CURVATURE TENSORS IN SP-KENMOTSU MANIFOLDS WITH RESPECT TO QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract

A conformal curvature tensor and con-circular curvature tensor in an SP-Kenmotsu manifold are derived in this article which admits a quarter-symmetric metric connection. Conclusively, we verified our results by considering a case of 3-D SP-Kenmotsu manifold.

Keywords: η-Einstein manifold,SP-Kenmotsu manifold, con-circular curvature tensor, Quarter-symmetric metric connection, Ricci tensor, conformal curvature tensor.
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I. Introduction

A M_n (Riemannian manifold) is symmetrical locally if $\nabla.R = 0$ and symmetric if $R(X,Y).R = 0$ where $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ appears as a derivation. If $R(X,Y).R = 0$, then M_n is turns to be the pseudo symmetric space that is defined with the criteria $R.R = L(g,R)$. A manifold M_n is conformally symmetric if $\nabla.C = 0$ and if $R.C = 0$, it is said to be Weyl semi symmetric which are characterised by the condition $R.C = L_C Q(g,C)$.

Schouten & Friedman proposed the concept of semi-symmetric linear connection on a differentiable manifold. Some of the semi-symmetric curvature criteria in Riemannian manifolds are given by Yano [12].

Semi symmetric metric connection plays a very significant part in the geometry of Riemannian manifolds. For instance, a semi-symmetric metric is the displacement of the earth's surface after a fixed point. A quarter-symmetric connection is a linear connection $\widetilde{\nabla}$ on an n-dimensional Riemannian manifold (M_n, g) if $\widetilde{T}$ is $\widetilde{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y$.

Sato [8] proposed concepts of almost para contact Riemannian manifold. In 1977, Matsumoto and Adati [1] characterized special para-Sasakian as well as para-Sasakian manifolds as a particular type of almost contact Riemannian manifolds. Before Sato, Kenmotsu [6] characterized a type of this manifold. In 1995, Sinha and Sai Prasad [9] characterized a type of almost para contact metric manifolds mainly para-Kenmotsu and special para-Kenmotsu manifolds. For the literature, on Para-Kenmotsu manifolds one can refer to Balga [2], Srivastava and Srivastava [10], Olszak [7] .

On the other hand, various geometers of Riemannian manifolds and specifically, SP-Sasakian manifolds were widely explored for the quarter-symmetric metric connections [3, 4, 5]. Inspired by these studies, in this work, we explore a class of special para-Kenmotsu manifolds that allowing the quarter-symmetric metric connection.

The current study is arranged as follows: Section 2 has certain prerequisites. In relation to the quarter symmetric metric connection in an SP-Kenmotsu manifold, we derive the equations for the Ricci tensor $\widetilde{S}$ & Riemannian curvature tensor $\widetilde{R}$ in Section 3. The equations in relation to quarter symmetric metric connection are also derived in an SP-Kenmotsu manifold M_n for concircular curvature tensor $\widetilde{Z}$ in Section 4. It is illustrated that the manifold M_n is η-Einstein given the concircular curvature tensor $\widetilde{Z}$ meets either of these conditions $\widetilde{R}(\xi, U).\widetilde{Z} = 0$, $\widetilde{Z}(\xi, U).\widetilde{R} = 0$, $\widetilde{Z}(\xi, U).\widetilde{Z} = 0$, $\widetilde{Z}(X, Y).\widetilde{S} = 0$. Section 5 is intended to define and analyse the curvature properties in the quarter-symmetric metric connection of the Weyl-conformal curvature tensor $\widetilde{C}$, of form $(0, 4)$, of SP-Kenmotsu manifold M_n. Finally, an illustration of a 3d SP-Kenmotsu manifold is considered in Section 6.

II. Preliminaries

Suppose M_n be an n-dimensional differentiable manifold provided with structure tensors (Φ, ξ, η) such that

$$\begin{aligned} &(a)\ \eta(\xi) = 1 \\ &(b)\ \Phi^2(X) = X - \eta(X)\xi;\ \overline{X} = \Phi X. \end{aligned} \tag{1}$$

M_n is called an almost para contact manifold.

