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Transmuted Sine-Dagum Distribution and its Properties

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Abstract

In this paper, we introduce a new four parameters continuous probability distribution called transmuted sine-Dagum distribution obtained through the transmuted Sine-G family introduced by Sakthivel et al. [13]. We have obtained some distributional properties including moments, inverted moments, incomplete moments, central moments and order statistics for proposed model. The reliability measures such as reliability function, hazard rate function, reversed hazard rate function, cumulative distribution function, mean waiting time and mean residual life time are studied in this paper. Further, we have discussed some income inequality measures including Lorenz curve, Bonferroni index and Zenga index. The maximum likelihood method is used to estimate the parameters of the proposed probability distribution. Finally, we demonstrated goodness of fit the proposed model with other suitable models in the literature using real life data sets.

Keywords: Dagum distribution, Sine-G family, Reliability function, Order statistics, Lorenz curve, Maximum likelihood method.

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I. Introduction

The lifetime model is playing a vital role in different fields such as life sciences, biological sciences, environmental sciences, medicine, finance and actuarial science. The last three decades, the development and applications of new probability distributions for lifetime data are remarkable in the literature. In this information era, the data generated from different fields are voluminous and dynamic in nature. Therefore, the need for generating new family of probability distributions is inevitable. As a result, the generating new family of probability distribution has attracted many researchers. The main advantage of generating new family of probability distributions is provide better flexibility and better fit at the cost of one or more additional parameters. The following are a list of few well-known generating new family of probability distributions: exponential family is introduced by Gupta et al. [6], Marshall-Olkin family is introduced by Marshal and Olkin [9], transmuted family is introduced by Shaw and Bucklay [14], Kumaraswamy-G family is introduced by Cordeiro and Castro [3], Topp-Leone family is introduced by Al-Shomrani et al. [1], Power Lindley-G family is introduced by Hassan and Nassr [7] and gamma-G family is introduced by Cordeiro et al. [4], to mention a few.

Dagum distribution is introduced by Camilo Dagum [5] in the year 1977 for modeling income data. It has been extensively used in various fields including wealth data, reliability analysis, survival analysis and meteorological data. The Dagum distribution is an alternative to log-normal, Pareto and generalized beta distributions. This distribution is also called Burr-XII distribution, particularly in the actuarial literature.

A continuous random variable X is said to have Dagum distribution with three parameters σ, θ and β if its probability density function and cumulative distribution function are given respectively as

$$f(x;\sigma,\theta,\beta) = \sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1};\ x>0,\sigma>0,\theta>0 \text{ and } \beta>0. \tag{1}$$

and

$$F(x;\sigma,\theta,\beta) = \left(1+\sigma x^{-\theta}\right)^{-\beta};\ x>0,\sigma>0,\theta>0 \text{ and } \beta>0. \tag{2}$$

where σ is scale parameter, while θ and β are shape parameters. It is to be noted that if σ=1 the Dagum distribution becomes Burr III distribution and if θ=1, the Dagum distribution becomes log-logistic or Fisk distribution.

In this paper, we introduce a new four parameter continious probability distribution namely transmuted Sine-Dagum distribution. This generating probability distribution provides better fit and flexibility for real life problem.

This paper is organized as follows: In Section 1, a brief introduction and need for the generating family of distributions is given. In Section 2, we present the transmuted Sine-G family and the proposed probability distribution namely transmuted Sine-Dagum distribution. In Section 3, we discuss some reliability measures like reliability function, hazard rate function, reversed hazard rate function, cumulative hazard function, mean waiting time, mean residual life function and mean past life time. In Section 4, we present some distributional properties including moments, inverted moments, incomplete moments, central moments and order statistics. The income inequality measures are discussed in Section 5. The method of maximum likelihood estimation is presented in Section 6. In Section 7, the real time application is presented. Finally, the concluding remarks are presented in Section 8.

II. Transmuted Sine-G Family

Transmuted Sine-G family is introduced by Sakthivel and Rajkumar [13]. This transmuted Sine-G family is the mixture of Sine function and quadratic rank transmuted map. The probability density function of transmuted Sine-G family of distributions is given by

$$f(x;\lambda) = \frac{\pi}{2}h(x)cos\left(\frac{\pi}{2}H(x)\right)\left[(1+\lambda)-2\lambda sin\left(\frac{\pi}{2}H(x)\right)\right];x>0,\lambda>0. \tag{3}$$

and the corresponding cumulative distribution function is given by

$$F(x;\lambda) = (1+\lambda)sin\left(\frac{\pi}{2}H(x)\right)-\lambda\left[sin\left(\frac{\pi}{2}H(x)\right)\right]^2;x>0,\lambda>0. \tag{4}$$

where, λ is the parameter of transmuted Sine-G family of distributions. If $\lambda=0$, the transmuted Sine-G family is becomes Sine-G family.

I. Transmuted Sine-Dagum Distribution

A continuous random variable X is said to be follow the transmuted Sine-Dagum distribution with parameters σ, θ, β and λ, (i.e.,) $X \sim TSD(X;\sigma,\theta,\beta,\lambda)$, then the probability density function of X is of the form

$$\begin{aligned} f(x;\sigma,\theta,\beta,\lambda) =& \frac{\pi}{2}\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\\ &\times\left[(1+\lambda)-2\lambda\, sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right];\\ &x>0,\sigma>0,\theta>0,\beta>0 \text{ and } -1\le\lambda\le 1 \end{aligned} \tag{5}$$

The above equation can be rewritten as

$$f(x;\sigma,\theta,\beta,\lambda) = (1+\lambda)\frac{\pi}{2}\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\pi\sigma\theta\beta\, x^{-\theta-1} \times \left(1+\sigma x^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) \quad (6)$$

The cumulative distribution function is given by

$$F(x;\sigma,\theta,\beta,\lambda) = \left[(1+\lambda) sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^{2}\right]; \quad x>0, \sigma>0, \theta>0, \beta>0 \text{ and } -1\leq\lambda\leq 1. \quad (7)$$

where θ is scale parameter; σ and β are shape parameters and λ is the parameter of quadratic rank transmutation map.

