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A Case Study to Analyze Ageing Phenomenon in Reliability Theory

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Abstract

Hazard rate, and ageing intensity (AI) are measures or functions required for qualitative and quantitative analysis of ageing phenomena of a system with a well defined statistical distribution respectively. In this paper, we reiterate upon the fact that in a few cases hazard rate and ageing intensity do not depict the same pattern as far as monotonicity is concerned. So, a question naturally arises which among hazard rate, and ageing intensity is a preferable measure for characterizing ageing phenomena of a system. As a consequence, an example involving two design systems are analyzed and is illustrated to answer the aforementioned question.

Keywords: Ageing phenomenon, hazard rate, ageing intensity function.

AMS 2020 Subject Classification: Primary 60E15, Secondary 62N05, 60E05

1. Introduction

The notion of ageing phenomena and its mathematical counterpart are established by Barlow and Proschan (1975), Shaked and Shanthikumar (2007), Deshpande and Purohit (2005), Nanda et al. (2010) to name a few. The measures (or functions) usually used in this context are many, namely, survival function, hazard rate function, reversed hazard rate function, mean residual function, reversed mean residual function (cf. Block et al. (1998), Nanda et al. (2003,2005)).

Jiang et al. (2003) came forward with ageing intensity function relevant in reliability analysis. He established that the quantitative analysis of ageing phenomena for a system can be done using ageing intensity (AI) function, whereas hazard rate does the qualitative analysis.

The ageing intensity function (AI), denoted by $L_X(t)$ of a random variable X at time $t > 0$, with probability density function $f_X(t)$, survival function $\bar{F}_X(t)$ and failure rate $\lambda_X(t) =$

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$f_X(t)/\bar{F}_X(t)$ is given by (cf. Jiang et al. (2003)),

$$L_X(t) = \frac{-tf_X(t)}{\bar{F}_X(t)\ln\bar{F}_X(t)}, \text{ where defined,}$$
$$= \frac{t\lambda_X(t)}{\int_0^t \lambda_X(u)du}. \tag{1.1}$$

Nanda et al. (2007) and Bhattacharjee et al. (2013), Giri et al. (2021) derive the AI function of a few distributions. Sunoj and Rasin (2017) introduce quantile-based ageing intensity function and study its various ageing properties. To learn more on ageing intensity function, one can refer to Misra and Bhattacharjee (2018), Szymkowiak (2018a,b) to name a few.

Stochastic orders play an important role in the theory of reliability as it helps in comparison of systems based on the functions, discussed in this section, namely survival function $\bar{F}(t)$, hazard rate function $\lambda(t)$, reversed hazard rate function $\mu(t)$, mean residual function $m(t)$, ageing intensity function $L(t)$ etc. giving rise to usual stochastic order (ST order), hazard rate order (HR order), reversed hazard rate order (RHR order), mean residual order (MRL order) and ageing intensity order(AI order) respectively. The stochastic orders are mathematically represented as given in the next definition.

Definition 1.1. *A random variable X is said to be smaller than another random variable Y in*

(i) usual stochastic order (denoted by $X \leq_{ST} Y$) if $\bar{F}_X(t) \leq \bar{F}_Y(t)$, for all $t \geq 0$.

(ii) hazard rate order (denoted by $X \leq_{HR} Y$) if $\lambda_X(t) \geq \lambda_Y(t)$, for all $t \geq 0$.

(iii) reversed hazard rate order (denoted by $X \leq_{RHR} Y$) if $\mu_X(t) \leq \mu_Y(t)$, for all $t \geq 0$.

(iv) mean residual life order (denoted by $X \leq_{MRL} Y$) if $m_X(t) \leq m_Y(t)$, for all $t \geq 0$.

(v) AI order (denoted by $X \leq_{AI} Y$) if $L_X(t) \geq L_Y(t)$, for all $t > 0$.

Based on the hazard rate function, an ageing class has been defined in the literature as follows.

Definition 1.2. *A random variable X is said to have increasing (decreasing) hazard rate function, denoted by $IFR(DFR)$, if $\lambda_X(t)$ is increasing (decreasing) in $t \geq 0$.*

The words 'failure rate' and 'hazard rate' have been synonymously used in this article. Throughout the article, the words increasing (decreasing) and non-decreasing (non-increasing) are used interchangeably.

Section 2 discuss the monotonic properties of failure rate and ageing intensity functions in a few statistical distributions. Section 3 simply highlights the estimator of functions appearing in this paper. Section 4 cites an example to illustrate the study of ageing phenomena through reliability function, hazard rate, reversed hazard rate and ageing intensity functions. Section 5 demonstrates the concluding remarks of the work.