Suppose that g be a Riemannian metric such that, for all vector fields X and Y on M_n

$$\begin{aligned} &(a)\ g(X, \xi) = \eta(X) \\ &(b)\ \Phi\xi = 0,\ \eta(\Phi X) = 0,\ \text{rank}\ \Phi = n - 1 \\ &(c)\ g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y). \end{aligned} \tag{2}$$

Then it is stated that the manifold [8] M_n accepts an almost para contact structure of Riemannian (Φ, ξ, η, g).

Furthermore, if (Φ, ξ, η, g) fulfils the equations

$$\begin{aligned} &(a)\ (\nabla_X \eta)Y - (\nabla_Y \eta)X = 0; \\ &(b)\ (\nabla_X \nabla_Y \eta)Z = [-g(X, Z) + \eta(X)\eta(Z)]\eta(Y) + [-g(X, Y) + \eta(X)\eta(Y)]\eta(Z); \\ &(c)\ \nabla_X \xi = \Phi^2 X = X - \eta(X)\xi; \\ &(d)\ (\nabla_X \Phi)Y = -g(X, \Phi Y)\xi - \eta(Y)\Phi X; \end{aligned} \tag{3}$$

then M_n is termed a para-Kenmotsu manifold or simply a P-Kenmotsu manifold [9].

A P-Kenmotsu manifold M_n permitting a 1-form η fulfilling

$$\begin{aligned} &(a)\ \ (\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y); \\ &(b)\ (\nabla_X \eta)Y = \varphi(\overline{X}, Y); \end{aligned} \tag{4}$$

here φ signifies Φ associate, is termed a special para-Kenmotsu manifold or shortly SP-Kenmotsu manifold [9].

Suppose $(M_n,$ g) be an n-dimensional, $n \geq 3$, differentiable manifold of class C^∞ and let ∇

be its connection Levi-Civita. Then curvature tensor R of class (1, 3) of the the Riemannian Christoffel is provided by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{5}$$

The (0,2)-tensor S^2 and the Ricci operator S are described as follows

$$g(SX,Y) = S(X,Y), \tag{6}$$

$$\text{and } S^2(X,Y) = S(SX,Y). \tag{7}$$

It is known [9] that the following relationship exist in the P-Kenmotsu manifold:

$$\begin{aligned} &(a)\ S(X,\xi) = -(n-1)\eta(X), \\ &(b)\ g[R(X,Y)Z,\xi] = \eta[R(X,Y,Z)] = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \\ &(c)\ R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \\ &(d)\ R(X,Y)\xi = \eta(Y)X - \eta(X)Y;\ \text{when } X \text{ is orthogonal to } \xi. \end{aligned} \tag{8}$$

Almost para-contact Riemannian manifold M_n is termed to be η-Einstein and form of its Ricci tensor

$$S(X,Y) = a\, g(X,Y) + b\, \eta(X)\, \eta(Y) \tag{9}$$

Fields X and Y for any vector; a and b are a few scalars on M_n. In specific, if $b = 0$ thus M_n is considered to be an Einstein manifold.

III. Curvature tensor

A linear connection $\tilde{\nabla}$ in a Riemannian manifold M_n is called a quarter-symmetric metric connection [4] if their torsion tensor $T(X,Y)$ meets

$$T(X,Y) = \eta(Y)\, \Phi X - \eta(X)\, \Phi Y, \tag{10}$$

and

$$(\tilde{\nabla}_X g)(Y,Z) = 0; \tag{11}$$

where Φ iindicates a tensor field of the form (1, 1) and η is a 1-form.

A quarter-symmetric metric connection $\tilde{\nabla}$ with torsion tensor (10) is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\, \Phi X\ - \varphi(X,Y)\xi \tag{12}$$

here, ∇ indicates Riemannian connection.