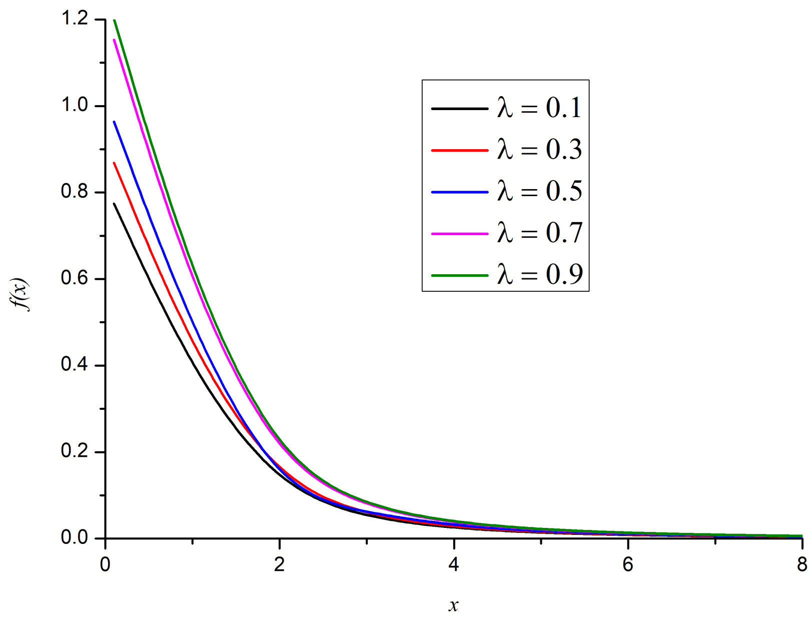

Figure 1: *Pdfs of transmuted Sine-Dagum distribution for fixed values of $\sigma = 0.5, \theta = 1, \beta = 2$ and different values of λ.*

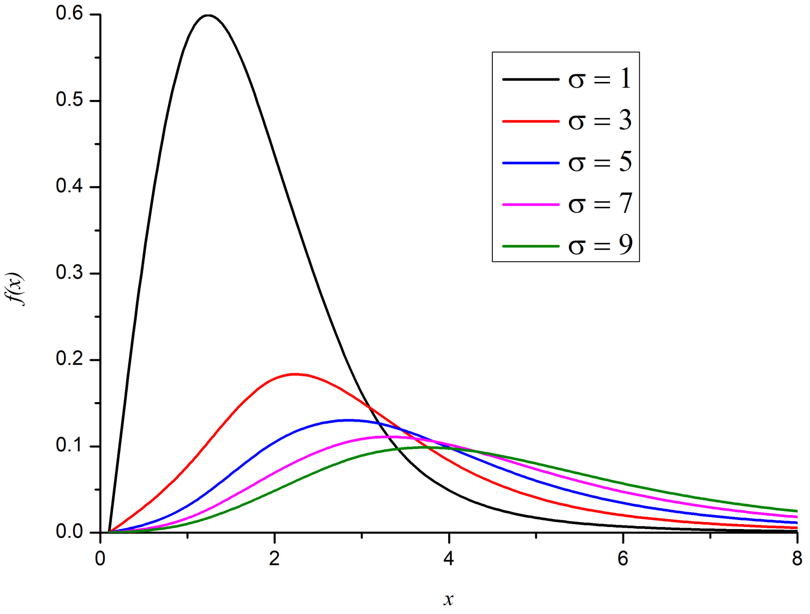

Figure 2: *Pdfs of transmuted Sine-Dagum distribution for fixed values of* $\theta = 2, \beta = 3, \lambda = 0.5$ *and different values of* σ.

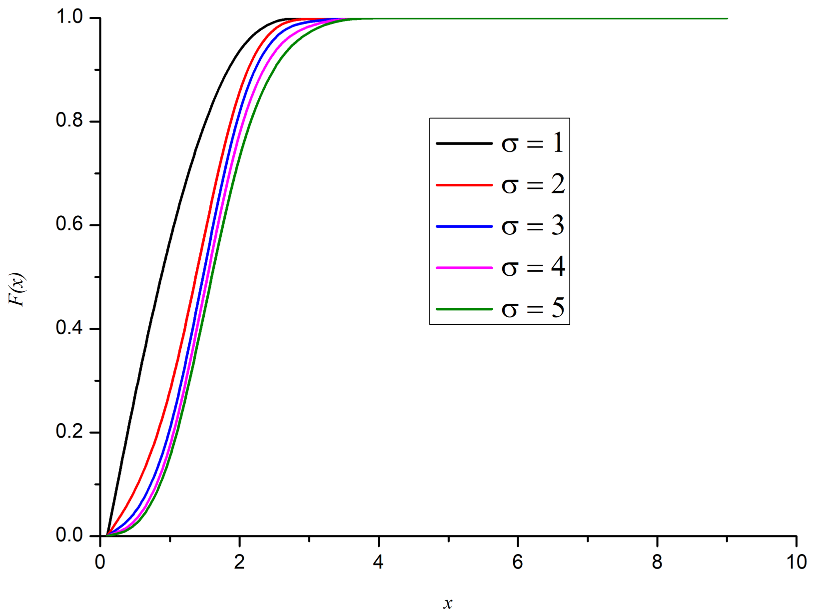

Figure 3: *Cdfs of transmuted Sine-Dagum distribution for fixed values of* $\theta = 4, \beta = 2, \lambda = 1$ *and different values of* σ.

