;\lambda)} < \alpha < \frac{\chi^2_{2m}(\frac{1-\sqrt{1-\nu}}{2})}{2k\sum_{i=1}^m (r_i+1)\Psi(x_i;\lambda)}\Big) = \\ P\Big(F_{2(m-j),2j}(\frac{1+\sqrt{1-\nu}}{2}) < \eta_j < F_{2(m-j),2j}(\frac{1-\sqrt{1-\nu}}{2})\Big) \\ P\Big(\chi^2_{2m}(\frac{1+\sqrt{1-\nu}}{2}) < \xi < \chi^2_{2m}(\frac{1-\sqrt{1-\nu}}{2})\Big) = \sqrt{1-\nu}\sqrt{1-\nu} = 1-\nu, \end{aligned}$$

and the first equality follows from the fact that η_j and ξ are independent. ■

It is observed that Theorems 1 and 2 provides the different confidence intervals and confidence regions, respectively for various j. We can derive optimal confidence interval and region based on different criteria such as shortest interval length and minimum region area.

4. Bayes Estimation

The Bayesian approach in statistical inference provides an alternative choice for parameters estimation. We consider the Bayesian estimates of the unknown parameters α and λ as well as reliability function $R(t)$ and hazard rate function $h(t)$ under symmetric and asymmetric loss functions.

The loss function plays a critical role in Bayes perspective. In many practical situations, usually symmetric loss function such as squared error loss function is taken into consideration to produce Bayes estimates. In most cases, it is done for convenience but may not be appropriate in many real life situations. Since under this loss function overestimation and underestimation are equally penalized which is not a good criteria from practical point of view. As an example, in reliability estimation overestimation is considered to be more serious than the underestimation. Thus, it is important to consider Bayes estimates under asymmetric loss function. The squared error loss function is defined as

$$L_1\left(f(\mu),\hat{f}(\mu)\right)=\left(\hat{f}(\mu)-f(\mu)\right)^2,$$

with $\hat{f}(\mu)$ begins an estimator of $f(\mu)$. Here $f(\mu)$ denotes some parametric function of μ. Bayes estimator, say $\hat{f}_{SB}(\mu)$ is evaluated by the posterior mean of $f(\mu)$.

One of the most commonly used asymmetric loss function is LINEX loss function which introduced first by [16] and further properties of this loss function have been investigated by [17] . It is defined as follows:

$$L_2\left(f(\mu),\hat{f}(\mu)\right)=e^{c\Delta}-c\Delta-1,\qquad c\neq 0,$$

where $\Delta=\hat{f}(\mu)-f(\mu)$. When c is negative, underestimation is more serious than overestimation and it is opposit for positive c. The Bayes estimator of $f(\mu)$ for the loss function L_2 can be obtained as $\hat{f}_{LB}(\mu)=-\frac{1}{c}\ln\left\{E_\mu\left(e^{-cf(\mu)}|data\right)\right\}$, provided that $E_\mu(.)$ exists and is finite. Another useful asymmetric loss function is the general entropy loss which is a generalization of the entropy loss and is given as

$$L_3\left(f(\mu),\hat{f}(\mu)\right)\propto\left(\frac{\hat{f}(\mu)}{f(\mu)}\right)^{-q}-q\ln\left(\frac{\hat{f}(\mu)}{f(\mu)}\right)-1,\quad q\neq 0.$$

For this loss function, overestimation is heavily penalized when q is positive, and vice versa. The Bayes estimator of $f(\mu)$ under general entropy loss function is obtained as

$$\hat{f}_{EB}(\mu)=\left\{E\left[(f(\mu))^{-q}|data\right]\right\}^{-\frac{1}{q}},$$

provided that $E_\mu(.)$ exists and is finite.