Suppose manifold M_n to be an SP-Kenmotsu manifold and $\Phi(X)$ as ΦX = $\overline{\text{X}}$. Therefore the (10) and (11) may be represented as:

$$T(X,Y) = \eta(Y)\overline{X} - \eta(X)\overline{Y} \tag{13}$$

$$(\tilde{\nabla}_X g)(Y,Z) = 0. \tag{14}$$

Let us choose the linear and Riemannian connection as $\tilde{\nabla}$ and ∇, respectively

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X,Y), U\ is\ a\ tensor\ of\ type\ (1,2) \tag{15}$$

We have [12], for $\tilde{\nabla}$ to be a quarter symmetric metric connection in M_n,

$$U(X,Y) = 1/2[T(X,Y) + T'(X,Y) + T'(Y,X)], \tag{16}$$

where

$$g(T'(X,Y),Z) = g(T(Z,X),Y)]. \tag{17}$$

Using (13) and (17), we get

$$T'(X,Y) = \eta(X)\overline{Y} - {}'F(X,Y)\xi; \tag{18}$$

here $'F(X,Y) = g(\overline{X},Y)$, η signifies a 1-form and ξ indicates the associated vector field.

From (13) and (16), in (18), we have

$$U(X,Y) = \eta(Y)\overline{X} - {}'F(X,Y)\xi, \tag{19}$$

and then (15) becomes

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\overline{X} - {}'F(X,Y)\xi; \tag{20}$$

which indicates $\widetilde{\nabla}$ in an SP-Kenmotsu manifold.

Suppose $\widetilde{R}$ and R be the curvature tensors of the connections $\widetilde{\nabla}$ and ∇ correspondingly, we get

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[}X,Y]Z \tag{21}$$

Using (20) and (5) in (21), we have

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y. \tag{22}$$

If we describe $\widetilde{R}(X,Y,Z,U)$ as $g(\widetilde{R}(X,Y)Z,U)$ and $R(X,Y,Z,U)$ as $g(R(X,Y)Z,U)$; then (22) becomes

$$\widetilde{R}(X,Y,Z,U) = R(X,Y,Z,U) + g(Y,Z)g(X,U) - g(X,Z)g(Y,U). \tag{23}$$

The above expression (23) denotes the relation between $\widetilde{R}(X,Y)Z$ of M_n w.r.t. $\widetilde{\nabla}$ and $R(X,Y)Z$ w..r.t. ∇.

Put X = U = e_i in (23), where e_i be an orthonormal basis of the tangent space at any point of the manifold and taking summation over i ($1 \leq i \leq$n), we get

$$\widetilde{S}(Y,Z) = S(Y,Z) + n\, g(Y,Z) - \eta(Y)\eta(Z); \tag{24}$$

here $\widetilde{S}$ and S signifies the Ricci tensors of $\widetilde{\nabla}$ and ∇.

From (24), by using Y = Z = e_i, we obtain

$$\widetilde{r} = r + n^2 - 1; \tag{25}$$

here $\widetilde{r}$ and r indicates the scalar curvatures of $\widetilde{\nabla}$ and ∇ correspondingly.

Theorem 3.1: Suppose that $\widetilde{S}$ be the Ricci tensor & $\widetilde{R}$ be the curvature tensor in an SP-Kenmotsu manifold M_n w.r.t. $\widetilde{\nabla}$, then

(a) $\widetilde{R}(X,Y)Z + \widetilde{R}(Y,Z)X + \widetilde{R}(Z,X)Y = 0,$

(b) $\widetilde{R}(X,Y,Z,U) + \widetilde{R}(X,Y,U,Z) = 0,$

(c) $\widetilde{R}(X,Y,Z,U) - \widetilde{R}(Z,U,X,Y) = 0,$

(d) $\widetilde{R}(X,Y,Z,\xi) = 2R(X,Y,Z,\xi),$

(e) $\widetilde{S}(X,\xi) = 2S(X,\xi).$

Proof: Using first Bianchi identity and eq.(22) w.r.t. the Riemannian connection, we obtain (a).

From eq. (23), we obtain (b) & (c). By putting U = ξ in (23) and by using (8) we have (d).

By using Y = Z = e_i in equation (d) as well as summation with i, we obtain (e).

Theorem 3.2: The Ricci tensor $\widetilde{S}$ in an SP-Kenmotsu manifold M_n w.r.t. the connection for the quarter-symmetric metric is symmetrical.

Proof: The theorem-proof is based on the eq. provided in (24).

IV. Concircular curvature tensor

The n-dimensional Riemannian manifold M_n is provided by the concircular curvature tensor $Z(X,Y)$ [11, 13]:

$$Z(X,Y)U = R(X,Y)U - \frac{r}{n(n-1)}[g(Y,U)X - g(X,U)Y] \tag{26}$$

for all X, Y, U $\in TM$.

