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POWER - EXPONENTIAL GEOMETRIC QUANTILE FUNCTION

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Abstract

In this article, we introduced a new quantile function which is the sum of quantile functions of Power and Exponential geometric distributions. Different distributional charactaristics and reliability properties are discussed and also simulation study is conducted by using R software. Finally the new model is applied to a real data set.

Keywords: Exponential geometric distribution; Hazard quantile function; L - moments; Mean residual quantile function; Percentile residual quantile function; Power distribution; Reversed hazard quantile function; Reversed mean residual quantile function.

1. Introduction

Reliability analysis can be done by using distribution functions or by using quantile functions, although both convey the same information about the distribution with different interpretations. In reliability analysis, quantile based methods are particularly useful. For a nonnegative random variable X with distribution function F(x), the quantile function Q(u) is defined by (see Nair and Sankaran(2009))

$$Q(u) = F^{-1}(u) = \inf\{x : F(x) \geq u\},\ 0 \leq u \leq 1 \tag{1}$$

If f(x) is the probability density function of X, then f(Q(u)) is called the density quantile function. If F(x) is right continuous and strictly increasing, we have,

$$F(Q(u)) = u$$

The derivative of Q(u) is known as the quantile density function of X and is denoted by q(u).i.e.,

$$q(u) = Q'(u) \tag{2}$$

When f(x) is the probability density function (pdf) of X, then by taking the derivative of F(Q(u))=u we get,

$$q(u)f(Q(u)) = 1$$

Quantile functions have several properties that are not shared by distribution functions. For example, the sum of two quantile functions is again a quantile function. Further, the product of two positive quantile functions is again a quantile function in the nonnegative setup. There are explicit general distribution forms for the quantile function of order statistics. It is easier to generate random numbers from the quantile function. A major development in portraying quantile functions to model statistical data is given by Hastings et al. (1947), who introduced a family of distributions by a quantile function. This was refined later by Tukey (1962) to form a symmetric distribution, called the Tukey lambda distribution. This model was generalized in different ways, referred as lambda distributions which include various forms of quantile functions discussed by Ramberg and Schmeiser (1972), Ramberg (1975), Ramberg et al. (1979) and Freimer et al. (1988).

Govindarajulu (1977) introduced a new quantile function by taking the weighted sum of quantile functions of two power distributions. Hankin and Lee (2006) presented a new Power - Pareto distribution by taking the product of power and Pareto quantile functions. Van Staden and Loots (2009) developed a four - parameter distribution, using a weighted sum of the generalized Pareto and its reflection quantile functions. Sankaran et al. (2016) developed a new quantile function based on the sum of quantile functions of generalized Pareto and Weibull quantile functions. Sankaran and Dileep (2016) introduced a new class of quantile functions by taking the sum of quantile functions of half logistic and exponential geometric distributions which is useful in reliability analysis. Also in (2018), they introduced another class of quantile function by taking the product of quantile functions of Pareto and Weibull distributions. The density and distribution functions for these models are not available in closed forms except for certain special cases. The great advantage of these models is that the simple forms of the quantile functions make it extremely straightforward to simulate random values, which is useful in inference problems.

The power exponential geometric quantile function is derived by taking the sum of quantile functions of power and exponential geometric distributions. The survival function and quantile function of power distribution are respectively given by,

$$S(x) = 1 - \left(\frac{x}{\alpha}\right)^{\beta}, 0 < x < \alpha \ and \ \alpha > 0, \beta > 0 \tag{3}$$

and

$$Q_1(u) = \alpha u^{\frac{1}{\beta}}, 0 < u < \alpha \ and \ \alpha > 0, \beta > 0 \tag{4}$$

Adamidis and Loukas (1998) introduced the exponential geometric (EG) distribution with applications to reliability modelling in the context of decreasing failure rate data. The survival function and quantile function of the EG distribution are given by,

$$S(x) = 1 - F(x) = (1-P)e^{-\frac{x}{\alpha}}\left(1 - Pe^{-\frac{x}{\alpha}}\right)^{-1}, 0 < P < 1 \ and \ \alpha, \lambda > 0 \tag{5}$$

and

$$Q_2(u) = \frac{1}{\lambda} \ log\left(\frac{1-Pu}{1-u}\right), 0 < P < 1 \ and \ \alpha > 0 \ \lambda > 0 \tag{6}$$