4.1. Prior distribution and posterior analysis

In this subsection, we need to assume some prior distributions for the unknown parameters. Under the assumption that two parameters α and λ are unknown, specifying a general conjugate joint prior forα and λ is not easy task. In this case, we develop the Bayesian set-up by considering the idea of [18] regarding the choice of prior distributions. We assume that for $j=1,2,\ldots,M$, λ has a discrete prior say,

$$P(\lambda=\lambda_j)=\theta_j,\ \sum_{j=1}^{M}\theta_j=1,\tag{21}$$

while the conditional distribution of α given λ_j has a conjugate prior distribution, with density

$$g(\alpha|\lambda_j)=\beta_j\exp\{-\alpha\beta_j\},\quad \alpha,\beta_j>0,\tag{22}$$

where $\beta_j,\ \ j=1,2,\ldots,M$, are hyper-parameters. Combining (6) and (22), the conditional posterior of the parameter α,

$$\pi(\alpha|\mathbf{x},\lambda_j)=\frac{g(\alpha|\lambda_j)L(\alpha,\lambda_j;\mathbf{x})}{\int_\alpha g(\alpha|\lambda_j)L(\alpha,\lambda_j;\mathbf{x})\,d\alpha},\tag{23}$$

takes the form

$$\pi(\alpha|\mathbf{x},\lambda_j)=\frac{1}{\Gamma(m+1)}c_j^{m+1}\alpha^m\exp\{-\alpha c_j\},\ \ j=1,2,\ldots,M,\tag{24}$$

where $c_j = k\sum_{i=1}^{m}(r_i+1)\Psi(x_i;\lambda_j)+\beta_j$. Also by applying (6), (21), (22) and the discrete version of Bayes theorem, the marginal posterior distribution of λ can be expressed as

$$p_j = P(\lambda=\lambda_j|\mathbf{x}) = \frac{\int_\alpha P(\lambda=\lambda_j)g(\alpha|\lambda_j)L(\alpha,\lambda_j;\mathbf{x})\,d\alpha}{\sum_{j=1}^{M}\int_\alpha P(\lambda=\lambda_j)g(\alpha|\lambda_j)L(\alpha,\lambda_j;\mathbf{x})\,d\alpha}$$
$$= \frac{\beta_j\theta_j c_j^{-(m+1)}\prod_{i=1}^{m}\psi(x_i;\lambda_j)}{\sum_{j=1}^{M}\beta_j\theta_j c_j^{-(m+1)}\prod_{i=1}^{m}\psi(x_i;\lambda_j)}, \quad j=1,2,\ldots,M. \tag{25}$$

Therefore, the Bayes estimators of α and λ under the squared error loss function L_1 are

$$\hat{\alpha}_{SB} = (m+1)\sum_{j=1}^{M}\frac{p_j}{c_j}, \tag{26}$$

$$\hat{\lambda}_{SB} = \sum_{j=1}^{M}p_j\lambda_j, \tag{27}$$

respectively. Also, the Bayes estimators of $R(t)$ and $h(t)$ against the loss function L_1 are given respectively, by

$$\hat{R}_{SB}(t) = \sum_{j=1}^{M}p_j\big[1+\frac{\Psi(t;\lambda_j)}{c_j}\big]^{-(m+1)}, \tag{28}$$

$$\hat{h}_{SB}(t) = (m+1)\sum_{j=1}^{M}\frac{p_j\psi(t;\lambda_j)}{c_j}. \tag{29}$$

For the loss function L_2, the Bayes estimators of α, λ, $R(t)$ and $h(t)$ are respectively obtained as

$$\hat{\alpha}_{LB} = -\frac{1}{c}\ln\Bigg[\sum_{j=1}^{M}p_j(1+\frac{c}{c_j})^{-(m+1)}\Bigg], \tag{30}$$

$$\hat{\lambda}_{LB} = -\frac{1}{c}\ln\Bigg[\sum_{j=1}^{M}p_je^{-c\lambda_j}\Bigg], \tag{31}$$

$$\hat{R}_{LB}(t) = -\frac{1}{c}\ln\Bigg[\sum_{j=1}^{M}\sum_{l=0}^{\infty}\frac{(-1)^l}{\Gamma(l+1)}p_jc^l(1+\frac{l\Psi(t;\lambda_j)}{c_j})^{-(m+1)}\Bigg], \tag{32}$$

$$\hat{h}_{LB}(t) = -\frac{1}{c}\ln\Bigg[\sum_{j=1}^{M}p_j(1+\frac{c\psi(t;\lambda_j)}{c_j})^{-(m+1)}\Bigg]. \tag{33}$$

