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On Transmuted Exponential-Topp Leon Distribution with Monotonic and Non-Monotonic Hazard Rates and its Applications

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Abstract

For the last decade, inspired by the increasing demand for probability distributions in numerous fields, many generalized distributions have been studied. Most of these distributions are developed by adding one or more parameter(s) to the standard probability distributions to make them flexible in capturing the sensitive parts of a dataset. The Topp-Leone distribution (TL) is one of the continuous probability distributions used in modelling lifetime datasets and sometimes is called J-shaped distribution. In this paper, we proposed a new lifetime distribution named transmuted Exponential- Topp Leon distribution in short (TE-TLD) which possessed different density shapes. Some properties of the distribution were presented in an explicit form and the parameters of the distribution are estimated by the method of maximum likelihood. The hazard function of the TE-TLD can be monotonic or non-monotonic failure rate which makes it more robust in terms of studying failure rates. The TE-TLD outperformed other distributions with the same underlying baseline distribution when applied to real datasets in the study. Furthermore, the likelihood ratio test (LRT) shows that the additional parameter(s) are significant which further proves the robustness of the TE-TLD over the nested distributions in the study.

Keywords: Topp Leon distribution, failure rate, Maximum Likelihood, generalized distributions

1. Introduction

In reliability and survival analysis, lifetime distributions such as Exponential, Weibull among others, play an important role in modelling lifetime data. Most of these distributions have infinite support in theory, as the lifetime of a system or item can be infinite. On the other hand, distributions with finite support will be appropriate in modelling data sets that are generated as a result of limited power supply, the design life of the system, among others [1]. For the past few years, inspired by the increasing demand for probability distributions in numerous fields, many generalized distributions have been studied. Most of these distributions are developed by adding one or more parameter(s) to the standard probability distributions to make them robust in capturing the sensitive parts of a dataset. For example, [2] proposed Beta Exponential distribution which has three parameters and was found to be more flexible than the classical Exponential distribution.

The Topp-Leone distribution (TL) is one of the continuous probability distributions used in modelling lifetime datasets and sometimes is called J-shaped distribution. This distribution has a closed-form and was proposed by [3]. However, the J-shaped distribution had not received much attention due to some of its complexity until [4] studied some properties of the distribution which include moments, central moments, and characteristic function. This work led to increasing interest in studying TL distribution. For instance, [1] studied and explored some reliability measures and their stochastic orderings, a comprehensive study on flat-toppedness of the TL distribution was studied by [5], study on record values by [6], the moments of the order statistics of the TL distribution was studied by [7], the goodness-of-fit tests for the TL distribution are evaluated by [8] and [9] proposed Topp-Leone-Exponential distribution which has two parameters and is skewed to the right.

Current kinds of literature pay more attention to propose more flexible distributions but give less concern to the hazard function of the distributions. In reliability analysis, hazard rate plays an important role to characterize life phenomena and as well guides in model selection [10]. Furthermore, many systems exhibit failure rates that are non-monotonic. For instance, the failure rate pattern of numerous electronic components comprises of three phases: initial phase (or burn-in) where failure is high at the start of the product life cycle due to design and manufacturing problems and decreases to a constant level, the middle phase (flat region) with an approximately constant hazard rate, and the final phase (or wear-out stage), from where the hazard rate starts to increase: This failure (hazard) rates are "U" or bathtub shaped. The Exponential, Weibull, Gamma among other distributions, and some of their extensions allow only monotone failure rates and are unable to produce bathtub shape and thus cannot appropriately describe the datasets with this feature. These have opened room for more research that can account for monotone and non-monotone hazard rate function.

In this research, we developed an extension of Topp Leon distribution named transmuted Exponential-Topp Leon distribution (TE-TLD) which possessed both monotonic and non-monotonic failure rate shapes and its density function can be left-skewed, right-skewed, Bathtub, or J-shape. The cumulative distribution and probability density function of Topp Leon distribution are respectively given as;

$$G(x,\alpha) = x^{\alpha}(2-x)^{\alpha} \tag{1}$$

and

$$g(x,\alpha) = 2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1} \tag{2}$$

where, $0 < x < 1\, and\, \alpha > 0$.

Based on the work of [11], the cdf and pdf of Transmuted Exponential-G family of distributions are respectively given by;

$$F(x;\lambda,\theta,\xi) = \left(1-(1-G(x,\xi))^{\lambda}\right)\left(1+\theta\,(1-G(x,\xi))^{\lambda}\right) \tag{3}$$

and

$$f(x;\lambda,\theta,\xi) = \frac{g(x,\xi)}{1-G(x,\xi)}\lambda\,(1-G(x,\xi))^{\lambda}\left(1-\theta+2\theta\,(1-G(x,\xi))^{\lambda}\right) \tag{4}$$

Where, $G(x,\xi)$ and $g(x,\xi)$ are the baseline cdf and pdf respectively depending on a vector parameter ξ whereas, $\lambda > 0$, $-1 \leq \theta \leq 1$ are two additional parameters i.e scale and transmuted (shape) parameter respectively.