2. Monotonicity of failure rate and ageing intensity functions

On the basis of the monotonicity of the AI function, Nanda et al. (2007) define ageing classes, namely increasing ageing intensity class (IAI) (decreasing ageing intensity class (DAI)) if the corresponding AI function $L(t)$ is increasing(decreasing) in $t \geq 0$. It was pointed out that the monotonic behavior of the failure rate function is not, in general, transmitted to the monotonicity of the AI function, which is established by the following examples.

Example 2.1. *(cf. Nanda et al. (2007)) Let X has Erlang distribution, with density function with* $f_X(t) = \lambda^2 te^{-\lambda t}, t \geq 0$. *Clearly,its failure rate function is* $r_X(t) = \lambda^2 t/(1+\lambda t)$ *which increases for* $t \geq 0$, *i.e., X has increasing failure rate (IFR). On the other hand,* $L_X(t) = \lambda^2 t^2/(1+\lambda t)(\lambda t - \ln(1+\lambda t))$, *decreases for* $t > 0$, *i.e., X is* DAI. *So, X is* IFR *but* DAI.

Example 2.2. *(cf. Nanda et al. (2007)) Let X be a random variable having uniform distribution over* $[a,b], 0 \leq a < b < \infty$, *i.e., Then, its failure rate* $r_X(t) = 1/(b-t), a < t < b$ *is increasing in* $t \in (a,b)$, *i.e., X is IFR. However,* $L_X(t) = t/(b-t)/\ln\big(b/b-t\big)$, *for* $a < t < b$, *is increasing in* $t, a < t < b$. *So, X is* IFR *and* IAI.

In the next example, we find that a random variable is DFR and DAI.

Example 2.3. *Let X be a random variable having Pareto distribution with density function or* $f_X(t) = ak^a/t^{a+1}$, *for* $t \geq k > 0$, *so that its failure rate* $r_X(t) = a/t$, *is decreasing in* $t \in (k,\infty)$. *i.e., X is DFR. However,* $L_X(t) = 1/(\ln t - \ln k)$, *is increasing in* $t \in (k,\infty)$. *Thus, X is DFR and* IAI.

Through these aforementioned examples, one concludes that an IFR random variable could be IAI or DAI. So, does a DFR random variable. The non-monotonic nature are also observed for some statistical distributions (cf. Nanda et al. (2007, 2013)).

Reliability analysts can obviously strive for a question, if a system (or a random variable) depicts different characteristics in terms of failure rate and ageing intensity function then which function should be used in the final conclusion of knowing the behavior of the system in terms of ageing phenomena. In this paper, we try to answer this question by giving a case study mentioned in Section 4 and analyzing it.

3. Estimator of functions

Nanda et al. (2013) gives the logical estimates of survival function $\bar{F}_X(t)$, probability density function $\bar{f}_X(t)$, hazard rate function $\lambda_X(t)$, reversed hazard rate $\mu_X(t)$ and ageing intensity function $L_X(t)$. Let n units be put to test at $t = 0$. Further, let the number of units having survived at ordered times t_j be $n_s(t_j)$. Then logical estimates of $\bar{F}_X(t), f_X(t), \lambda_X(t), \mu_X(t)$ and $L_X(t)$ for $t_j < t < t_j + \Delta t_j$, are respectively given by

$$\hat{\bar{F}}_X(t) = \frac{n_s(t_j)}{n},$$

$$\hat{f}_X(t) = \frac{n_s(t_j) - n_s(t_j + \Delta t_j)}{n\Delta t_j},$$

$$\hat{\lambda}_X(t) = \frac{\Big\{n_s(t_j) - n_s(t_j + \Delta t_j)\Big\}}{n_s(t_j)\Delta t_j},$$

$$\hat{\mu}_X(t) = \frac{\Big\{n_s(t_j) - n_s(t_j + \Delta t_j)\Big\}}{(n - n_s(t_j))\Delta t_j}.$$

Thus, logical estimate of $L_X(t)$ is

$$\hat{L}_X(t) = \frac{-t\Big\{n_s(t_j) - n_s(t_j + \Delta t_j)\Big\}}{n_s(t_j)\Delta t_j \ln \frac{n_s(t_j)}{n}},$$

for $t_j < t < t_j + \Delta t_j$.

4. An Example to illustrate the study of ageing phenomena through reliability function, hazard rate, reversed hazard rate and ageing intensity functions

A good number of life testing data can be found for analysis in Shooman (1968), Ebeling (1997) and others.