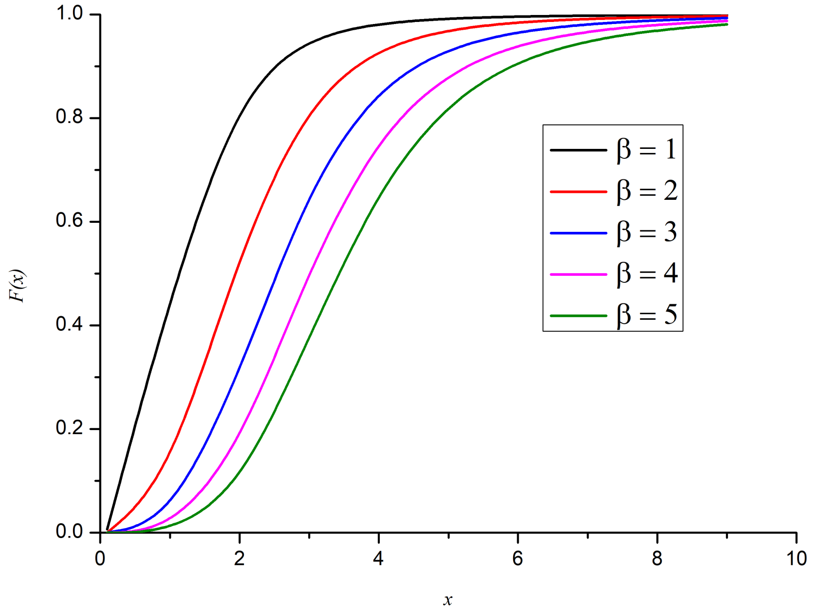

Figure 4: *Cdfs of transmuted Sine-Dagum distribution for fixed values of $\sigma = 4, \theta = 2, \lambda = 0.7$ and different values of β.*

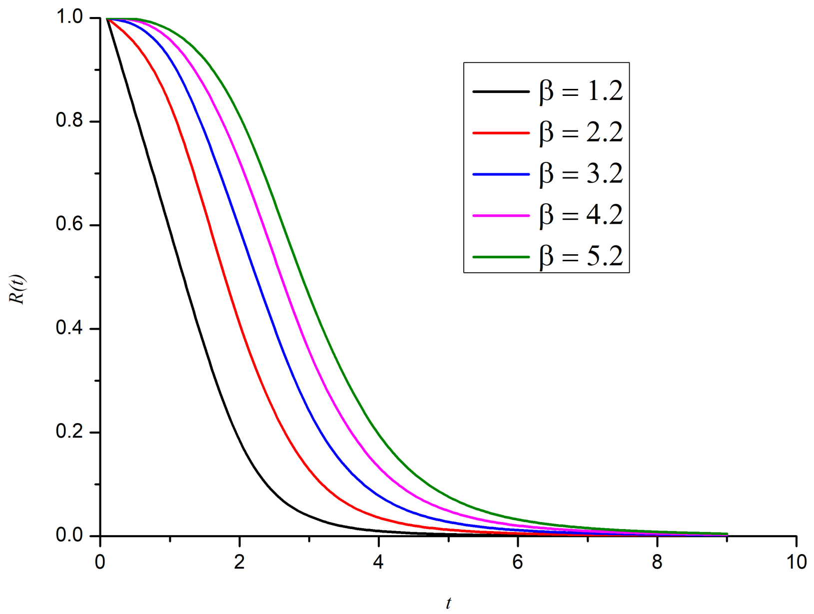

Figure 5: *Reliability function of transmuted Sine-Dagum distribution for fixed values of $\sigma = 3.5, \theta = 2.2, \lambda = 0.8$ and different values of β.*

III. Reliability Measures

In this Section, we deal with some reliability measures for transmuted Sine-Dagum distribution. If X∼TSD (X;σ,θ,β,λ), then the reliability measures of random variable X are given by;

I. Reliability function

The reliability function is defined by

$$R(x;\sigma,\theta,\beta,\lambda) = 1 - F(x;\sigma,\theta,\beta,\lambda)$$

$$= 1 - \left[(1+\lambda)sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^{2}\right]. \tag{8}$$

II. Hazard rate function

The hazard rate function is defined by

$$h(x,\sigma,\theta,\beta,\lambda) = \frac{f(x,\sigma,\theta,\beta,\lambda)}{1 - F(x,\sigma,\theta,\beta,\lambda)}$$

$$= \frac{\frac{\pi}{2}\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\left[(1+\lambda) - 2\lambda\, sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right]}{1 - \left[(1+\lambda)sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^{2}\right]}. \tag{9}$$

III. Reversed hazard rate function

The reversed hazard rate function is given by

$$r(x,\sigma,\theta,\beta,\lambda) = \frac{f(x,\sigma,\theta,\beta,\lambda)}{F(x,\sigma,\theta,\beta,\lambda)}$$

$$= \frac{\frac{\pi}{2}\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\left[(1+\lambda) - 2\lambda sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right]}{\left[(1+\lambda)sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^{2}\right]}. \tag{10}$$

IV. Cumulative hazard function

The cumulative hazard function is defined by

$$H(x,\sigma,\theta,\beta,\lambda) = -log\ R(x,\sigma,\theta,\beta,\lambda)$$

$$= -log\left[1 - \left((1+\lambda)sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^{2}\right)\right]. \tag{11}$$

V. Mean waiting time

The mean waiting time is defined by

$$\varphi(x) = x - \left[\frac{1}{F(x)}\int_0^x sf(s)ds\right]$$

$$\begin{aligned}\varphi(x) =& x - \left[\frac{1}{F(x)}\int_0^x \frac{\pi}{2}\sigma\theta\beta\, s^{-\theta}\left(1+\sigma s^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma s^{-\theta}\right)^{-\beta}\right)\right.\\ &\left.\times\left((1+\lambda) - 2\lambda sin\left(\frac{\pi}{2}\left(1+\sigma s^{-\theta}\right)^{-\beta}\right)\right)\right] ds\end{aligned}$$