2. Transmuted Exponential-Topp Leon Distribution

Substituting equations (1) and (2) into (3) yields cumulative distribution function (cdf) of the transmuted Exponential-Topp Leon distribution (TE-TLD).

$$F(x) = \left[1-(1-x^{\alpha}(2-x)^{\alpha})^{\lambda}\right]\left[1+\theta\,(1-x^{\alpha}(2-x)^{\alpha})^{\lambda}\right] \tag{5}$$

and the associated probability density function (pdf) is given by;

$$f(x) = 2\alpha\lambda x^{\alpha-1}(1-x)(2-x)^{\alpha-1}\,(1-x^{\alpha}(2-x)^{\alpha})^{\lambda-1}\left[1-\theta+2\theta\,(1-x^{\alpha}(2-x)^{\alpha})^{\lambda}\right] \tag{6}$$

where, $0 < x < 1,\ \alpha\,\lambda > 0\,and\ -1 \leq \theta \leq 1.$

A useful linear representation for the pdf of TE-TLD is given as;

$$f(x) = 2\sum_{j=0}^{n} A_j x^{\alpha(1+j)-1}(1-x)(2-x)^{\alpha(1+j)-1} \tag{7}$$

$$where\ A_j = (-1)^j \alpha\lambda \left\{(1-\theta)\left(\begin{array}{c}\lambda-1\\ j\end{array}\right) + 2\theta\left(\begin{array}{c}2\lambda-1\\ j\end{array}\right)\right\}$$

2.1. Distribution validity check

Fact 1: The TE-TLD is a valid density function.

$\int\limits_0^1 f(x;\alpha\,\lambda\,\theta)dx = 1$

Proof:

$\int\limits_0^1 \left(2\alpha\lambda x^{\alpha-1}(1-x)(2-x)^{\alpha-1}\left(1-x^{\alpha}(2-x)^{\alpha}\right)^{\lambda-1}\left[1-\theta+2\theta\left(1-x^{\alpha}(2-x)^{\alpha}\right)^{\lambda}\right]\right)dx$

$let\, u = 1 - x^{\alpha}(2-x)^{\alpha},\ \ as\, x \to 0,\ u \to 1\, and\, x \to 1,\ u \to 0$

$\frac{du}{dx} = -\left(x^{\alpha}\alpha(2-x)^{\alpha-1}(-1) + (2-x)^{\alpha}\alpha x^{\alpha-1}\right) = -\left(x^{\alpha}\alpha(2-x)^{\alpha}\left(\frac{-1}{(2-x)} + \frac{1}{x}\right)\right)$

$\frac{du}{dx} = -x^{\alpha}\alpha(2-x)^{\alpha}\left(\frac{-x+2-x}{(2-x)x}\right) = \frac{-2\alpha x^{\alpha}(2-x)^{\alpha}(1-x)}{(2-x)x}$

$dx = \frac{-du}{2\alpha x^{\alpha-1}(2-x)^{\alpha-1}(1-x)}$

$\int\limits_0^1 \left(2\alpha\lambda x^{\alpha-1}(1-x)(2-x)^{\alpha-1}u^{\lambda-1}\left[1-\theta+2\theta u^{\lambda}\right]\right)\frac{du}{2\alpha x^{\alpha-1}(2-x)^{\alpha-1}(1-x)}$

$\int\limits_0^1 \lambda u^{\lambda-1}(1-\theta+2\theta u^{\lambda})du$

$let\, v = u^{\lambda},\ \frac{dv}{du} = \lambda u^{\lambda-1},\ du = \frac{dv}{\lambda u^{\lambda-1}}$

$\int\limits_0^1 \lambda u^{\lambda-1}(1-\theta+2\theta v)\frac{dv}{\lambda u^{\lambda-1}}$

Finally,

$\int\limits_0^1 (1-\theta+2\theta v)dv = \left(v - \theta v + \theta v^2\right)_0^1 = 1$