Finally, against the loss function L_3, the Bayes estimators of α, λ, $R(t)$ and $h(t)$ can be expressed as

$$\hat{\alpha}_{EB} = \Bigg[\frac{\Gamma(m+1-q)}{\Gamma(m+1)}\sum_{j=1}^{M}p_jc_j^q\Bigg]^{-\frac{1}{q}}, \quad m+1>q, \tag{34}$$

$$\hat{\lambda}_{EB} = \Bigg[\sum_{j=1}^{M}p_j\lambda_j^{-q}\Bigg]^{-\frac{1}{q}}, \tag{35}$$

$$\hat{R}_{EB}(t) = \Bigg[\sum_{j=1}^{M}p_j(1-\frac{q\Psi(t;\lambda_j)}{c_j})^{-(m+1)}\Bigg]^{-\frac{1}{q}}, \tag{36}$$

$$\hat{h}_{EB}(t) = \Bigg[\frac{\Gamma(m+1-q)}{\Gamma(m+1)}\sum_{j=1}^{M}p_j(\frac{c_j}{\psi(t;\lambda_j)})^q\Bigg]^{-\frac{1}{q}}, \quad m+1>q, \tag{37}$$

respectively.

4.2. The choice of hyper-parameters

The priors specification are completed by specifying λ_j, θ_j and hyper-parameters β_j, for $j = 1, 2, \ldots, M$, in practice. The values of λ_j and θ_j are fairly straightforward to specify, but sometimes it is not always possible to know the values of the hyper-parameters β_j, in prior (22). In practice, the values of β_j are difficulty to know, since it is necessary to condition prior beliefs about α on each λ_j, $j = 1, 2, \ldots, M$. Thus, the estimation problem for hyper-parameters β_j, $j = 1, 2, \ldots, M$, is considered in this subsection.

There are different methods to estimate the hyper-parameters $\beta_j, j = 1, 2, \ldots, M$. First, we consider the maximum likelihood type-II method (see [19, pp. 99]).

Let $U_i = \Psi(X_i; \lambda), i = 1, 2, \ldots, m$. It can be shown that $U_1 < U_2 < \cdots < U_m$, are the progressive first-failure censored order statistics with censoring scheme $(r_1, r_2, \ldots, r_m)$, from conditional density

$$f_U(u; \alpha) = \alpha e^{-\alpha u}, \quad u > 0. \tag{38}$$

For given λ_j, the marginal PDF and CDF of U are given by

$$f_U(u) = \int_\alpha g(\alpha|\lambda_j) f_U(u; \alpha)\, d\alpha = \frac{\beta_j}{(\beta_j + u)^2}, \quad u > 0, \tag{39}$$

$$F_U(u) = 1 - \frac{\beta_j}{\beta_j + u}, \quad u > 0, \tag{40}$$

respectively. From (1), the log-likelihood function of $\mathbf{U} = (U_1, U_2, \ldots, U_m)$, can be written as

$$logL(\beta_j; \mathbf{u}) = \ln(Ak^m) + nk \ln(\beta_j) - \sum_{i=1}^{m} (k(r_i + 1) + 1) \ln(\beta_j + u_i). \tag{41}$$

By setting the derivative of the log-likelihood function with respective to β_j to zero, the MLE of β_j, is the solution to the likelihood equation $\frac{1}{\beta_j} = H(\beta_j)$, where

$$H(\beta_j) = \frac{1}{nk} \sum_{i=1}^{m} \frac{k(r_i + 1) + 1}{\beta_j + u_i}, \tag{42}$$

and it is unique (see Appendix). Most of the standard iterative process can be used for finding the MLE. We propose a simple iterative scheme to finding the MLE of β_j. Start with an initial guess of β_j, say $\beta_j^{(0)}$, then obtain $\beta_j^{(1)} = 1/H(\beta_j^{(0)})$, and proceeding in this way iteratively to obtain $\beta_j^{(N)} = 1/H(\beta_j^{(N-1)})$. Stop the iterative procedure, when $|\beta_j^{(N)} - \beta_j^{(N-1)}| < \varepsilon$, some pre-assigned tolerance limit.