Therefore, the mean waiting time of transmuted Sine-Dagum distribution is given by

$$= x - \frac{\begin{array}{c}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n}}{(2n)!}\left[B\left(1-\frac{1}{\theta},2\beta n+\beta+\frac{1}{\theta};y\right)\right.\\ \left.-\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n+1}}{(2n+1)!}2\lambda\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},4\beta n+2\beta+\frac{1}{\theta};y\right)\right]\end{array}}{\left[(1+\lambda)sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)-\lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^2\right]}. \quad (12)$$

VI. Mean residual life function

The mean residual life function is defined by

$$\phi(x) = \frac{1}{S(x)}\int_x^{\infty} xf(x)dx - x$$

$$= \frac{\begin{array}{c}\int_0^{\infty} x\left[\frac{\pi}{2}\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right.\\ \left.\times\left((1+\lambda)-2\lambda sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right)\right] dx - x\end{array}}{1-\left[(1+\lambda)sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)-\lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^2\right]}$$

Therefore, the mean residual life function of transmuted Sine-Dagum distribution is given by

$$= \frac{\begin{array}{c}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n}}{(2n)!}\left[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},2\beta n+\beta+\frac{1}{\theta}\right)\right.\\ \left.-\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n+1}}{(2n+1)!}\lambda\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},4\beta n+2\beta+\frac{1}{\theta}\right)\right]\end{array}}{1-\left[(1+\lambda)sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)-\lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^2\right]} - x. \quad (13)$$

VII. Mean past lifetime

The mean past lifetime of the component can be defined as

$$K(x) = E\left[x - X|X \le x\right] = \frac{\int_0^x F(t)dt}{F(x)} = x - \frac{\int_0^x tf(t)dt}{F(x)}$$

$$K(x) = x - \frac{\begin{array}{c}\int_0^x t\left[\frac{\pi}{2}\sigma\theta\beta\, t^{-\theta-1}\left(1+\sigma t^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma t^{-\theta}\right)^{-\beta}\right)\right.\\ \left.\times\left((1+\lambda)-2\lambda sin\left(\frac{\pi}{2}\left(1+\sigma t^{-\theta}\right)^{-\beta}\right)\right)\right] dt\end{array}}{\left[(1+\lambda)sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)-\lambda\left(sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right)^2\right]}$$

Therefore, the mean past life time is given by

$$= x - \frac{\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 2\beta n+\beta+\frac{1}{\theta}; y\right) - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} \lambda\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 4\beta n+2\beta+\frac{1}{\theta}; y\right) \right]}{\left[(1+\lambda) sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^2 \right]}. \tag{14}$$

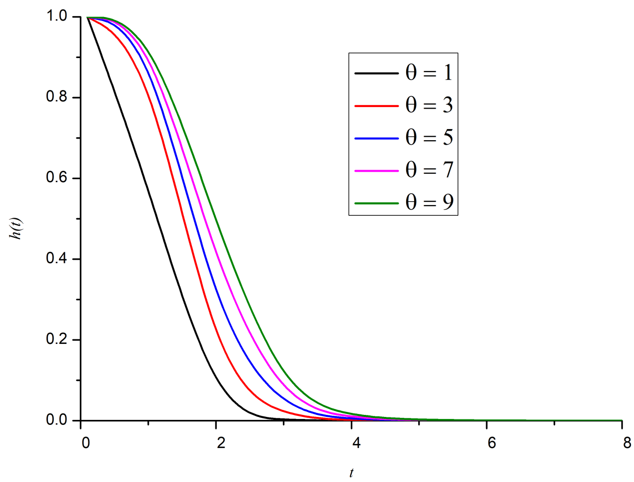

Figure 6: *Reliability function of transmuted Sine-Dagum distribution for fixed values of $\sigma = 1.4, \beta = 2.4, \lambda = 0.7$ and different values of θ.*

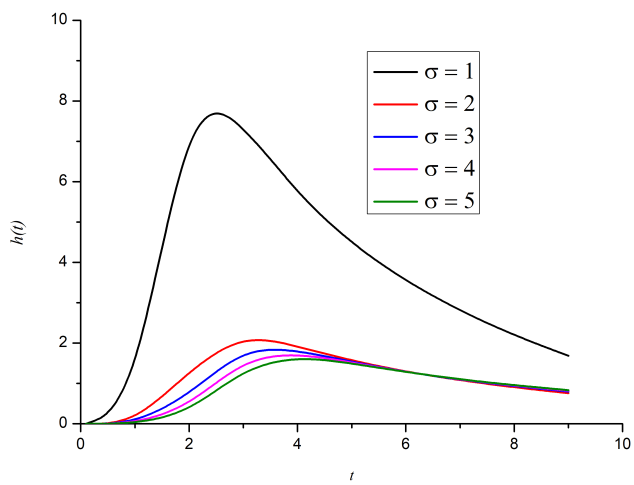

Figure 7: *Hazard rate function of transmuted Sine-Dagum distribution for fixed values of $\theta = 4, \beta = 6, \lambda = 0.5$ and different values of σ.*

IV. Distributional Properties

I. Moments

The r^{th} moment of transmuted Sine-Dagum distribution of the random variable X is obtained as

$$\mu_r^{'} = \int_0^\infty x^r \Bigg[(1+\lambda)\frac{\pi}{2}\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)$$
$$-\lambda\pi\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\Bigg] dx$$

Using the Taylor series of the sine and cosine functions for moments, we have

$$sin\ x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad cos\ x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