2.2. Graphical illustration of the pdf and cdf of TE-TLD

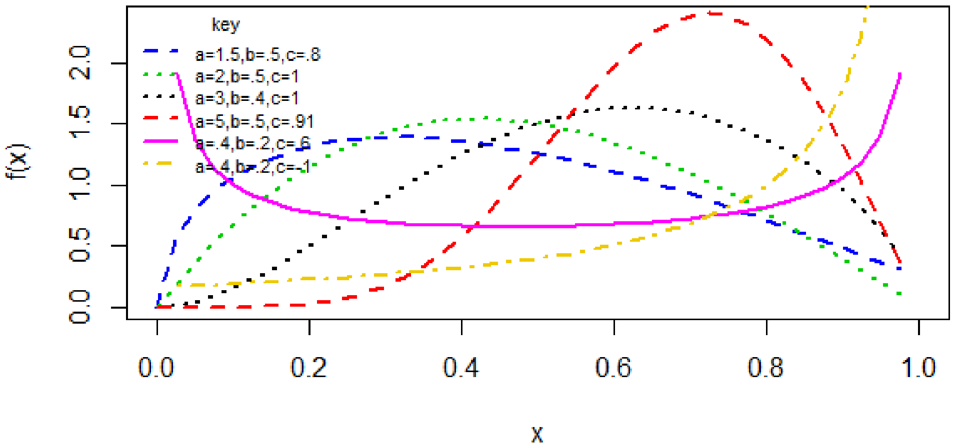

Figure 1: *The pdf plot of TE-TLD*

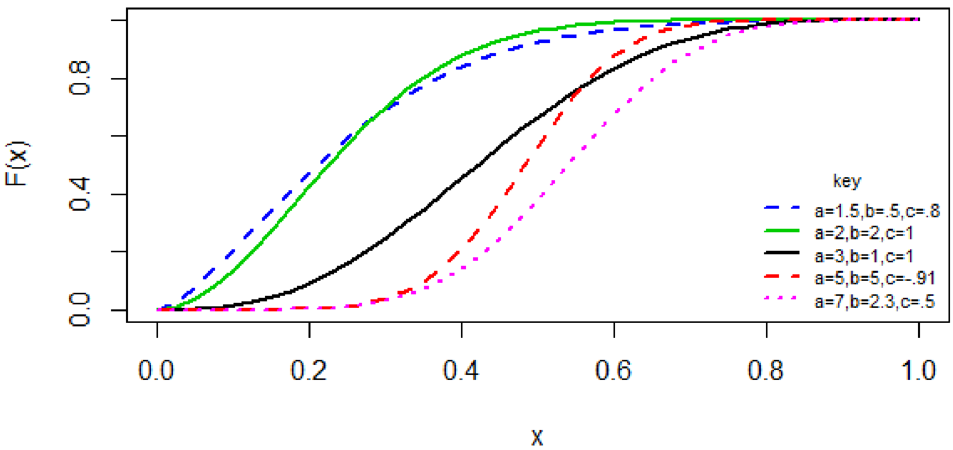

Figure 2: *The cdf plot of TE-TLD*

The shape of the density function corresponding to the TE-TLD may be characterized as follows;

- 1. For $\alpha = 1.5,\ \lambda = .5,\ \theta = 0.8\, and\, \alpha = 2,\ \lambda = .5,\ \theta = 1$, f(x) has a positive skewed shape.
- For $\alpha = 3,\ \lambda = .4,\ \theta = 1\, and\, \alpha = 5,\ \lambda = .5,\ \theta = .91$, f(x) has a negative skewed shape.
- For $\alpha = .4,\ \lambda = .2,\ \theta = .6$, f(x) has a bathtub shape.
- For $\alpha = .4,\ \lambda = .2,\ \theta = -1$, f(x) has a J-shaped.

3. Statistical properties of TE-TLD

In this section, some basic properties of TE-TLD are provided in an explicit form.

3.1. Moments and Moment generating function

The r^{th} Moments of TE-TLD is given by;

$$\mu_r' = 2^r \sum_{j=0}^{\infty} A_j 4^{\alpha(1+j)} Be\left(r+\alpha(1+j), \alpha(1+j); \frac{1}{2}\right) - 2^{r+1} \sum_{j=0}^{\infty} A_j 4^{\alpha(1+j)} Be\left(r+\alpha(1+j)+1, \alpha(1+j); \frac{1}{2}\right) \tag{8}$$

$$where\ A_j = (-1)^j \alpha\lambda \left\{(1-\theta)\binom{\lambda-1}{j} + 2\theta\binom{2\lambda-1}{j}\right\}$$