Another useful alternative method to estimate the hyper-parameters $\beta_j, j = 1, 2, \ldots, M$, is based on the idea of [20]. By applying (22), the expected value of the reliability function $R(t)$ conditional on $\lambda = \lambda_j$, can be written as

$$E(R(t)) = \int_\alpha R(t) g(\alpha|\lambda_j)\, d\alpha = \frac{\beta_j}{\beta_j + \Psi(t; \lambda_j)}, \quad j = 1, 2, \ldots, M. \tag{43}$$

For given time t, by considering $E(R(t)) = \hat{R}(t)$, the estimate of β_j is

$$\hat{\beta}_j = \frac{\hat{R}(t)\Psi(t; \lambda_j)}{1 - \hat{R}(t)}, \quad j = 1, 2, \ldots, M. \tag{44}$$

Similarly, we can use the expected value of the hazard rate function $h(t)$ conditional on $\lambda = \lambda_j, j = 1, 2, \ldots, M$. It can be shown that

$$E(h(t)) = \int_\alpha h(t) g(\alpha|\lambda_j)\, d\alpha = \frac{\psi(t; \lambda_j)}{\beta_j}, \tag{45}$$

and

$$\hat{\beta}_j = \frac{\psi(t;\lambda_j)}{\hat{h}(t)}, \quad j = 1,2,\ldots,M. \tag{46}$$

It is obviously the second method to estimate the hyper-parameters $\beta_j, j = 1,2,\ldots,M$, depend on the value of MLEs $\hat{\alpha}$ and $\hat{\lambda}$. Therefore, the author recommends the first method.

5. Data analysis

To illustrate the above procedures, we present the analysis of one real data set. Also, we report some numerical experiments performed to evaluate behavior of the different estimators.

Example 1.(Real Data) In this example, we analyze a data set from [21], which represents the number of 1000s of cycles to failure for electrical appliances in a life test. The complete data have been used earlier by [22]. They showed that the bathtub-shaped distribution is suitable to fitting the data. The CDF of the bathtub-shaped distribution is form(3), where $\Psi(t;\lambda) = e^{t^\lambda} - 1, t > 0$. It can be shown that $\frac{\Psi'(t;\lambda)}{\Psi(t;\lambda)}$, is strictly increasing in t (see [14]).

Table 1: *progressively first-failure censored sample of size 8 out of 20 groups.*

i	1	2	3	4	5	6	7	8
x_i	0.014	0.034	0.059	0.061	0.069	0.142	0.165	1.270
r_i	4	0	3	0	0	2	3	0

The data are randomly grouped into 20 groups with $k = 3$ items within each group. The progressively first-failure censored sample is given in Table 1. For this example, 12 groups of failure times are censored, and 8 first-failures are observed. By applying, (19) and (20), the 95% exact confidence intervals (CI) for λ, confidence regions (CR) for (α,λ), are obtained and the length of confidence intervals (LCI) and area for confidence regions (ACR) are presented in Table 2, where $A(\lambda) = \sum_{i=1}^{8}(r_i+1)(e^{x_i^\lambda} - 1)$.

Table 2: *The* 95% *confidence intervals and regions and their some properties for* λ *and* (α,λ).

j	CI	CR	LCI	ACR
1	$0.3933 < \lambda < 1.7034$	$0.3397 < \lambda < 1.8545$, $\frac{1.0114}{A(\lambda)} < \alpha < \frac{5.2012}{A(\lambda)}$	1.3101	1.3904
2	$0.3694 < \lambda < 1.4175$	$0.3192 < \lambda < 1.5198$, $\frac{1.0114}{A(\lambda)} < \alpha < \frac{5.2012}{A(\lambda)}$	1.0481	1.0334
3	$0.3538 < \lambda < 1.3167$	$0.3044 < \lambda < 1.4039$, $\frac{1.0114}{A(\lambda)} < \alpha < \frac{5.2012}{A(\lambda)}$	0.9629	0.9081
4	$0.2317 < \lambda < 1.0946$	$0.1920 < \lambda < 1.1696$, $\frac{1.0114}{A(\lambda)} < \alpha < \frac{5.2012}{A(\lambda)}$	0.8629	0.6805
5	$0.1391 < \lambda < 0.9320$	$0.1092 < \lambda < 1.0014$, $\frac{1.0114}{A(\lambda)} < \alpha < \frac{5.2012}{A(\lambda)}$	0.7929	0.5232
6	$0.1302 < \lambda < 0.9750$	$0.0963 < \lambda < 1.0462$, $\frac{1.0114}{A(\lambda)} < \alpha < \frac{5.2012}{A(\lambda)}$	0.8448	0.5708
7	$0.0212 < \lambda < 0.7932$	$0.0109 < \lambda < 0.8646$, $\frac{1.0114}{A(\lambda)} < \alpha < \frac{5.2012}{A(\lambda)}$	0.7720	0.4094