Example 4.1. *(cf. Ebeling (1997)) Fifteen units each of two different deadbolt locking mechanisms were tested under accelerated conditions until 10 failures of each were observed. The following failure times in thousands of cycles were recorded as in Table 1. Which design appears to provide the best function?*

Note that, estimator of probability density function for $t_i \leq t \leq t_{i+1}$ is

$$\begin{aligned}\hat{f}(t) &= -\frac{\hat{R}(t_{i+1}) - \hat{R}(t_i)}{(t_{i+1} - t_i)} \\ &= \frac{1}{(t_{i+1} - t_i)(n+1)} \end{aligned} \qquad (4.2)$$

that of failure rate function is

$$\begin{aligned}\hat{\lambda}(t) &= \frac{\hat{f}(t)}{\hat{R}(t)} \\ &= \frac{1}{(t_{i+1} - t_i)(n+1-i))}. \end{aligned} \qquad (4.3)$$

The estimator of reversed hazard rate is given by,

$$\begin{aligned}\hat{\mu}(t) &= (\hat{f}(t))/(\hat{F}(t)) \\ &= \frac{1/(t_{i+1} - t_i)(n+1)}{i/(n+1)} \\ &= \frac{1}{i(t_{i+1} - t_i)} \end{aligned} \qquad (4.4)$$

Now, for the ageing intensity, it is given by,

$$\begin{aligned}\hat{L}(t) &= \frac{-t\hat{f}(t)}{\hat{\bar{F}}(t) \ln \hat{\bar{F}}(t)} \\ &= \frac{-t/(t_{i+1} - t_i)(n+1)}{\{(n+1-i)/(n+1)\} \ln\{(n+1-i)/(n+1)\}} \\ &= \frac{-t}{(t_{i+1} - t_i)(n+1-i) \ln(n+1-i)/(n+1)} \end{aligned} \qquad (4.5)$$

The detailed analysis of the example considered in this Section are given in Table 2, Table 3, Table 4, Table 5 and Table 6. The Plots are also displayed in Figure 1, Figure 2, Figure 3 and Figure 4.

5. Conclusion

According to the literature on stochastic orders, we know that any system, say, here, Design-A is said to be better than design-B, if design-A has less ageing intensity, less hazard rate and higher reliability than that of design-B. The concluding remarks as noted in Table 6 at a certain interval of time are summarized as follows:

(i) Design A is better than design B in terms of the function being doubly underlined in a time interval.

(ii) Design B is better than design A on the basis of the function being singly underlined during a certain time interval.

(iii) However, the function being starred in a time interval denotes the fact that we cannot specify which among A or B is of the better design.

(iv) For example, in the interval (56.8,77], design B is better in terms of ageing intensity, whereas according to hazard rate, design A is better during (56.8,63] and design B is better in the interval (63,77]. Also, the analyzing the systems in terms of reliability reveal that, both the designs A and B have equal reliabilities during (56.8,63] but design-A is better on (63,77].

(v) It is evident that Table 6 contains more singly underlined cells than than that of doubly underlined cells.

(vi) In a nutshell, design B is more efficient than that of design A.

(vii) We attempt to identify the function which should be preferred in determining the ageing behaviour of a system.

In Table 6, one can observe that if at some interval of time the ageing intensity, hazard rate and the reliability have the same nature (either single underlined or doubly underlined) or (doubly underlined with starred) or (singly underlined with starred),then all the three measures give the same conclusion in choosing the best system design. But if one function is doubly underlined and another is singly underlined, then it gives different conclusion with regard to the performance of the systems.

(viii) For example, on the interval (56.8,63], the ageing intensity and the hazard rate show different behaviour, whereas on the interval (63,77] hazard rate and reliability show different behaviour. And on (897.8,1043.6], all the three measures show same behaviour.

(ix) Clearly, from Table 6 we can see that, hazard rate doesn't have opposite behaviour with the other two measures simultaneously. For example, on the interval (56.8,63], hazard rate shows opposite behaviour to ageing intensity function only, but not to reliability. Also, it shows opposite behaviour to reliability on (63,77], but not to the ageing intensity function in that interval. We note that, hazard rate doesn't have any doubtful situations ($\lambda_1 = \lambda_2$), which are in the case of ageing intensity or reliability at some intervals. (as, the equality sign doesn't say anything about which design is better, so these are the doubtful situations.)

Therefore, we conclude that, hazard rate should be preferred as a measure of ageing phenomena, while comparing the two systems in the problem concerned.