The rest of the paper is designed as follows. In section 2, we define the Power Exponential geometric (PEG) Quantile function and the members of this family are discussed in section 3. Distributional characteristics are studied in section 4, L - moments in section 5 and density function of r^{th} order statistic in section 6. In section 7, reliability properties like hazard quantile function, mean residual quantile function, percentile residual quantile function, etc. are studied. A simulation study is conducted in section 8 and concluded in section 9.

2. Power – Exponential Geometric (PEG) Quantile Function

We introduce a new quantile function, which is the sum of quantile functions of power and exponential geometric distributions.

let X and Y be two nonnegative random variables with distribution functions F(x) and G(x) with quantile functions $Q_1(u)$ and $Q_2(u)$, respectively . Then

$$Q(u) = Q_1(u) + Q_2(u) \tag{7}$$

is also a quantile function (see Nair et al. (2013)). We now introduce a new quantile function,

$$Q(u) = \alpha u^{\frac{1}{\beta}} + \frac{1}{\lambda} \ log\left(\frac{1-Pu}{1-u}\right), \ \alpha, \beta, \ \lambda > 0 \ and \ 0 < P < 1. \tag{8}$$

is the sum of (4) and (6). The support of the new model is (0, ∞). The quantile density function is obtained as,

$$q(u) = \frac{\beta(1-P)+\lambda(1-u-Pu+Pu^2)\alpha u^{\frac{1}{\beta}-1}}{\beta\lambda(1-u-Pu-Pu^2)},\ \alpha,\beta,\ \lambda > 0\ and\ 0 < P < 1. \quad (9)$$

For the PEG quantile function, the density function f(x) can be written in terms of the distribtion as,

$$f(x) = \frac{\beta\lambda(1-F(x)-PF(x)+P(F(x))^2)}{\beta-\beta P+\lambda\alpha F(x)^{\frac{1}{\beta}-1}(1-F(x)-PF(x)+P(F(x))^2)}, \alpha > 0, \beta > 0,\ \lambda > 0\ ,\ 0 < P < 1. \quad (10)$$

The quantile function (8) represents a family of distributions which have various shapes for different values of parameters. The shapes of density function for different values of parameters are given below.

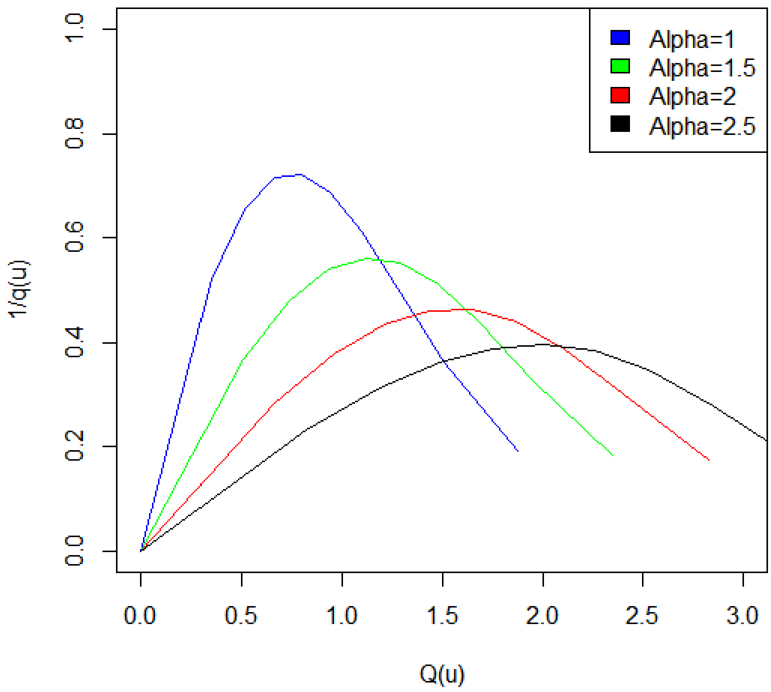

Figure 1: *Plot of density function for different values of α with β= 2, λ=2 and P=0.4.*

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3. Members of The Family

The PEG quantile function includes several well - known quantile functions for various values of the parameters. We can derive some well - known quantile functions from the proposed model by making use of various transformations.