The r^{th} Moment about the Mean is given by;

$$\begin{aligned} E(x-\mu)^r = \quad & \alpha\lambda \sum_{j=0}^{\infty}\sum_{k=0}^{r} \psi_{j,k} 2^{r+1+2\alpha(1+j)-k-1} Be\left(r+\alpha(1+j)-k, \alpha(1+j); \tfrac{1}{2}\right) \\ & - \alpha\lambda \sum_{j=0}^{\infty}\sum_{k=0}^{r} \psi_{j,k} 2^{r+1+2\alpha(1+j)-k} Be\left(r+\alpha(1+j)-k+1, \alpha(1+j); \frac{1}{2}\right) \end{aligned} \tag{9}$$

where,

$$\psi_{j,k} = (-1)^{j+k}\binom{r}{k}\mu^k\left\{(1-\theta)\binom{\lambda-1}{j} + 2\theta\binom{2\lambda-1}{j}\right\}$$

and, The moment generating function of TE-TLD is given by;

$$M_X(t) = \sum_{r,j=0}^{\infty} \frac{2^r t^r}{r!} A_j 4^{\alpha(1+j)} Be\left(r+\alpha(1+j), \alpha(1+j); \frac{1}{2}\right) - \sum_{r,j=0}^{\infty} \frac{2^r t^r}{r!} A_j 2^{2\alpha(1+j)+1} Be\left(r+\alpha(1+j)+1, \alpha(1+j); \frac{1}{2}\right) \tag{10}$$

$where\ A_j = (-1)^j \alpha\lambda \left\{(1-\theta)\binom{\lambda-1}{j} + 2\theta\binom{2\lambda-1}{j}\right\} and Be(.,.,u)$ is an incomplete Beta function .

3.2. Survival and Hazard function of TE-TLD

The survival and hazard function are respectively given as;

$$S(x) = 1 - \left[1-(1-x^{\alpha}(2-x)^{\alpha})^{\lambda}\right]\left[1+\theta\,(1-x^{\alpha}(2-x)^{\alpha})^{\lambda}\right] \tag{11}$$

and,

$$h(x) = \frac{2\alpha\lambda x^{\alpha-1}(1-x)(2-x)^{\alpha-1}\left(1-x^{\alpha}(2-x)^{\alpha}\right)^{\lambda-1}\left[1-\theta+2\theta\left(1-x^{\alpha}(2-x)^{\alpha}\right)^{\lambda}\right]}{1-\left[1-\left(1-x^{\alpha}(2-x)^{\alpha}\right)^{\lambda}\right]\left[1+\theta\left(1-x^{\alpha}(2-x)^{\alpha}\right)^{\lambda}\right]} \tag{12}$$

The plots of the survival and hazard function for some selected values of parameters are respectively displayed in Figures 3 and 4 as shown below;