The concircular curvature tensor w.r.t. $\widetilde{\nabla}$ in an SP-Kenmotsu manifold is $\widetilde{Z}$.

Therefore, using the equations (22) and (26), we get

$$\widetilde{Z}(X,Y)U = Z(X,Y)U - \frac{1}{n}[g(Y,U)X - g(X,U)Y], \tag{27}$$

which denotes the relation between the concircular curvature tensors w.r.t. $\widetilde{\nabla}$ and ∇.

Theorem 4.1: If $\widetilde{Z}$ w.r.t. $\widetilde{\nabla}$ in an SP-Kenmotsu manifold satisfies $\widetilde{R}(\xi,U).\widetilde{Z} = 0$, the manifold is η-Einstein.

Proof: Suppose $\widetilde{R}(\xi,U).\widetilde{Z}(X,Y)\xi = 0$, in an SP-Kenmotsu manifold.

Then

$$(\widetilde{R}(\xi,U).\widetilde{Z}(X,Y)\xi) - \widetilde{Z}(\widetilde{R}(\xi,U)X,Y)\xi - \widetilde{Z}(X,\widetilde{R}(\xi,U)Y)\xi - \widetilde{Z}(X,Y).\widetilde{R}(\xi,U)\xi = 0. \tag{28}$$

Also, from (8) and (22), we get

$$\widetilde{R}(X,Y)\xi = 2[\eta(Y)X - \eta(X)Y] \ and \tag{29}$$

$$\widetilde{R}(\xi,X)U = 2[g(X,U)\xi - \eta(U)X]. \tag{30}$$

Then, by using (28), (29) and (30), we get

$$\widetilde{Z}(X,Y)U = 0. \tag{31}$$

Now, using the equations (26) and (27), the equation (31) reduces to

$$\mathbb{R}(X,Y,U) = \frac{r+n-1}{n(n-1)}[g(Y,U)X - g(X,U)Y]. \tag{32}$$

We obtain with the above equation w.r.t. X,

$$S(Y,U) = \frac{r+n-1}{n(n-1)}[ng(Y,U)X - \eta(Y)\eta(U)], \tag{33}$$

which on further contracting, we get

$$r = 1 - n^2. \tag{34}$$

Using (34), the expression (33) becomes

$$S(Y, U) = \eta(Y)\eta(U) - ng(Y, U); \tag{35}$$

which proves η-Einstein manifold.

Theorem 4.2: If $\widetilde{Z}$ with respect to $\widetilde{\nabla}$ in an SP-Kenmotsu manifold satisfies $\widetilde{Z}(\xi, U).\widetilde{R} = 0$, the manifold is an η-Einstein.

Proof: Suppose that $\widetilde{Z}(\xi, U).\widetilde{R}(X, Y)\xi = 0$, in an SP-Kenmotsu manifold.

Then

$$(\widetilde{Z}(\xi, U).\widetilde{R}(X, Y)\xi) - \widetilde{R}(\widetilde{Z}(\xi, U)X, Y)\xi - \widetilde{R}(X, \widetilde{Z}(\xi, U)Y)\xi - \widetilde{R}(X, Y).\widetilde{Z}(\xi, U)\xi = 0 \tag{36}$$

Also, from (8), (26) and (27), we have

$$\widetilde{Z}(\xi, U)Y = \Big[\frac{r}{n(n-1)} + \frac{1}{n} - 1\Big]\Big[g(U, Y)\xi - \eta(Y)U\Big] \tag{37}$$

and

$$\widetilde{Z}(X, Y)\xi = \Big[\frac{r}{n(n-1)} + \frac{1}{n} - 1\Big]\Big[\eta(X)Y - \eta(Y)X\Big]. \tag{38}$$

By substituting the values from (29), (30), (37) and (38) in the expression (36), we obtain

$$\widetilde{R}(X, Y)U = g(U, Y)X - g(U, X)Y + \eta(U)[1 - \eta(X)]Y. \tag{39}$$

Using (22), the above eq. becomes

$$R(X, Y)U = \eta(U)[1 - \eta(X)]Y; \tag{40}$$

and it proves.