Therefore, we have

$$cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n}\left(1+\sigma x^{-\theta}\right)^{-2\beta n}}{(2n)!}$$

$$sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n+1}\left(1+\sigma x^{-\theta}\right)^{-2n\beta-\beta}}{(2n+1)!}$$

Hence, the r^{th} moment of transmuted Sine-Dagum distribution is given by

$$\mu_r^{'} = \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n}}{(2n)!}\Bigg[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{r}{\theta}} B\left(1-\frac{r}{\theta}, 2\beta n+\beta+\frac{r}{\theta}\right)$$
$$-\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n+1}}{(2n+1)!}\lambda\beta\sigma^{\frac{r}{\theta}} B\left(1-\frac{r}{\theta}, 4\beta n+2\beta+\frac{r}{\theta}\right)\Bigg] \tag{15}$$

We have obtained the mean and variance of this distribution as

$$\mu_1^{'} = \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n}}{(2n)!}\Bigg[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 2\beta n+\beta+\frac{1}{\theta}\right)$$
$$-\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n+1}}{(2n+1)!}\lambda\pi\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 4\beta n+2\beta+\frac{1}{\theta}\right)\Bigg]$$

and

$$v(x) = \Bigg[\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n}}{(2n)!}\Bigg((1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{2}{\theta}} B\left(1-\frac{2}{\theta}, 2\beta n+\beta+\frac{2}{\theta}\right)$$
$$-\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n+1}}{(2n+1)!}2\lambda\pi\beta\sigma^{\frac{2}{\theta}} B\left(1-\frac{2}{\theta}, 4\beta n+2\beta+\frac{2}{\theta}\right)\Bigg)\Bigg] -$$
$$\Bigg[\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n}}{(2n)!}\Bigg((1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 2\beta n+\beta+\frac{1}{\theta}\right)$$
$$-\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n+1}}{(2n+1)!}\lambda\pi\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 4\beta n+2\beta+\frac{1}{\theta}\right)\Bigg)\Bigg]^2 \tag{16}$$

The moment generating function of transmuted Sine-Dagum distribution is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left((1+\lambda) \frac{\pi}{2} \beta \sigma^{\frac{r}{\theta}} B\left(1 - \frac{r}{\theta}, 2\beta n + \beta + \frac{r}{\theta}\right) \right.\right.$$
$$\left.\left. - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} 2\lambda\pi\beta\sigma^{\frac{r}{\theta}} B\left(1 - \frac{r}{\theta}, 4\beta n + 2\beta + \frac{r}{\theta}\right) \right) \right] \tag{17}$$

The characteristic function is given by

$$\Phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left((1+\lambda) \frac{\pi}{2} \beta \sigma^{\frac{r}{\theta}} B\left(1 - \frac{r}{\theta}, 2\beta n + \beta + \frac{r}{\theta}\right) \right.\right.$$
$$\left.\left. - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} 2\lambda\beta\sigma^{\frac{r}{\theta}} B\left(1 - \frac{r}{\theta}, 4\beta n + 2\beta + \frac{r}{\theta}\right) \right) \right] \tag{18}$$

and the cumulant generating function is given by

$$K_X(t) = log \left[\sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left[(1+\lambda) \frac{\pi}{2} \beta \sigma^{\frac{r}{\theta}} B\left(1 - \frac{r}{\theta}, 2\beta n + \beta + \frac{r}{\theta}\right) \right.\right.\right.$$
$$\left.\left.\left. - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} \lambda\pi\beta\sigma^{\frac{r}{\theta}} B\left(1 - \frac{r}{\theta}, 4\beta n + 2\beta + \frac{r}{\theta}\right) \right] \right] \right]. \tag{19}$$

II. Inverted moments

The inverted moment is defined by

$$\mu_r^* = \int_{-\infty}^{\infty} x^{-r} f(x) dx$$

Thus, the inverted moment for this distribution is given by

$$\mu_r^* = \int_0^{\infty} x^{-r} \left[(1+\lambda) \frac{\pi}{2} \sigma\theta\beta\, x^{-\theta-1} \left(1 + \sigma x^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1 + \sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\pi\sigma\theta\beta\, x^{-\theta-1} \right.$$
$$\left. \times \left(1 + \sigma x^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1 + \sigma x^{-\theta}\right)^{-\beta}\right) sin\left(\frac{\pi}{2}\left(1 + \sigma x^{-\theta}\right)^{-\beta}\right) \right] dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left[(1+\lambda) \frac{\pi}{2} \beta \sigma^{\frac{r}{\theta}} B\left(1 + \frac{r}{\theta}, 2\beta n + \beta - \frac{r}{\theta}\right) \right.$$
$$\left. - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} \lambda\pi\beta\sigma^{-\frac{r}{\theta}} B\left(1 + \frac{r}{\theta}, 4\beta n + 2\beta - \frac{r}{\theta}\right) \right]. \tag{20}$$

III. Incomplete moments

The r^{th} incomplete moment is defined by

$$m_r(x) = \int_0^x s^r f(s) ds$$

$$m_r(x) = \int_0^x s^r \left[\frac{\pi}{2} \sigma\theta\beta\, s^{-\theta-1} \left(1 + \sigma s^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1 + \sigma s^{-\theta}\right)^{-\beta}\right) - \lambda\pi\sigma\theta\beta\, s^{-\theta-1} \right.$$
$$\left. \times \left(1 + \sigma s^{-\beta-1}\right) cos\left(\frac{\pi}{2}\left(1 + \sigma s^{-\theta}\right)^{-\beta}\right) sin\left(\frac{\pi}{2}\left(1 + \sigma s^{-\theta}\right)^{-\beta}\right) \right] ds$$

Hence, the r^{th} incomplete moments of transmuted Sine-Dagum distribution is given by