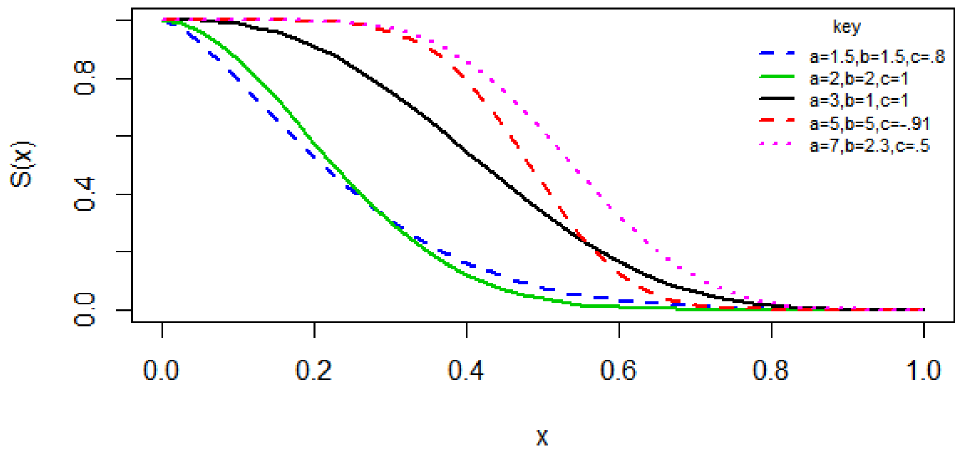

Figure 3: *Survival plot of TE-TLD*

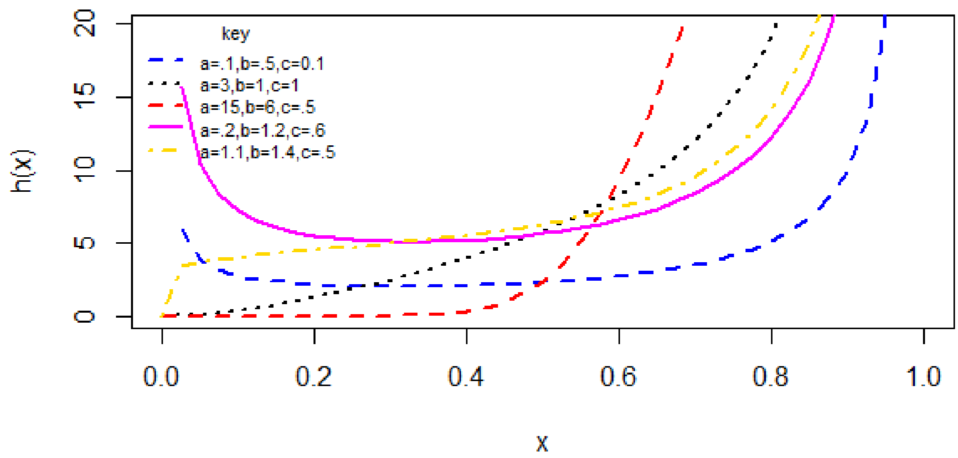

Figure 4: *Hazard plot of TE-TLD*

Figure 4 reveals that the hazard function of the TE-TLD possesses not only monotonic and non-monotonic failure rate shapes but also includes modified increasing failure rates. The monotonicity of the TE-TLD implies that the distribution may be a better choice when modelling age-dependent events where the risk increases with age. The resulting bathtub curve describes not only the behaviour of engineering components but also the lifetimes of human populations. Furthermore, the non-monotonicity of the TE-TLD implies that early failure or "infant mortality" is called the first stage of the bathtub curve and it is characterized by a decreasing component of the hazard rate. The weak members of the population are failing during this period. This section of the curve is based on the widely used testing practice of obviously defective components as well as weak ones with high failure potential. Products must survive some sort of initial stress during screening processes (e.g., burn-in at high temperature, application of electrical overstress, temperature cycling). Furthermore, the second stage is a roughly flat part called the intrinsic failure period. The hazard rate here is approximately constant and the failures occur at random in this area and most of the useful life of a component or system is spent here. The last stage on the curve is called the wear-out failure period and the hazard rate increase in this phase.

3.3. Quantile function of TE-TLD

For a non-negative continuous random variable X, that follows the TE-TLD, the quantile function is given by;

$$Q(u) = 1 - \sqrt{1 - \left(1 - \left(\frac{\theta - 1 + \sqrt{(\theta-1)^2 + 4\theta(1-u)}}{2\theta}\right)^{1/\lambda}\right)^{1/\alpha}} \tag{13}$$

3.3.1 Simulation Study

Numerical results are obtained by generating N=1000 random samples of size n=200 from $TE - TL(\alpha, 0.4, 0.8)$, $TE - TL(0.5, \lambda, 0.8)$ & $TE - TL(0.5, 0.4, \theta)$, where $\alpha = 0.5, 1, 1.5, 2, 3\,\&\,10$; $\lambda = 0.2, 0.5, 1, 1.5, 2\&\,2.5$; $\theta = -1, -0.5, -0.2, 0.2, 0.5\,\&\,1$. From the numerical results in Table 1, it was observed that the mean increases while variance, skewness and kurtosis decrease when α increases. While from Table 2, it was observed that for a constant value of $\alpha\, and\, \theta$, the mean and variance decrease both skewness and kurtosis increase as λ increases. Additionally, from Table 3, it was observed that as θ increases, both mean and variance decrease while at some certain point the variance increases and the skewness changes direction from negative to positive and the kurtosis decreases as well as increases at some point.

Table 1: *Mean, variance, skewness and kurtosis of TE-TLD for $\lambda = 0.4$, $\theta = 0.8$ and different values of α*

	Parameter (α)					
	0.5	1	1.5	2	3	10
Mean	0.3039 (0.0198)	0.4192 (0.019)	0.4910 (0.0178)	0.5414 (0.0167)	0.6091 (0.0149)	0.7710 (0.0094)
Variance	0.0805 (0.0069)	0.0755 (0.0053)	0.0668 (0.0046)	0.0590 (0.0041)	0.0472 (0.0035)	0.019 (0.0017)
Skewness	0.8090 (0.1226)	0.3622 (0.1042)	0.1389 (0.1005)	0.0011 (0.1008)	-0.1623 (0.1050)	0.4556 (0.1282)
Kurtosis	2.4942 (0.2688)	2.0439 (0.1382)	2.0151 (0.1067)	2.0665 (0.1063)	2.2004 (0.1331)	2.6575 (0.2797)

Table 2: *Mean, variance, skewness and kurtosis of TE-TLD for $\alpha = 0.5$, $\theta = 0.8$ and different values of λ*

	Parameter (λ)					
	0.2	0.5	1	1.5	2	2.5
Mean	0.4899 (0.0231)	0.2506 (0.0179)	0.1214 (0.0114)	0.0729 (0.0079)	0.0490 (0.0059)	0.0353 (0.0046)
Variance	0.1128 (0.0060)	0.0660 (0.0068)	0.0265 (0.0047)	0.0128 (0.0031)	0.0070 (0.0021)	0.0042 (0.0015)
Skewness	0.0556 (0.0060)	1.0672 (0.1363)	1.9809 (0.2547)	2.6254 (0.4628)	3.1268 (0.7152)	3.5226 (0.9585)
Kurtosis	1.6151 (0.0674)	3.1405 (0.3954)	7.0270 (1.5151)	11.4423 (3.9941)	15.8680 (7.5188)	19.9265 (11.2741)