Case 1 : $\alpha = 0$, $\lambda > 0$ and P = 0
The quantile function of the PEG model reduces to the form

$$Q(u) = \frac{1}{\lambda}\ (-log\,(1-u)) \quad (11)$$

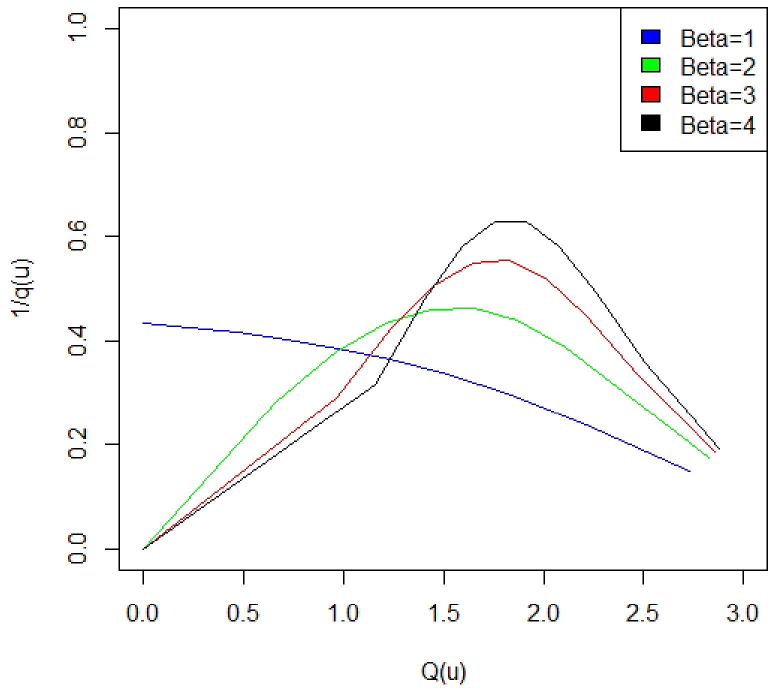

Figure 2: *Plot of density function for different values of β with α= 2, λ=2 and P=0.4.*

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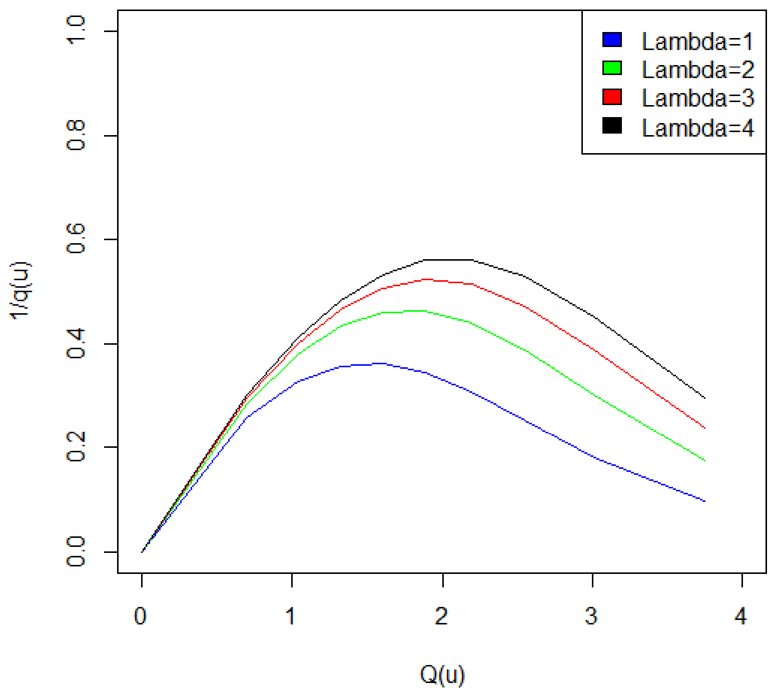

Figure 3: *Plot of density function for different values of λ with α= 2, β=2 and P=0.4.*

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which is the quantile function of exponential distribution with mean $\frac{1}{\lambda}$.