From Table 2, It is observed that, the 95% optimal confidence interval for λ is $(0.0212, 0.7932)$, and the optimal confidence region for (α,λ) is given by

$$0.0109 < \lambda < 0.8646\ ,\ \frac{1.0114}{A(\lambda)} < \alpha < \frac{5.2012}{A(\lambda)},$$

and $ACR = \int_{0.0109}^{0.8646} \frac{4.1898}{A(\lambda)}\, d\lambda = 0.4094$.

Since there is no prior information about α, to compute the Bayes estimates, we estimate the hyper-parameters $\beta_j, j = 1,2,\ldots,8$, using the maximum likelihood type-II method. The values of β_j and p_j, for each given λ_j and θ_j, $j = 1,2,\ldots,8$, are summarized in Table 3. The MLEs as well as Bayes estimates of α, λ, reliability function $R(t)$, and hazard rate function $h(t)$, for $t = 0.5$, are presented in Table 4.

Table 3: *Prior information, hyper-parameter values and the posterior probabilities.*

j	1	2	3	4	5	6	7	8
λ_j	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75
θ_j	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125
β_j	3.5605	3.1398	2.7814	2.4735	2.2073	1.9756	1.7727	1.5942
p_j	0.0308	0.0549	0.0859	0.1206	0.1532	0.1778	0.1897	0.1871

Table 4: *The ML and the Bayes estimates of $\alpha, \lambda, R(t)$ and $h(t)$, with $t = 0.5, c = 1$ and $q = 1$.*

$\hat{\alpha}$	$\hat{\alpha}_{SB}$	$\hat{\alpha}_{LB}$	$\hat{\alpha}_{EB}$	$\hat{\lambda}$	$\hat{\lambda}_{SB}$	$\hat{\lambda}_{LB}$	$\hat{\lambda}_{EB}$
0.4800	0.4252	0.4132	0.3674	0.7200	0.6268	0.6220	0.6099
$\hat{R}(t)$	$\hat{R}_{SB}(t)$	$\hat{R}_{LB}(t)$	$\hat{R}_{EB}(t)$	$\hat{h}(t)$	$\hat{h}_{SB}(t)$	$\hat{h}_{LB}(t)$	$\hat{h}_{EB}(t)$
0.6697	0.6871	0.6833	0.6753	0.7700	0.6584	0.6267	0.5570

Table 5: *The estimated MSE values of the estimators of α and λ.*

n	m	k	$C.S$	$\hat{\alpha}$	$\hat{\alpha}_{SB}$	$\hat{\alpha}_{LB}$	$\hat{\alpha}_{EB}$	$\hat{\lambda}$	$\hat{\lambda}_{SB}$	$\hat{\lambda}_{LB}$	$\hat{\lambda}_{EB}$
20	10	1	I	0.0026	0.0018	0.0017	0.0018	0.0300	0.0088	0.0082	0.0069
			II	0.0034	0.0025	0.0024	0.0023	0.0089	0.0054	0.0052	0.0047
			III	0.0027	0.0021	0.0021	0.0019	0.0111	0.0059	0.0056	0.0051
		5	I	0.0216	0.0031	0.0029	0.0020	0.0919	0.0088	0.0081	0.0063
			II	0.0025	0.0020	0.0020	0.0016	0.0262	0.0080	0.0075	0.0064
			III	0.0040	0.0025	0.0024	0.0018	0.0342	0.0082	0.0077	0.0065
30	10	1	I	0.0023	0.0018	0.0018	0.0016	0.0463	0.0089	0.0082	0.0067
			II	0.0034	0.0025	0.0024	0.0023	0.0079	0.0049	0.0048	0.0043
			III	0.0024	0.0021	0.0020	0.0018	0.0106	0.0058	0.0056	0.0051
		5	I	0.0658	0.0041	0.0038	0.0024	0.0955	0.0086	0.0079	0.0061
			II	0.0025	0.0021	0.0020	0.0016	0.0216	0.0077	0.0072	0.0062
			III	0.0066	0.0029	0.0028	0.0020	0.0291	0.0079	0.0074	0.0063
30	15	1	I	0.0016	0.0011	0.0011	0.0012	0.0156	0.0073	0.0069	0.0060
			II	0.0021	0.0017	0.0017	0.0016	0.0051	0.0037	0.0036	0.0034
			III	0.0016	0.0013	0.0013	0.0013	0.0059	0.0042	0.0040	0.0038
		5	I	0.0037	0.0016	0.0016	0.0012	0.0390	0.0080	0.0074	0.0061
			II	0.0011	0.0010	0.0010	0.0009	0.0139	0.0068	0.0065	0.0057
			III	0.0018	0.0013	0.0013	0.0011	0.0167	0.0070	0.0066	0.0058