Table 3: *Mean, variance, skewness and kurtosis of TE-TLD for $\alpha = 0.5$, $\lambda = 0.4$ and different values of θ*

	Parameter (θ)					
	-1	-0.5	-0.2	0.2	0.5	1
Mean	0.6416 (0.0.0187)	0.5469 (0.0216)	0.4901 (0.0223)	0.4151 (0.0223)	0.3594 (0.0215)	0.2667 (0.0063)
Variance	0.0757 (0.0055)	0.0994 (0.0058)	0.1054 (0.0058)	0.1040 (0.0063)	0.0955 (0.0068)	0.0666 (0.0063)
Skewness	-0.5251 (0.1064)	-0.2082 (0.1028)	0.0232 (0.1042)	0.3386 (0.1095)	0.5780 (0.1151)	0.9236 (0.3261)
Kurtosis	2.1826 (0.1715)	1.7621 (0.3954)	1.6593 (0.0691)	1.7660 (0.1167)	2.0456 (0.1837)	2.7921 (0.3261)

4. Maximum likelihood estimation

Let x_1, x_2, ..., x_n be a sample of size (n) from $TE - TL(\alpha, \lambda, \theta)$ distribution. Then, the log-likelihood function (LL) for the parameter vector $\Omega = (\alpha, \lambda, \theta)^T$ is given as;

$$LL(\Omega) = n\log 2 + n\log\alpha + n\log\lambda + (\alpha-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log(1-x_i) + (\alpha-1)\sum_{i=1}^{n}\log(2-x_i)$$
$$+ (\lambda-1)\sum_{i=1}^{n}\log(1-x_i^{\alpha}(2-x_i)^{\alpha}) + \sum_{i=1}^{n}\log\left[1-\theta+2\theta\left(1-x_i^{\alpha}(2-x_i)^{\alpha}\right)^{\lambda}\right] \quad (14)$$

To get the MLE of the unknown parameters, find the first partial derivative with respect to each of the parameter of the distribution.

$$\frac{\delta LL(\Omega)}{\delta\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log(2-x_i) + (\lambda-1)\sum_{i=1}^{n}\frac{x_i^{\alpha}(2-x_i)^{\alpha}\ln(x_i(2-x_i))}{(1-x_i^{\alpha}(2-x_i)^{\alpha})}$$
$$-2\lambda\theta\sum_{i=1}^{n}\frac{\left(1-x_i^{\alpha}(2-x_i)^{\alpha}\right)^{\lambda-1}x_i^{\alpha}(2-x_i)^{\alpha}\ln(x_i(2-x_i))}{\left[1-\theta+2\theta\left(1-x_i^{\alpha}(2-x_i)^{\alpha}\right)^{\lambda}\right]}$$
$$\frac{\delta LL(\Omega)}{\delta\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log(1-x_i^{\alpha}(2-x_i)^{\alpha}) + 2\theta\sum_{i=1}^{n}\frac{\left(1-x_i^{\alpha}(2-x_i)^{\alpha}\right)^{\lambda}\ln\left(1-x_i^{\alpha}(2-x_i)^{\alpha}\right)}{\left[1-\theta+2\theta\left(1-x_i^{\alpha}(2-x_i)^{\alpha}\right)^{\lambda}\right]}$$
$$\frac{\delta LL(\Omega)}{\delta\theta} = \sum_{i=1}^{n}\frac{2\left(1-x_i^{\alpha}(2-x_i)^{\alpha}\right)^{\lambda}-1}{\left[1-\theta+2\theta\left(1-x_i^{\alpha}(2-x_i)^{\alpha}\right)^{\lambda}\right]}$$

Finally, setting this system of non-linear equations to zero and solving them simultaneously gives the MLE $\widehat{\Omega} = (\widehat{\alpha}, \widehat{\lambda}, \widehat{\theta})^T$. Furthermore, these non-linear equations cannot be solved analytically and as such a numerical method of optimization should be employed.

5. APPLICATIONS

In this section, we demonstrate empirically the flexibility of the TE-TLD using an application to both real and simulated datasets and provide a comparison with other competing distributions based on some goodness-of-fit statistics.

5.1. Real-Life Datasets

The first data is about the total milk production in the first birth of 107 cows from the SINDI race. These cows are property of the Carnauba farm which belongs to the Agropecuaria Manoel Dantas Ltda (AMDA), located in Taperoa City, Paraiba (Brazil) [12]. The second data set was used by [13] and more recently by [14] which consist of n=50 observations on burr (in millimeter), with hole diameter and sheet thickness 12 mm and 3.15 mm respectively. The competing distributions are transmuted Topp Leon (TTL), Topp Leon Exponential (TLE), and Topp Leon (TL) distribution.