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{r}{\theta}} B\left(1-\frac{r}{\theta}, 2\beta n+\beta+\frac{r}{\theta}; y\right) - \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n+1}}{(2n+1)!} 2\lambda\beta\sigma^{\frac{r}{\theta}} B\left(1-\frac{r}{\theta}, 4\beta n+2\beta+\frac{r}{\theta}; y\right) \right]. \quad (21)$$

IV. Central moments

The central moment is defined by

$$\mu_r = \int_{-\infty}^{\infty} (x-\mu_1')^r f(x)dx = \sum_{m=0}^{r} \binom{r}{m} (-1)^m (\mu_1')^m \mu_{r-m}'$$

Therefore, the central moments of transmuted Sine-Dagum distribution is given by

$$\mu_r = \sum_{m=0}^{r} \binom{r}{m} (-1)^m \times \left[\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 2\beta n+\beta+\frac{1}{\theta}\right) - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} \lambda\pi\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 4\beta n+2\beta+\frac{1}{\theta}\right) \right] \right] \times \left[\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{r-m}{\theta}} B\left(1-\frac{r-m}{\theta}, 2\beta n+\beta+\frac{r-m}{\theta}\right) - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} \lambda\pi\beta\sigma^{\frac{r-m}{\theta}} B\left(1-\frac{r-m}{\theta}, 4\beta n+2\beta+\frac{r-m}{\theta}\right) \right] \right]. \quad (22)$$

V. Order statistics

The pdf of the j^{th} order statistics for transmuted Sine-Dagum distribution is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)(n-j)!} \left[\frac{\pi}{2}\sigma\theta\beta\, x^{-\theta-1} \left(1+\sigma x^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) \times \left((1+\lambda) - 2\lambda sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right) \right] \times \left[(1+\lambda) sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right)^2 \right]^{j-1} \times \left[1 - \left((1+\lambda) sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right)^2 \right) \right]^{n-1} \quad (23)$$

The pdf of the smallest order statistics $X_{(1)}$ is given by

$$f_{X_{(1)}}(x) = n \left[1 - \left((1+\lambda) sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right)^2 \right) \right]^{n-1} \times \left[\frac{\pi}{2}\sigma\theta\beta\, x^{-\theta-1} \left(1+\sigma x^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) \times \left((1+\lambda) - 2\lambda sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right) \right] \quad (24)$$

The pdf of the largest order statistics $X_{(n)}$ is given by

$$\begin{aligned} f_{X_{(n)}}(x) =& n\left[\left((1+\lambda)sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)-\lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^2\right)\right]^{n-1} \\ &\times\left[\frac{\pi}{2}\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right. \\ &\times\left.\left((1+\lambda)-2\lambda sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right)\right] \end{aligned} \tag{25}$$

The pdf of the median order statistics is given by

$$\begin{aligned} f_{m+1:n}(x) =& \frac{(2m+1)}{m!m!}\left[\left((1+\lambda)sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)-\lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^2\right)\right]^{m} \\ &\times\left[1-\left((1+\lambda)sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)-\lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^2\right)\right]^{m} \\ &\times\left[\frac{\pi}{2}\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}cos\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right. \\ &\times\left.\left((1+\lambda)-2\lambda sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right)\right]. \end{aligned} \tag{26}$$

V. Income inequality measures

In this section, we deal with different basic income inequality measures including Lorenz curve, Bonferroni index and Zenga index. The following measures are given below.

I. Lorenz curve

The lorenz curve was introduced by Lorenz [8] in the year 1905. it is widely used in economic and many other fields and its defined by

$$L(x)=\frac{1}{\mu}\int_0^x sf(s)ds$$

Hence, the Lorenz curve of transmuted Sine-Dagum distribution is given by

$$\begin{aligned} L(x) =& \frac{1}{\mu}\int_0^x s\left[\frac{\pi}{2}\sigma\theta\beta\, s^{-\theta-1}\left(1+\sigma s^{-\theta}\right)^{-\beta-1}cos\left(\frac{\pi}{2}\left(1+\sigma s^{-\theta}\right)^{-\beta}\right)\right. \\ &\times\left.\left((1+\lambda)-2\lambda sin\left(\frac{\pi}{2}\left(1+\sigma s^{-\theta}\right)^{-\beta}\right)\right)\right]dx \end{aligned}$$

$$=\frac{\begin{array}{c}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n}}{(2n)!}\left[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{r}{\theta}}B\left(1-\frac{r}{\theta},2\beta n+\beta+\frac{r}{\theta};y\right)\right. \\ \left.-\sum_{n=0}^{\infty}\frac{(-1)^n(\pi/2)^{2n+1}}{(2n+1)!}2\lambda\beta\sigma^{\frac{r}{\theta}}B\left(1-\frac{r}{\theta},4\beta n+2\beta+\frac{r}{\theta};y\right)\right]\end{array}}{\begin{array}{c}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\pi}{2})^{2n}}{(2n)!}\left[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},2\beta n+\beta+\frac{1}{\theta}\right)\right. \\ \left.-\sum_{n=0}^{\infty}\frac{(-1)^n(\pi/2)^{2n+1}}{(2n+1)!}2\lambda\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},4\beta n+2\beta+\frac{1}{\theta}\right)\right].\end{array}} \tag{27}$$

II. Bonferroni index

The Bonferroni index was introduced by Bonferroni [2] in the year 1930. it is ratio of Lorenz curve and cumulative distribution function of the distribution. The Bonferroni index is defined as

$$B(x)=\frac{L(x)}{F(x)}$$

Hence, the Bonferroni index of transmuted Sine-Dagum distribution is given by

$$B(x) = \frac{\delta}{\eta}$$

where

$$\delta = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{r}{\theta}} B\left(1-\frac{r}{\theta}, 4\beta n + 2\beta + \frac{r}{\theta}; y\right) - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} \lambda\pi\beta\sigma^{\frac{r}{\theta}} B\left(1-\frac{r}{\theta}, 4\beta n + 2\beta + \frac{r}{\theta}; y\right)\right]$$

$$\eta = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 2\beta n + \beta + \frac{1}{\theta}\right) - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} 2\lambda\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 4\beta n + 2\beta + \frac{1}{\theta}\right)\right] \times \left[(1+\lambda) sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^2 \right].$$