Table 1: *Failure Times*

Design A	44	77	218	251	317	380	438	739	758	1115
Design B	32	63	211	248	327	404	476	877	903	1416

Table 2: *Analysis of Design A*

i	t_i	$R_1(t)$	$\lambda_1(t)$	$\mu_1(t)$	$L_1(t)$
0	0	1	0.002066		
1	44	0.909	0.00303	0.022727	0.3179t
2	77	0.8182	0.000788	0.015152	$0.00393t$
3	218	0.7273	0.003788	0.002364	$0.01189t$
4	251	0.6364	0.002165	0.007576	$0.00479t$
5	317	0.5455	0.002646	0.00303	$0.00436t$
6	380	0.4546	0.003448	0.002646	$0.00437t$
7	438	0.3636	0.000831	0.002463	$0.00082t$
8	739	0.2727	0.017544	0.000415	$0.0135t$
9	758	0.1818	0.001401	0.005848	$0.00082t$
10	1115	0.0909		0.00028	

Table 3: *Analysis of Design B*

i	t_i	$R_2(t)$	$\lambda_2(t)$	$\mu_2(t)$	$L_2(t)$
0	0	1	0.002841		
1	32	0.909	0.002933	0.03125	0.0339t
2	63	0.8182	0.000614	0.016129	$0.00374t$
3	211	0.7273	0.002457	0.002252	$0.0106t$
4	248	0.6364	0.001151	0.006757	$0.004t$
5	327	0.5455	0.001181	0.002532	$0.00357t$
6	404	0.4546	0.001263	0.002165	$0.00352t$
7	476	0.3636	0.000227	0.001984	$0.00062t$
8	877	0.2727	0.003497	0.000312	$0.00987t$
9	903	0.1818	0.000177	0.004274	$0.00057t$
10	1416	0.0909		0.000195	

Table 4: *Comparison of $R(t), \lambda(t), \mu(t)$*

Time	$R_1(t)$	$R_2(t)$	Order $R(t)$	$\lambda_1(t)$	$\lambda_2(t)$	Order $\lambda(t)$	$\mu_1(t)$	$\mu_2(t)$	Order $\mu(t)$
$(0, 32]$	1	1	$R_1 = R_2$	0.002066	0.002841	$\lambda_1 < \lambda_2$	0.022727	0.03125	$\mu_1 < \mu_2$
$(32, 44]$	1	0.909	$R_1 > R_2$	0.002066	0.003226	$\lambda_1 < \lambda_2$	0.022727	0.016129	$\mu_1 > \mu_2$
$(44, 63]$	0.909	0.909	$R_1 = R_2$	0.00303	0.003226	$\lambda_1 < \lambda_2$	0.015152	0.016129	$\mu_1 > \mu_2$
$(63, 77]$	0.909	0.8182	$R_1 > R_2$	0.00303	0.000751	$\lambda_1 < \lambda_2$	0.015152	0.002252	$\mu_1 > \mu_2$
$(77, 211]$	0.8182	0.8182	$R_1 = R_2$	0.000788	0.000751	$\lambda_1 < \lambda_2$	0.002364	0.002252	$\mu_1 > \mu_2$
$(211, 218]$	0.8182	0.7273	$R_1 > R_2$	0.000788	0.003378	$\lambda_1 < \lambda_2$	0.002364	0.006757	$\mu_1 > \mu_2$
$(218, 248]$	0.7273	0.7273	$R_1 = R_2$	0.003788	0.003378	$\lambda_1 < \lambda_2$	0.007576	0.006757	$\mu_1 > \mu_2$
$(248, 251]$	0.7273	0.6364	$R_1 > R_2$	0.003788	0.001808	$\lambda_1 < \lambda_2$	0.007576	0.002532	$\mu_1 > \mu_2$
$(251, 317]$	0.6364	0.6364	$R_1 = R_2$	0.002165	0.001808	$\lambda_1 < \lambda_2$	0.00303	0.002532	$\mu_1 > \mu_2$
$(317, 327]$	0.5455	0.6364	$R_1 < R_2$	0.002646	0.001808	$\lambda_1 < \lambda_2$	0.002646	0.002532	$\mu_1 > \mu_2$
$(327, 380]$	0.5455	0.5455	$R_1 = R_2$	0.002646	0.002165	$\lambda_1 < \lambda_2$	0.002646	0.002165	$\mu_1 > \mu_2$
$(380, 404]$	0.4546	0.5455	$R_1 < R_2$	0.003448	0.002165	$\lambda_1 < \lambda_2$	0.002463	0.002165	$\mu_1 > \mu_2$
$(404, 438]$	0.4546	0.4546	$R_1 = R_2$	0.003448	0.002778	$\lambda_1 < \lambda_2$	0.002463	0.001984	$\mu_1 > \mu_2$
$(438, 476]$	0.3636	0.4546	$R_1 < R_2$	0.000831	0.002778	$\lambda_1 < \lambda_2$	0.000415	0.001984	$\mu_1 > \mu_2$
$(476, 739]$	0.3636	0.3636	$R_1 = R_2$	0.000831	0.000623	$\lambda_1 < \lambda_2$	0.000415	0.000312	$\mu_1 > \mu_2$
$(739, 758]$	0.2727	0.3636	$R_1 < R_2$	0.017544	0.000623	$\lambda_1 < \lambda_2$	0.005848	0.000312	$\mu_1 > \mu_2$
$(758, 877]$	0.1818	0.3636	$R_1 < R_2$	0.001401	0.000623	$\lambda_1 < \lambda_2$	0.00028	0.000312	$\mu_1 > \mu_2$
$(877, 903]$	0.1818	0.2727	$R_1 < R_2$	0.001401	0.012821	$\lambda_1 < \lambda_2$	0.00028	0.004274	$\mu_1 > \mu_2$
$(903, 1115]$	0.1818	0.1818	$R_1 = R_2$	0.001401	0.000975	$\lambda_1 < \lambda_2$	0.00028	0.000195	$\mu_1 > \mu_2$
$(1115, 1416]$	0.0909	0.1818	$R_1 < R_2$		0.000975				