Case 2 : $\alpha = 0$, $\lambda > 0$ and $0 < P < 1$.
Then the corresponding quantile function is,

$$Q(u) = \frac{1}{\lambda}\ log\left(\frac{1-Pu}{1-u}\right) \tag{12}$$

which belongs to the class of distributions with linear hazard quantile functions defined by Midhu et al. (2014), with quantile function

$$Q(u) = \frac{1}{a(1+\theta)}\ log\left(\frac{1+\theta u}{1-u}\right) \tag{13}$$

where θ = - P, $-1 < \theta < 0$ and λ = a(1 - P)

Case 3 : $\alpha > 0$, β = 1, $0 < P < 1$ and λ tends to ∞
The quantile function of the PEG model is reduced to,

$$Q(u) = \alpha u \tag{14}$$

which is the quantile function of uniform U(0,$\frac{1}{\alpha}$)

Case 4 : We can apply the power transformation on (11) with $\alpha = 0$, $\lambda > 0$ and P = 0 to form the quantile function of Weibull distribution with parameters $\frac{1}{\lambda}$ and K.

$$Q(u) = \frac{1}{\lambda}\ \left(-log\left(1-u\right)\right)^{K} \tag{15}$$

where K is the power.
There are some theorems that are applicable in PEG quantile function.

Theorem 1. If X follows Power distribution with distribution function $F_X(x) = \left(\frac{x}{\alpha}\right)^{\beta}$; $0 \leq$ X $\leq\alpha$, $\beta > 0$, then the random variable Z = X + $\frac{1}{\lambda}log\left(\frac{\alpha^{\beta}-PZ^{\beta}}{\alpha^{\beta}-Z^{\beta}}\right)$ will follow PEG(α,β,λ,P) distribution.

Proof: Let T and V be the two random variables with $Q_T(u)$ and $Q_V(u)$ be the corresponding quantile functions and $F_T(x)$ and $F_V(x)$ be the corresponding distribution functions respectively. Now suppose Q*(u) is defined by $Q_T(u) + Q_V(u)$.
Then the random variable that corresponds to the quantile function Q*(u) is $T + Q_V(F_T(T))$ or $V + Q_T(F_V(V))$ (Sankaran et al. 2016). Now let Y follow exponential geometric distribution with distribution function $F_Y(x) = (1-e^{-\lambda x})(1-Pe^{-\lambda x})^{-1}$ and X follows power distribution with distribution function $F_X(x) = \left(\frac{x}{\alpha}\right)^{\beta}$ then X + $Q_Y(F_X(X))$ has PEG(α, β, λ, P) distribution. Since $Q_Y(u)$= $\frac{1}{\lambda}log\left(\frac{1-Pu}{1-u}\right)$ and $F_X(x) = \left(\frac{x}{\alpha}\right)^{\beta}$, we get,

$$X + Q_Y(F_X(X)) = X + \frac{1}{\lambda}log\left(\frac{\alpha^{\beta}-PZ^{\beta}}{\alpha^{\beta}-Z^{\beta}}\right)$$

Hence the proof.

Theorem 2. Let Z follows EG ($\frac{1}{\lambda}$, P), then a random variable X = $Z + \alpha(1-e^{-\lambda x})^{\frac{1}{\beta}}(1-Pe^{-\lambda x})^{\frac{-1}{\beta}}$ will follow PEG(α, β, λ, P) distribution.
The proof is similar to that of Theorem 1.