Example 2.(Simulation study) To evaluate the performance of the MLEs and Bayes estimators, a simulation study using Monte Carlo method is performed. In this example, we exclusively focus on the bathtub-shaped distribution. For comparison purpose different n, m, k, and censoring schemes(C.S) are considered. We present the results for $\alpha = 0.1$ and $\lambda = 0.5$. For generating progressively first-failure censored samples, we use the algorithm suggested in [23]. We take into consideration that the progressive first-failure censored order statistics $X^{\mathbf{r}}_{1;m,n,k}, X^{\mathbf{r}}_{2;m,n,k}, ..., X^{\mathbf{r}}_{m;m,n,k}$ is a progressively type-II censored sample from a population with distribution function $1 - (1 - F(x))^k$. We considered the following censoring schemes:

- Scheme I: $r_m = n - m, r_i = 0$, for $i \neq m$.
- Scheme II: $r_1 = n - m, r_i = 0$, for $i \neq 1$.
- Scheme III: $r_{\frac{m}{2}} = n - m, r_i = 0$, for $i \neq \frac{m}{2}$ if m is even, and $r_{\frac{m+1}{2}} = n - m, r_i = 0$, for $i \neq \frac{m+1}{2}$ if m is odd.

The Bayes estimates are obtained for $c = 1, q = 1$, and λ_j and θ_j were given in previous example. The performance of all estimators has been compared numerically in terms of their mean squared errors (MSEs). In each case, for a particular censoring scheme the estimated MSEs are computed over 10,000 simulations. The simulation study was conducted in R software (R x64 4.0.3) and the R code can be obtained on request from the author. Based on tabulated the estimated MSEs, following conclusions can be drawn from Tables 5 and 6.

1. For all censoring schemes, it can be observed that the Bayes estimators are superior to MLE for the parameters α, λ. We also observe that Bayes estimators of $h(t)$ perform better than MLEs of $h(t)$.
2. It is clearly observed that the performance of all estimators of $R(t)$ are very fine in respect to MSE in all situations.
3. In the case of λ, when n and m are fixed, the censoring scheme $(n - m, 0, \ldots, 0)$ posses the smallest estimated MSE values.

4. For fixed n and k, it is observed that as m increases the performance of all estimators improve in terms of the estimated MSE values.

Table 6: *The estimated MSE values of the estimators of $R(t)$ and $h(t)$, fot $t = 0.75$.*