Figure 1: *Plot of R_1 and R_2 versus time t.*

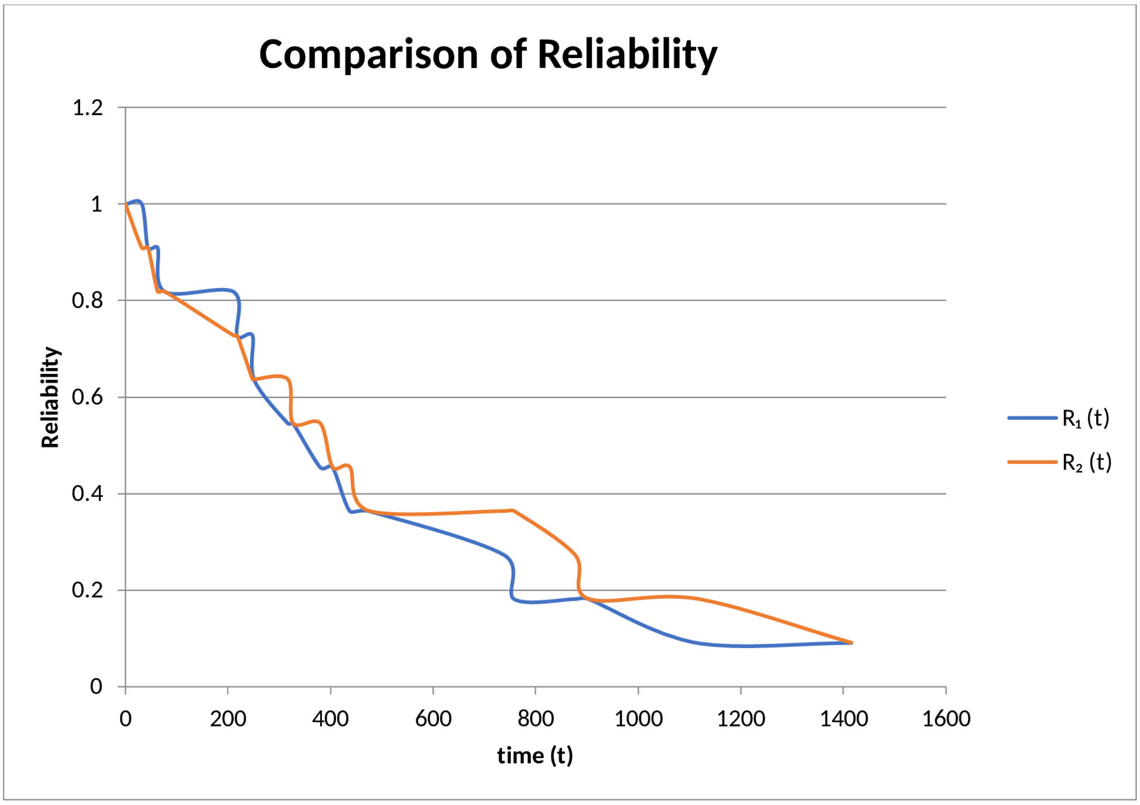

Figure 2: *Plot of HR_1 and HR_2 versus time t*

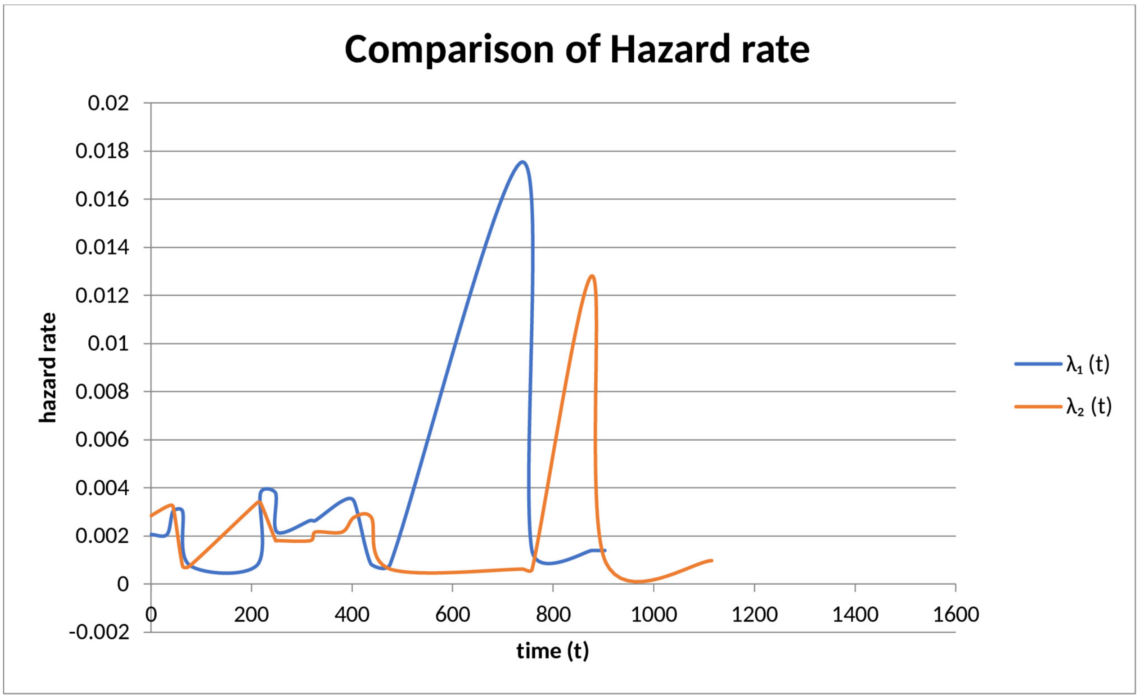

Table 5: *$L_1(t)$ and $L_2(t)$*

Design A		Design B	
t	$L_1(t)$	t	$L_2(t)$
32	1.0848	44	1.39876
38.2	1.29498	50.6	1.608574
44.4	1.50516	57.2	1.818388
50.6	1.71534	63.8	2.028202
56.8	1.92552	70.4	2.238016
63	0.23562	77	0.30261
92.6	0.346324	105.2	0.413436
122.2	0.457028	133.4	0.524262
151.8	0.567732	161.6	0.635088
181.4	0.678436	189.8	0.745914
211	2.2366	218	2.59202
218.4	2.31504	224.6	2.670494
225.8	2.39348	231.2	2.748968
233.2	2.47192	237.8	2.827442
240.6	2.55036	244.4	2.905916
248	0.992	251	1.20229
263.8	1.0552	264.2	1.265518
279.6	1.1184	277.4	1.328746
295.4	1.1816	290.6	1.391974
311.2	1.2448	303.8	1.455202
327	1.16739	317	1.38212
342.4	1.222368	329.6	1.437056
357.8	1.277346	342.2	1.491992
373.2	1.332324	354.8	1.546928
388.6	1.387302	367.4	1.601864
404	1.42208	380	1.6606
418.4	1.472768	391.6	1.711292
432.8	1.523456	403.2	1.761984
447.2	1.574144	414.8	1.812676
461.6	1.624832	426.4	1.863368
476	0.29512	438	0.35916
556.2	0.344844	498.2	0.408524
636.4	0.394568	558.4	0.457888
716.6	0.444292	618.6	0.507252
796.8	0.494016	678.8	0.556616
877	8.65599	739	9.9765
882.2	8.707314	742.8	10.0278
887.4	8.758638	746.6	10.0791
892.6	8.809962	750.4	10.1304
897.8	8.861286	754.2	10.1817
903	0.51471	758	0.62156
1005.6	0.573192	829.4	0.680108
1108.2	0.631674	900.8	0.738656
1210.8	0.690156	972.2	0.797204
1313.4	0.748638	1043.6	0.855752
1416		1115	