III. Zenga index

Zenga index is introduced by Zenga [15] in the year 1980. The Zenga index is defined by

$$Z = 1 - \frac{\bar{\mu}_{(x)}}{\mu^{+}_{(x)}}$$

where

$$\bar{\mu}_{(x)} = \frac{1}{F(x)} \int_0^x s f(s) ds$$
$$\mu^{+}_{(x)} = \frac{1}{1-F(x)} \int_0^{\infty} x f(x) dx$$

Therefore, we get

$$\bar{\mu}_{(x)} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{r}{\theta}} B\left(1-\frac{r}{\theta}, 2\beta n + \beta + \frac{r}{\theta}; y\right) - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} \lambda\pi\beta\sigma^{\frac{r}{\theta}} B\left(1-\frac{r}{\theta}, 4\beta n + 2\beta + \frac{r}{\theta}; y\right)\right]}{\left[(1+\lambda) sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right)^2 \right]}$$

$$\mu^{+}_{(x)} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 2\beta n + \beta + \frac{1}{\theta}\right) - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} \lambda\pi\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 4\beta n + 2\beta + \frac{1}{\theta}\right)\right]}{1 - \left[(1+\lambda) sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right)^2 \right]}$$

Hence, the Zenga index of transmuted Sine-Dagum distribution is given by

$$Z = 1 - \frac{\gamma}{\delta}$$

where

$$\gamma = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left[(1+\lambda)\frac{\pi}{2}\beta\sigma^{\frac{r}{\theta}} B\left(1-\frac{r}{\theta}, 2\beta n + \beta + \frac{r}{\theta}; y\right) \right.$$
$$\left. - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} \lambda\pi\beta\sigma^{\frac{r}{\theta}} B\left(1-\frac{r}{\theta}, 4\beta n + 2\beta + \frac{r}{\theta}; y\right) \right]$$
$$\times \left[1 - \left((1+\lambda) sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right)^2 \right) \right]$$

$$\delta = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!} \left[\frac{(1+\lambda)\pi}{2}\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 2\beta n + \beta + \frac{1}{\theta}\right) \right.$$
$$\left. - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} \lambda\pi\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, 4\beta n + 2\beta + \frac{1}{\theta}\right) \right]$$
$$\times \left[(1+\lambda) sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right) - \lambda\left(sin\left(\frac{\pi}{2}\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)\right)^2 \right].$$

VI. Parameter Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample from the transmuted Sine-Dagum distribution then the likelihood function is given by

$$L(\sigma,\theta,\beta,\lambda;\underline{x}) = \prod_{i=1}^{n} \left[\frac{\pi}{2}\sigma\theta\beta\, x_{(i)}^{-\theta-1} \left(1+\sigma_{(i)}^{-\theta}\right)^{-\beta-1} cos\left(\frac{\pi}{2}\left(1+\sigma_{(i)}^{-\theta}\right)^{-\beta}\right) \right.$$
$$\left. \times \left((1+\lambda) - 2\lambda sin\left(\frac{\pi}{2}\left(1+\sigma_{(i)}^{-\theta}\right)^{-\beta}\right)\right) \right] \tag{28}$$

Hence, the log likelihood function is given by

$$L(\sigma,\theta,\beta,\lambda;\underline{x}) = n\ log\frac{\pi}{2} + n\ log\sigma + n\ log\theta + n\ log\beta + (-\theta-1)\sum_{i=1}^{n} logx_{(i)}$$
$$+ (-\beta-1)\sum_{i=1}^{n} log\left(1+\sigma x_{(i)}^{-\theta}\right) + \sum_{i=1}^{n} log\ cos\left(\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta}\right)$$
$$+ \sum_{i=1}^{n} log\ cos\left((1+\lambda) - 2\lambda\ sin\left(\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta}\right)\right)$$

The MLE of parameters σ, θ, β and λ are obtained from the following equations

$$\frac{\partial logL}{\partial\sigma} = 0, \frac{\partial logL}{\partial\theta} = 0, \frac{\partial logL}{\partial\beta} = 0 \text{ and } \frac{\partial logL}{\partial\lambda} = 0$$

That is,

$$\frac{\partial logL}{\partial\sigma} = \frac{n}{\sigma} + \sum_{i=1}^{n} \frac{(-\beta-1)x_{(i)}^{-\theta}}{(1+\sigma x_i^{-\theta})} + \sum_{i=1}^{n} \frac{sin\left(\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta}\right)\frac{\pi}{2}\beta\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta-1} x_{(i)}^{-\theta}}{cos\left(\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta}\right)} = 0 \tag{29}$$

$$\frac{\partial logL}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} logx_{(i)} + \sum_{i=1}^{n} \frac{(-\beta-1)x_{(i)}^{-\theta} log\ x_{(i)}}{\left(1+\sigma x_{(i)}^{-\theta}\right)}$$
$$+ \sum_{i=1}^{n} \frac{sin\left(\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta}\right)\frac{\pi}{2}\beta\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta-1}\sigma x_{(i)}^{-\theta} log\ x_{(i)}}{cos\left(\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta}\right)}$$
$$- \sum_{i=1}^{n} \frac{2\lambda cos\left(\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta}\right)\beta\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta-1}\sigma x_{(i)}^{-\theta} logx_{(i)}}{cos\left((1+\lambda) - 2\lambda\ sin\left(\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta}\right)\right)} = 0 \tag{30}$$

$$\frac{\partial logL}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} log\left(1+\sigma x_{i}^{-\theta}\right)^{-\beta}$$
$$- \sum_{i=1}^{n} \frac{sin\left(\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta}\right)\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta} log\left(1+\sigma x_{(i)}^{-\theta}\right)}{cos\left(\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta}\right)} = 0 \tag{31}$$

and

$$\frac{\partial logL}{\partial\lambda} = \frac{1-2sin\left(\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta}\right)}{(1+\lambda) - 2\lambda sin\left(\frac{\pi}{2}\left(1+\sigma x_{(i)}^{-\theta}\right)^{-\beta}\right)} = 0. \tag{32}$$