n	m	k	$C.S$	$\hat{R}(t)$	$\hat{R}_{SB}(t)$	$\hat{R}_{LB}(t)$	$\hat{R}_{EB}(t)$	$\hat{h}(t)$	$\hat{h}_{SB}(t)$	$\hat{h}_{LB}(t)$	$\hat{h}_{EB}(t)$
20	10	1	I	0.0040	0.0038	0.0040	0.0043	0.0039	0.0039	0.0037	0.0033
			II	0.0048	0.0053	0.0056	0.0062	0.0046	0.0036	0.0034	0.0034
			III	0.0038	0.0049	0.0051	0.0055	0.0040	0.0036	0.0034	0.0031
		5	I	0.0060	0.0072	0.0075	0.0082	0.1122	0.0114	0.0100	0.0054
			II	0.0028	0.0048	0.0050	0.0054	0.0084	0.0060	0.0056	0.0040
			III	0.0033	0.0058	.0060	0.0065	0.0176	0.0084	0.0076	0.0048
30	10	1	I	0.0032	0.0042	0.0043	0.0046	0.0058	0.0050	0.0047	0.0036
			II	0.0047	0.0055	0.0058	0.0064	0.0047	0.0038	0.0036	0.0035
			III	0.0034	0.0048	0.0050	0.0054	0.0040	0.0037	0.0036	0.0031
		5	I	0.0118	0.0091	0.0095	0.0107	0.4210	0.0161	0.0135	0.0064
			II	0.0028	0.0050	0.0052	0.0055	0.0078	0.0062	0.0058	0.0040
			III	0.0043	0.0066	0.0069	0.0075	0.0320	0.0103	0.0091	0.0056
30	15	1	I	0.0026	0.0027	0.0027	0.0029	0.0022	0.0020	0.0020	0.0020
			II	0.0032	0.0043	0.0044	0.0048	0.0029	0.0022	0.0022	0.0022
			III	0.0024	0.0035	0.0037	0.0039	0.0023	0.0019	0.0019	0.0018
		5	I	0.0025	0.0047	0.0048	0.0051	0.0210	0.0065	0.0059	0.0036
			II	0.0015	0.0031	0.0031	0.0033	0.0028	0.0029	0.0028	0.0022
			III	0.0018	0.0039	0.0040	0.0042	0.0071	0.0046	0.0043	0.0030

6. Conclusions

Lifetime studies are very important to assess the reliability of products. This article investigates the problem of reliability analysis for a class of an exponential distribution based on progressive first failure censoring. It is note that many well-known and useful lifetime distributions which have wide application in reliability theory and failure time modeling as well as other related fields, are included in this class of exponential distribution. Both classical and Bayesian point estimations have been developed. Additionally, the exact confidence interval and region respectively for α and (α, λ) have been conducted. In the future, we can study the problem of predicting times to failure of units censored in multiple stages in progressive first failure censored sample based on model (2).

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Appendix

We show that the equation

$$\frac{1}{\beta_j} - H(\beta_j) = 0, \tag{47}$$

where $H(\beta_j)$ is defined in (42), has only one root. By considering

$$Q_1(\beta_j) = \sum_{i=1}^{m} k(r_i+1)\frac{u_i}{\beta_j(\beta_j+u_i)},$$
$$Q_2(\beta_j) = \sum_{i=1}^{m} \frac{1}{\beta_j+u_i},$$

the equation (47) is equivalent to $Q_1(\beta_j) - Q_2(\beta_j) = 0$. The functions $Q_1(\beta_j)$ and $Q_2(\beta_j)$ are strictly decreasing and convex, since

$$\begin{aligned}
\frac{\partial Q_1(\beta_j)}{\partial \beta_j} &= -\sum_{i=1}^{m} k(r_i+1)\frac{u_i}{\beta_j^2(\beta_j+u_i)^2} < 0,\\
\frac{\partial^2 Q_1(\beta_j)}{\partial \beta_j^2} &= 2\sum_{i=1}^{m} k(r_i+1)\frac{u_i^2}{\beta_j^3(\beta_j+u_i)^3} > 0,\\
\frac{\partial Q_2(\beta_j)}{\partial \beta_j} &= -\sum_{i=1}^{m} \frac{1}{(\beta_j+u_i)^2} < 0,\\
\frac{\partial^2 Q_2(\beta_j)}{\partial \beta_j^2} &= 2\sum_{i=1}^{m} \frac{1}{(\beta_j+u_i)^3} > 0.
\end{aligned}$$

Also,

$$\begin{aligned}
&\lim_{\beta_j\to 0} Q_1(\beta_j) = +\infty, \quad \lim_{\beta_j\to +\infty} Q_1(\beta_j) = 0,\\
&\lim_{\beta_j\to 0} Q_2(\beta_j) = \sum_{i=1}^{m}\frac{1}{u_i}, \quad \lim_{\beta_j\to +\infty} Q_2(\beta_j) = 0,\\
&\lim_{\beta_j\to \infty} \frac{Q_2(\beta_j)}{Q_1(\beta_j)} = +\infty,
\end{aligned}$$

thus the equation (47), has only one root.