Table 6: *Interval-wise Study*

Interval	Compare $L(t)$	Interval	Compare $\lambda(t)$	Interval	Compare $R(t)$
$(56.8, 77]$	$L_1 > L_2$	$(56.8, 63]$	$\lambda_1 < \lambda_2$	$(56.8, 63]$	$R_1 = R_2^*$
		$(63, 77]$	$\lambda_1 > \lambda_2$	$(63, 77]$	$R_1 > R_2$
$(77, 211]$	$L_1 = L_2^*$	$(77, 211]$	$\lambda_1 > \lambda_2$	$(77, 211]$	$R_1 = R_2^*$
$(211, 240.6]$	$L_1 > L_2$	$(211, 218]$	$\lambda_1 < \lambda_2$	$(211, 218]$	$R_1 > R_2$
		$(218, 240.6]$	$\lambda_1 > \lambda_2$	$(218, 240.6]$	$R_1 = R_2^*$
$(240.6, 248]$	$L_1 = L_2^*$	$(240.6, 248]$	$\lambda_1 > \lambda_2$	$(240.6, 248]$	$R_1 = R_2^*$
$(248, 418.4]$	$L_1 > L_2$	$(248, 418.4]$	$\lambda_1 > \lambda_2$	$(248, 251]$	$R_1 > R_2$
				$(251, 317]$	$R_1 = R_2^*$
				$(317, 327]$	$R_1 < R_2$
				$(327, 380]$	$R_1 = R_2^*$
				$(380, 404]$	$R_1 < R_2$
				$(404, 418.4]$	$R_1 = R_2^*$
$(418.4, 476]$	$L_1 < L_2$	$(418.4, 438]$	$\lambda_1 > \lambda_2$	$(418.4, 438]$	$R_1 = R_2^*$
		$(438, 476]$	$\lambda_1 < \lambda_2$	$(438, 476]$	$R_1 = R_2^*$
$(476, 636.4]$	$L_1 = L_2^*$	$(476, 636.4]$	$\lambda_1 > \lambda_2$	$(476, 636.4]$	$R_1 = R_2^*$
$(636.4, 796.8]$	$L_1 > L_2$	$(636.4, 796.8]$	$\lambda_1 > \lambda_2$	$(636.4, 739]$	$R_1 = R_2^*$
				$(739, 758]$	$R_1 < R_2$
				$(758, 796.8]$	$R_1 < R_2$
$(796.8, 897.8]$	$L_1 < L_2$	$(796.8, 877]$	$\lambda_1 > \lambda_2$	$(796.8, 877]$	$R_1 < R_2$
		$(877, 897.8]$	$\lambda_1 < \lambda_2$	$(877,897.8]$	$R_1 < R_2$
$(897.8, 1043.6]$	$L_1 > L_2$	$(897.8, 1043.6]$	$\lambda_1 > \lambda_2$	$(897.8, 903]$	$R_1 < R_2$
				$(903, 1043.6]$	$R_1 = R_2^*$