4. Distributional Characteristics

The quantile based measures of the distributional characteristics of location, dispersion, skewness and kurtosis are popular in statistical analysis. These measures are also useful for estimating parameters of the model by matching population characteristics with corresponding sample characteristics.
Median (M) of the PEG model is,

$$\begin{aligned} M &= Q(0.5) \\ &= \alpha (0.5)^{\frac{1}{\beta}} \end{aligned} \tag{16}$$

Interquartile range (IQR) of the PEG model is,

$$\begin{aligned} IQR &= Q_3 - Q_1 \\ &= \alpha \left((0.75)^{\frac{1}{\beta}} - (0.25)^{\frac{1}{\beta}} \right) + \frac{1}{\lambda} \log \left(\frac{4-3P}{1.33-0.33P} \right) \end{aligned} \tag{17}$$

Galton's coefficient of skewness (S) of the PEG model is,

$$\begin{aligned} S &= \frac{Q_1 + Q_3 - 2M}{Q_3 - Q_1} \\ &= \frac{\alpha \left((0.25)^{\frac{1}{\beta}} + (0.75)^{\frac{1}{\beta}} - 2(0.5)^{\frac{1}{P}} \right) + \left(\log(1.33 - 0.33P) + \log \left(\frac{4-3P}{2-P} \right) \right)}{\alpha \left((0.75)^{\frac{1}{\beta}} - (0.25)^{\frac{1}{\beta}} \right) + \frac{1}{\lambda} \log \left(\frac{4-3P}{1.33-0.33P} \right)} \end{aligned} \tag{18}$$

Moor's coefficient of kurtosis (T) of the PEG model is,

$$\begin{aligned} T &= \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{IQR} \\ &= \frac{\alpha \left(0.875^{\frac{1}{\beta}} - 0.625^{\frac{1}{\beta}} + 0.375^{\frac{1}{\beta}} - 0.125^{\frac{1}{\beta}} \right) + \frac{1}{\lambda} \log \left(\frac{0.107P^2 - 0.41P + 0.328}{0.006P^2 - 0.039P + 0.078} \right)}{\alpha \left(0.75^{\frac{1}{\beta}} - 0.25^{\frac{1}{\beta}} \right) + \frac{1}{\lambda} \log \left(\frac{4-3P}{1.33-0.33P} \right)} \end{aligned} \tag{19}$$

5. L - Moments

L- moments are the expected values of linear function of order statistics. The L- moments are often found to be more desirable than the conventional moments in describing the characteristics of the distributions as well as for inference. L-moments can be used as summary measures (statistics) of probability distributions (samples) to identify distributions and to fit models to data. A unified theory and a systematic study on L - moments have been presented by Hosking (1990).

The r^{th} L-moment is given by,

$$L_r = \int_0^1 \sum_{k=0}^{r-1} (-1)^{r-1-k} \binom{r-1}{k} \binom{r-1+k}{k} u^k Q(u) du \tag{20}$$

The first L moment is the mean of the distribution. For the PEG model, L_1 is obtained as,

$$\begin{aligned} L_1 &= \int_0^1 Q(u) du \\ &= \frac{\alpha\beta}{1+\beta} + \frac{(P-1)\log(1-P)}{\lambda P} \end{aligned} \tag{21}$$

The second L moment is obtained as,

$$\begin{aligned} L_2 &= \int_0^1 (2u-1)Q(u)du \\ &= \frac{\alpha\beta}{1+2\beta^2+3\beta} + \frac{(P-1)\,(P+log(1-P))}{\lambda P^2} \end{aligned} \tag{22}$$

The third L moment is,

$$\begin{aligned} L_3 &= \int_0^1 (6u^2-6u+1)Q(u)du \\ &= \frac{\alpha\beta}{\beta+1} - \frac{6\alpha\beta^2}{1+5\beta+6\beta^2} + \frac{2(P-1)}{\lambda P^2} - \frac{(P-1)(P-2)log(1-P)}{\lambda P^3} \end{aligned} \tag{23}$$