Theorem 4.3: If the $\widetilde{Z}$ w.r.t. $\widetilde{\nabla}$ in an SP-Kenmotsu manifold meets $\widetilde{Z}(\xi, U).\widetilde{Z} = 0$, the manifold is η-Einstein.

Proof: The theorem-proof is trivial by the use of the the fact that $\widetilde{Z}(\xi, U).\widetilde{Z}$ indicates $\widetilde{Z}(\xi, U)$ was acting on $\widetilde{Z}$ as a derivation.

Theorem 4.4: If $\widetilde{Z}$ (concircular curvature tensor) with respect to $\widetilde{\nabla}$(quarter symmetric metric connection) in an SP-Kenmotsu manifold fulfills $\widetilde{Z}(X, Y).\widetilde{S} = 0$, the manifold signifies η-Einstein.

Proof: Let $\widetilde{Z}(X, Y).\widetilde{S}(U, V) = 0$ in an SP-Kenmotsu manifold.

Then it means

$$\widetilde{S}(\widetilde{Z}(X, Y)U, V) + \widetilde{S}(U, \widetilde{Z}(X, Y)V) = 0. \tag{41}$$

By choosing X = ξ in (41) and on using the equations (37) and (24), we obtain

$$\begin{aligned}\Big[\frac{r}{n(n-1)} + \frac{1}{n} - 1\Big]\Big[&- \eta(U)S(Y, V) - n\eta(U)g(Y, V) + 2\eta(U)\eta(V)\eta(Y) \\ &- \eta(V)S(U, Y) - n\eta(V)g(U, Y)\Big] = 0.\end{aligned} \tag{42}$$

Again by using $U = \xi$ in the eq. (42), we get

$$S(Y, V) = \eta(Y)\eta(V) - ng(Y, V); \tag{43}$$

which provides the required result.

V. Conformal curvature tensor

The Weyl conformal curvature tensor C of the type $(0,4)$ of a manifold M_n w.r.t. a Riemannian connection provided by [12, 13]:

$$\begin{aligned} C(X,Y,Z,U) = \; & R(X,Y,Z,U) - \frac{1}{n-2}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ & + g(Y,Z)S(X,U) - g(X,Z)S(Y,U)] \\ & + \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{44}$$

Analogous to this, we define $\widetilde{C}$ i.e.Weyl conformal curvature tensor of the type $(0,4)$, of an SP-Kenmotsu manifold w.r.t. the quarter-symmetric metric connection as:

$$\begin{aligned} \widetilde{C}(X,Y,Z,U) = & \widetilde{R}(X,Y,Z,U) - \frac{1}{n-2}[\widetilde{S}(Y,Z)g(X,U) - \widetilde{S}(X,Z)g(Y,U) \\ & + g(Y,Z)\widetilde{S}(X,U) - g(X,Z)\widetilde{S}(Y,U)] \\ & + \frac{\widetilde{r}}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{45}$$

Then, using the equations (23), (24), (25), (44) and (45), we get

$$\widetilde{C}(X,Y,Z,U) = C(X,Y,Z,U), \tag{46}$$

which implies the following statement:

Theorem 5.1: The conformal curvature tensors of $\widetilde{\nabla}$ and ∇ are equal in an SP-Kenmotsu manifold.

Suppose that $\widetilde{R} = 0$. Then $\widetilde{S} = 0$ and $\widetilde{r} = 0$.

From (45) we get that $\widetilde{C} = 0$ and hence using (46), we get $C = 0$.

Therefore, we provide the following theorem.

Theorem 5.2: The manifold is conformally flat in an SP-Kenmotsu manifold if the conformal curvature tensor $\widetilde{C}$ of $\widetilde{\nabla}$ vanishes.

Let $\widetilde{S} = 0$. Then $\widetilde{r} = 0$. Hence from (24) and (25), we get

$$S(Y,Z) = \eta(Y)\eta(Z) - n\, g(Y,Z) \tag{47}$$

and

$$r = 1 - n^2. \tag{48}$$

Then by using (23), (44), (47) and (48), we obtain

$$\widetilde{R}(X,Y,Z,U) = C(X,Y,Z,U). \tag{49}$$

From (49), we state that

Theorem 5.3: Conformal curvature tensor C of the manifold is identical in an SP-Kenmotsu manifold if $\widetilde{S}$ (Ricci tensor) of $\widetilde{\nabla}$ i.e quarter-symmetric metric connection vanishes, then $\widetilde{R}$ i.e. curvature tensor of $\widetilde{\nabla}$.