Table 4: *Goodness-of-fit statistics for dataset I*

Distribution	-LL	AIC	CAIC	HQIC	W*	A*	KS
TE-TL	-25.9741	-45.9481	-45.7151	-42.6975	0.1371	0.9035	0.0701
TTL	-22.3121	-40.6241	-40.5087	-38.4570	0.1542	1.0103	0.1086
TLE	-5.0388	-6.0775	-5.9621	-3.9104	0.7291	4.4011	0.1477
TL	-21.5262	-41.0524	-41.0143	-39.9689	0.2333	1.4792	0.0972

Table 5: *Likelihood ratio test statistic for Dataset I*

Distribution	Hypotheses	LRT	P-value
TE-TL vs TTL	$H_0 : \lambda = 1 \ vs \ H_1 : H_0 \ is \ false$	7.324	0.0068
TE-TL vs TLE	$H_0 : \theta = 0 \ vs \ H_1 : H_0 \ is \ false$	41.871	0.00001
TE-TL vs TL	$H_0 : \lambda = 1 \ and \ \theta = 0 \ vs \ H_1 : H_0 \ is \ false$	8.896	0.011

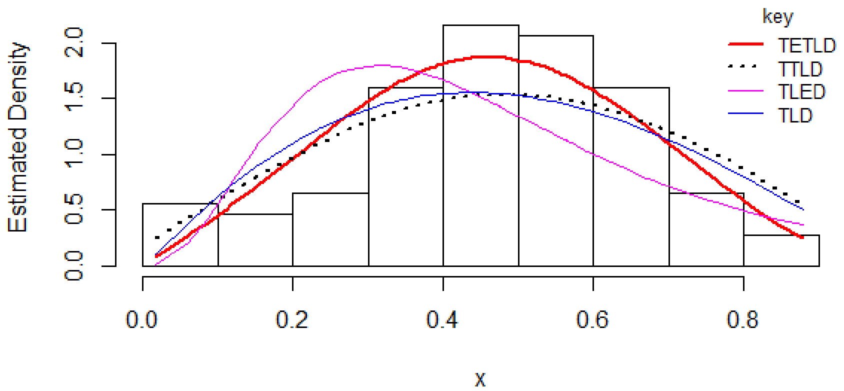

Figure 5: *The plot of the estimated densities for dataset I*

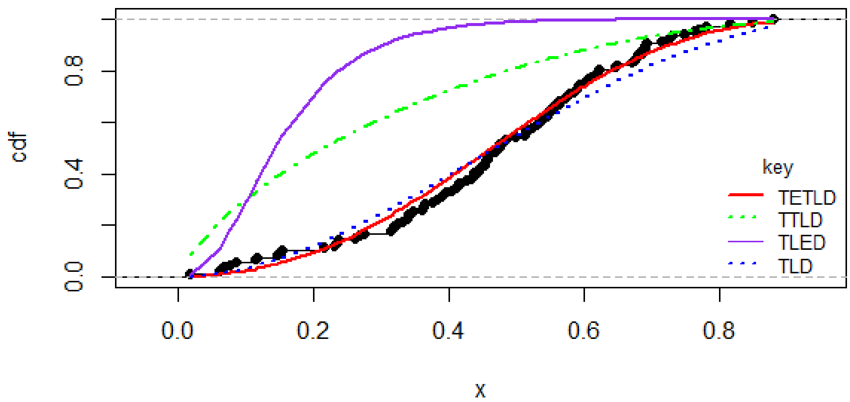

Figure 6: *The plot of the ecdf for dataset I*

The estimated densities and the ecdf for dataset I are respectively displayed in Figures 5 and 6 and the important aspect of these figures is to provide illustration and explanation on the flexibility of the competing distributions in the study in terms of capturing the sensitive parts of the dataset and draw a possible conclusion regarding the performance of the distributions. From Figures 5 and 6, we can observe that the TE-TLD shows a greater performance in capturing the sensitive part of the dataset as compared to other distributions in the study.