The above mentioned four nonlinear equations are difficult to solve analytically. Therefore, these equations can be solved through iteration methods like Newton-Raphshon method etc., However, we estimate the parameters using R software.

VII. Applications

The data set is about the time-to-failure of a 100 cm polyster/viscose in a textile experiment to evaluate the tensile fatigue characteristics of the yarn when its strain level is 2.3 percentage. This data set is used early by Quesenberry and Kent [12], Pal and Tiensuwan [11] and Nasiru et al. [10].

We have fitted the model based on the minimum value of different goodness of fit measures values represented by -2log likelihood, corrected Akaike Information Criterion (CAIC), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC). We have compared the Topp-Leone generated Dagum distribution with other competitive statistical models like Exponentiated generalized exponential Dagum distribution (EGEDD), Exponentiated generalized Dagum distribution (EGDD), Dagum distribution (DD), Exponentiated generalized exponential Burr distribution (EGEBD), Exponentiated generalized Burr distribution (EGBD), Exponentiated generalized exponential Frechet distribution (EGEFD), Exponentiated generalized Frechet distribution (EGFD), Mc-Dagum distribution (McD), exponentiated Kumaraswamy Dagum distribution (EKDD) for this yarn data. The transmuted Sine-Dagum distribution provides better fit for tensile fatigue characteristics of the yarn data compared to other above mentioned competitive statistical models. The details are given in the following tables.

Table 1: *Summary of statistics for tensile fatigue characteristic of the yarn data*

n	Mean	Median	Minimum	Maximum	Q_1	Q_3
100	222.0	195.5	15.0	829.0	129.2	283.0

Table 2: *The values of estimated parameters for tensile fatigue characteristic of the yarn data*

Model	Estimated value of the parameters
TSDD	$\hat{\sigma}$=10868.26, $\hat{\theta}$=1.732, $\hat{\beta}$=1.0574, λ=-0.473
EGEDD	$\hat{\sigma}$=0.026, $\hat{\sigma}$=75 .310, $\hat{\beta}$=0.017, $\hat{\theta}$=3.513, $\hat{c}$=45.692, $\hat{d}$=0.090
EGDD	$\hat{\alpha}$=1.992, $\hat{\beta}$=10.480, $\hat{\theta}$=4.733, $\hat{c}$=75.487, $\hat{d}$=0.223
DD	$\hat{\alpha}$=19.749, $\hat{\beta}$=11.599, $\hat{\theta}$=1.126
EGEBD	$\hat{\lambda}$=35.463, $\hat{\beta}$=35.965, $\hat{\theta}$=4.859, $\hat{c}$=15.667, $\hat{d}$=0.070
EGBD	$\hat{\beta}$=24.801, $\hat{\theta}$=4.196, $\hat{c}$=73.9120, $\hat{d}$=0.258
EGEFD	$\hat{\alpha}$=20.662, $\hat{\lambda}$=34.477, $\hat{\theta}$=5.217, $\hat{c}$=16.438, $\hat{d}$=0.65
EGFD	$\hat{\alpha}$=10.537, $\hat{\theta}$=5.239, $\hat{c}$=21.341, $\hat{d}$=0.140
McD	$\hat{\lambda}$=0.027, $\hat{\delta}$=0.600, $\hat{\beta}$=98.780, $\hat{a}$=0.333, $\hat{b}$=25.042, $\hat{c}$=46.276
EKD	$\hat{\alpha}$=546.109, $\hat{\lambda}$=39.413, $\hat{\delta}$=5.188, $\hat{\phi}$=0.203, $\hat{\theta}$=31.169

Table 3: *Statistical model selection for tensile fatigue characteristic of the yarn data*

Model	-2LL	AIC	AICC	BIC
TSDD	1252.689	1260.689	1261.11	1271.11
EGEDD	1256.34	1268.336	1269.553	1283.967
EGDD	1306.14	1316.137	1317.040	1329.163
DD	1298.52	1304.517	1304.938	1312.333
EGEBD	1261.74	1271.745	1272.648	1284.771
EGBD	1306.06	1314.056	1314.694	1324.447
EGEFD	1261.52	1271.523	1272.426	1284.549
EGFD	1333.76	1341.757	1342.395	1352.177
McD	1256.4	1268.399	1269.616	1284.030
EKD	1307.92	1317.913	1318.816	1330.938

VIII. Conclusion

In this paper, we have presented a new transmuted Sine-Dagum distribution using transmuted Sine-G family of distributions. We have studied some reliability measures like reliability function, hazard rate function, reverse hazard rate function, cumulative hazard rate function, second failure rate function, mean waiting time, mean past lifetime and mean residual life. We have obtained some distributional properties like moments, moment generating function, characteristic function, cumulant generating function, incomplete moments, central moments and order statistics. We have also investigated some income inequality measures including Lorenz curve, Bonferroni index and Zenga index for proposed new probability distribution. The maximum likelihood method is used to estimate the parameters of proposed new probability distribution. Finally, we have analysed a real lifetime data set for proposed probability distribution. The proposed distribution fits well for this data compared to other competitive models.

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