Using theorem (5.2) and (5.3), we state that

Theorem 5.4: If $\widetilde{S}$ of $\widetilde{\nabla}$ in an SP-Kenmotsu manifold disappears, then the manifold is conformally flat if $\widetilde{R}$ of $\widetilde{\nabla}$ vanishes.

VI. Example of a 3d SP-Kenmotsu manifold admitting the quarter-symmetric metric connection

Example 6.1: Suppose that 3d manifold $M = \{(x, y, u) \in R^3\}$, where (x, y, u) indicates "standard coordinates" in R^3. Considering e_1, e_2 & e_3 be fields of vector in M as

$$e_1 = e^{-u}\frac{\partial}{\partial x}, \quad e_2 = e^{-u}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial u}. \tag{50}$$

for each point of M are linearly independent vectors and constitute a basis of $\chi(M)$.

Riemannian metric $g(X, Y)$ is

$$g(e_i, e_j) = \begin{cases} 1, & if \;\; i = j \\ 0, & if \;\; i \neq j; \; i, \; j = 1, 2, 3, 4, 5. \end{cases}$$

$$Let \eta(Z) = g(Z, e_3), for \; any \; Z \; \in \; \chi(M)$$

Let η be a 1-form & (1, 1)-tensor field on M expressed by Φ defined as

$$\Phi^2(e_1) = e_1, \Phi^2(e_2) = e_2, \Phi^2(e_3) = 0.$$

The $g(X, Y)$ and linearity of Φ yields that

$$\eta(e_3) = 1, \; \Phi^2(X) = X - \eta(X)e_3 \; ; and$$
$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields $X, \; Y \; \in \; \chi(M)$.

Thus for $e_3 = \xi$, (Φ, ξ, η, g) describes an almost para-contact structure in M.

Let ∇ be a Riemannian connection in regard to the Riemannian metric g.

$$[e_1, e_2] = 0, [e_1, e_3] = e_1, [e_2, e_3] = e_2.$$

The formula of Koszul's is

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{51}$$

By taking $e_3 = \xi$ in (51), one can get

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = e_1;$$
$$\nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = -e_3, \nabla_{e_2} e_3 = e_2;$$
$$\nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = 0.$$

Therefore manifold under consideration satisfies $\nabla_X \xi = \Phi^2 X = X - \eta(X)\xi$, $\eta(\xi) = 1$ and the expression (3)d.

The above expressions satisfy all the properties of SP-Kenmotsu manifold with (Φ, ξ, η, g) . Thus $M(\Phi, \xi, \eta, g)$ is a 3-dimensional manifold.

Further from (20), we get

$$\widetilde{\nabla}_{e_1} e_1 = -2e_3, \widetilde{\nabla}_{e_1} e_2 = 0, \widetilde{\nabla}_{e_1} e_3 = 2e_1;$$
$$\widetilde{\nabla}_{e_2} e_1 = 0, \widetilde{\nabla}_{e_2} e_2 = -2e_3, \widetilde{\nabla}_{e_2} e_3 = 2e_2;$$
$$\widetilde{\nabla}_{e_3} e_1 = 0, \widetilde{\nabla}_{e_3} e_2 = 0, \widetilde{\nabla}_{e_3} e_3 = 0;$$

Therefore $T(X,Y)$ of $\tilde{\nabla}$ can be expressed as:

$$T(e_i, e_i) = 0, \ for\ i = 1,2,3;\ and$$
$$T(e_1, e_2) = 0, T(e_1, e_3) = e_1, T(e_2, e_3) = e_2.$$

Also, we get

$$(\tilde{\nabla}_{e_1} g)(e_2, e_3) = 0, (\tilde{\nabla}_{e_2} g)(e_3, e_1) = 0, (\tilde{\nabla}_{e_3} g)(e_1, e_2) = 0,$$

which proves that the manifold M under consideration admits $\tilde{\nabla}$.

Thus it proves that M under consideration is an SP-Kenmotsu manifold and allows $\tilde{\nabla}$.

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