Figure 3: *Plot of RHR_1 and RHR_2 versus time t*

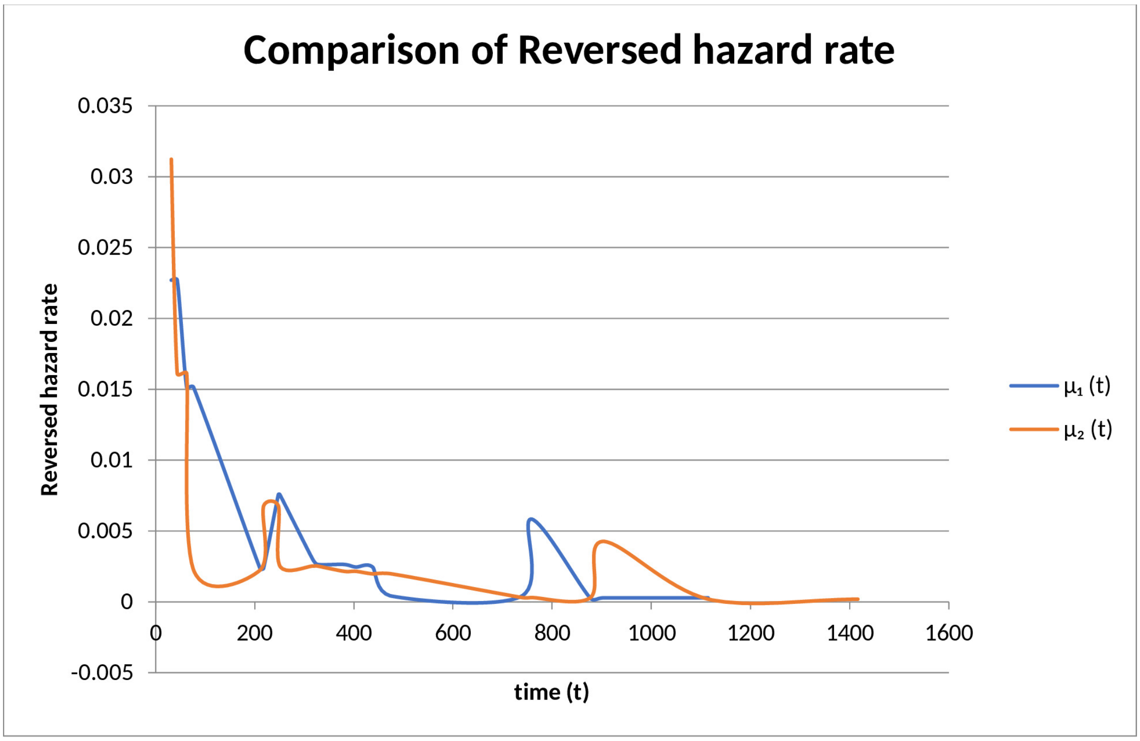

Figure 4: *Plot of AI_1 and AI_2*

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References

[1] Barlow, R.E. and Proschan, F. (1975), *Statistical Theory of Reliability and Life Testing, Probability Models.* Holt Rinehart and Winston Inc. New York.

[2] Bhattacharjee, S., Nanda A.K. and Misra, S.K. (2013), Reliability Analysis using ageing intensity function. *Statistics and Probability Letters*, 83, 1364–1371.

[3] Block, H.W., Savits, T. and Singh, H. (1998), Reversed hazard rate functions. *Probability in the Engineering and Informational Sciences*, 12, 69–90.

[4] Ebeling C.E. (1997), An Introduction to Reliability and Maintainability Engineering Michigan: Mcgraw Hill.

[5] Deshpande, J.V. and Purohit, S.G. (2005), *Life Time Data: Statistical Models and Methods.* Series in Quality, Reliability and Engineering Statistics, Vol 11, World scientific Publishing Co. Pte Ltd.

[6] Giri, R. L., Nanda, A. K., Dasgupta, M., Misra, S. Kr. and Bhattacharjee, S., On ageing intensity function of some Weibull models, *Communications in Statistics-Theory and Methods,* DOI: 10.1080/03610926.2021.1910845, Published online on 26 April 2021.

[7] Jiang, R., Ji, P., and Xiao, X. (2003), Ageing property of univariate failure rate models, *Reliability Engineering and System Safety,* 79, 113-116.

[8] Misra, S.K. and Bhattacharjee, S. (2018), A case study of ageing intensity funtion on censored data, *Alexendria Engineering Journal,* 57, 3931–3952.

[9] Nanda, A. K., Bhattacharjee, S. and Balakrishnan, N. (2010), Mean residual life fuction, associated orederings and properties, *IEEE Transactions on Reliability,* 59(1), 55-65.

[10] Nanda, A.K., Singh, H., Misra, N. and Paul, P. (2003), Reliability properties of reversed residual lifetime, *Communications in Statistics-Theory and Methods,* 32(10), 2031-2042. [Addendum: *Communications in Statistics-Theory and Methods*, (2004), 33(4), 991–992].

[11] Nanda, A.K. and Sengupta, D. (2005), Discrete life distributions with decreasing reversed hazard. *Sankhyā* 67, Part 1, 1-19.

[12] Nanda, A. K., Bhattacharjee, S., Alam, S.S. (2007), Properties of ageing intensity function, *Statistics and Probability Letters,* 77, 365–373.

[13] Nanda, A.K., Bhattacharjee, S. and Balakrishnan,N. (2010), Mean Residual Life Function, Associated Orderings and Properties, *IEEE Transactions on Reliability,* 59(1),55–65.

[14] Nanda, A.K., Bhattacharjee, S. and Alam, S.S. (2006), On Up Shifted Reversed Mean Residual Life Order, *Communications in Statistics: Theory and Methods,* 35(8), 1513–1523.

[15] Shaked, M., and Shanthikumar, J.G. (2007), Stochastic Orders. New York: Springer.

[16] Shooman, M. L. (1968), Probabilistic Reliability: An Engineering Approach. McGraw-Hill, Inc.

[17] Sunoj, S. M., and Rasin, R. S. (2018). A quantile-based study on ageing intensity function, *Communications in Statistics: Theory and Methods,* 47 (22), 5474–5484.

[18] Szymkowiak, M. (2018a), Characterizations of Distributions Through Ageing Intensity, *IEEE Transactions on Reliability,* 67, 446–458.

[19] Szymkowiak, M. (2018 b), Lifetime Analysis by Ageing Intensity Functions Studies in Systems, Decision and Control Volume 196 Series editor, Springer.