Table 6: *Goodness-of-fit statistics for dataset II*

Distribution	-LL	AIC	CAIC	HQIC	W*	A*	KS
TE-TL	-55.9914	-105.9827	-105.4610	-103.7984	0.1015	0.6246	0.1068
TTL	-28.3701	-52.7403	-52.4850	-51.2841	0.1653	0.9917	0.3676
TLE	-52.2863	-100.5725	-100.3172	-99.1163	0.2121	1.2634	0.1653
TL	-28.4078	-54.8156	-54.7323	-54.0875	0.1654	0.9919	0.3623

Table 7: *Likelihood ratio test statistic for Dataset II*

Distribution	Hypotheses	LRT	P-value
TE-TL vs TTL	$H_0 : \lambda = 1 \; vs \; H_1 : H_0 \; is \; false$	55.243	0.00001
TE-TL vs TLE	$H_0 : \theta = 0 \; vs \; H_1 : H_0 \; is \; false$	7.410	0.0065
TE-TL vs TL	$H_0 : \lambda = 1 \; and \; \theta = 0 \; vs \; H_1 : H_0 \; is \; false$	55.167	0.00001

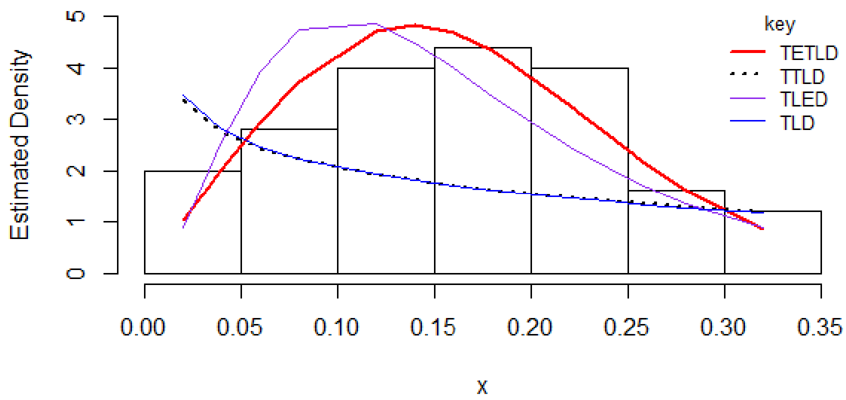

Figure 7: *The plot of the estimated densities for dataset II*

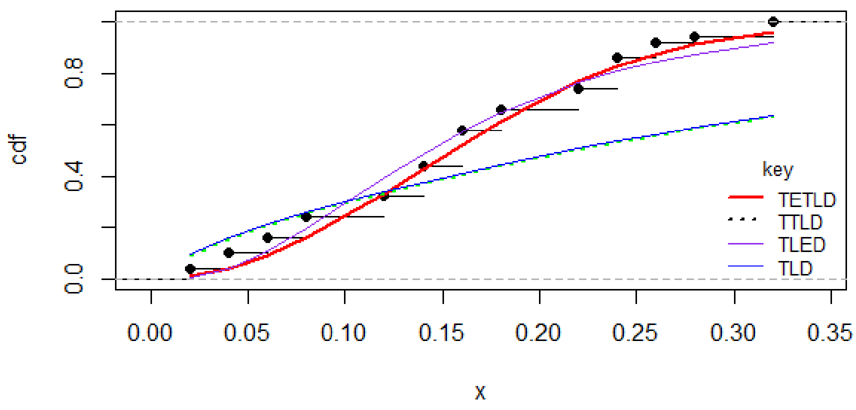

Figure 8: *The plot of the ecdf for dataset II*

The estimated densities and the ecdf for dataset II are respectively displayed in Figures 7 and 8 and the important aspect of these figures is to provide illustration and explanation on the flexibility of the competing distributions in the study in terms of capturing the sensitive parts of the dataset and draw a possible conclusion regarding the performance of the distributions. From Figures 7 and 8, we can observe that the TE-TLD shows a greater performance in capturing the sensitive part of the dataset as compared to other distributions in the study.

Some adequacy measures for the distributions are presented in Tables 4 and 6 for data sets I, II respectively. Hence, it is observed that the proposed distribution has the lowest values of the goodness-of-fit statistics and therefore, outperformed the other competing distributions in the study. Figures 5 and 6 are respectively displayed the estimated densities and the ecdf for datasets I, Figures 7 and 8 are respectively displayed the estimated densities and the ecdf for datasets II.

Likelihood ratio test is carried out to assess the significance of the additional parameter(s) of the TE-G family of distributions. For datasets I and II, since all the P-values are less than $\alpha = 0.05$, we therefore reject H_0 and conclude that the additional parameter(s) are significant which further prove the robustness of the TE-TLD over the nested distributions in the study.

6. Conclusion

In this paper, we proposed a new probability distribution by inducing Topp Leon distribution into transmuted Exponential-G family of distributions. The proposed distribution named transmuted Exponential- Topp Leon distribution in short (TE-TLD) which possessed different density shapes. Some properties of the distribution were presented in an explicit form and the parameters of the distribution are estimated by the method of maximum likelihood. The hazard function of the TE-TLD can be monotonic or non-monotonic failure rate which makes it more robust in terms of studying failure rates. The proposed distribution was found to be more robust as compared to other competing distributions with the same underlying baseline distribution when applied to real datasets in the study.

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