Fourth L moment is obtained as,

$$\begin{aligned} L_4 &= \int_0^1 (20u^3-30u^2+12u-1)Q(u)du \\ &= \frac{20\alpha\beta}{1+4\beta} - \frac{30\alpha\beta}{1+3\beta} + \frac{11\alpha\beta+10\alpha\beta^2}{1+3\beta+2\beta^2} + \frac{\alpha(P-1)(P^3-15P^2+30P+6\left((P-5)P+5\right)log(1-P))}{6P^4} \end{aligned} \tag{24}$$

The L - coefficient of variation, analogous to the coefficient of variation based on ordinary moments for model (8) is given by,

$$\tau_2 = \frac{L_2}{L_1} = \frac{\frac{\alpha\beta}{1+\beta} + \frac{(P-1)(P+log(1-P))}{\lambda P^2}}{\frac{\alpha\beta}{1+\beta} + \frac{(P-1)\log(1-P)}{\lambda P}} \tag{25}$$

L - coefficient of skewness is obtained as,

$$\tau_3 = \frac{L_3}{L_2} = \frac{\frac{\alpha\beta}{\beta+1} - \frac{6\alpha\beta^2}{1+5\beta+6\beta^2} + \frac{2(P-1)}{\lambda P^2} - \frac{(P-1)(P-2)log(1-P)}{\lambda P^3}}{\frac{\alpha\beta}{1+2\beta^2+3\beta} + \frac{(P-1)(P+log(1-P))}{\lambda P^2}} \tag{26}$$

L - coefficient of kurtosis of PEG function is,

$$\tau_4 = \frac{L_4}{L_2} = \frac{\frac{20\alpha\beta}{1+4\beta} - \frac{30\alpha\beta}{1+3\beta} + \frac{11\alpha\beta+10\alpha\beta^2}{1+3\beta+2\beta^2} + \frac{\alpha(P-1)(P^3-15P^2+30P+6((P-5)P+5)log(1-P))}{6P^4}}{\frac{\alpha\beta}{1+2\beta^2+3\beta} + \frac{(P-1)(P+log(1-P))}{\lambda P^2}} \tag{27}$$

6. Order Statistics

There are several topics in reliability of analysis in which order statistics appear quite naturally. If $X_{r:n}$ is the r^{th} order statistic in a random sample of size n, then the density function of $X_{r:n}$ can be written as,

$$f_r(x) = \frac{1}{B(r,n-r+1)} f(x)(F(x))^{r-1}(1-F(x))^{n-r} \tag{28}$$

From (10) the equation will be,

$$f_r(x) = \frac{1}{B(r,n-r+1)} \frac{\beta\lambda(1-F(x)-PF(x)+PF(x)^2)(F(x))^{r-1}(1-F(x))^{n-r}}{\beta-\beta P+\lambda\alpha F(x)^{\frac{1}{\beta}-1}(1-F(x)-PF(x)+PF(x)^2)}$$

Hence,

$$E(X_{r:n}) = \frac{1}{B(r,n-r+1)} \int_0^\infty x \frac{\beta\lambda(1-F(x)-PF(x)+PF(x)^2)(F(x))^{r-1}(1-F(x))^{n-r}}{\beta-\beta P+\lambda\alpha F(x)^{\frac{1}{\beta}-1}(1-F(x)-PF(x)+PF(x)^2)} dx$$

In quantile terms, it can be written as,

$$E(X_{r:n}) = \frac{1}{B(r, n-r+1)} \int_0^1 Q(u) \frac{\beta\lambda(1-u-Pu+Pu^2)u^{r-1}(1-u)^{n-r}}{\beta - \beta P + \lambda\alpha u^{\frac{1}{\beta}-1}(1-u-Pu+Pu^2)} dx$$

For the class of distributions (8), the first - order statistic $X_{1:n}$ has the quantile function

$$\begin{aligned} Q_{(1)}(u) &= Q(1-(1-u)^{\frac{1}{n}}) \\ &= \alpha\left(1-(1-u)^{\frac{1}{n}}\right)^{\frac{1}{\beta}} + \frac{1}{\lambda}\log\left(P+(1-P)(1-u)^{\frac{-1}{n}}\right) \end{aligned} \tag{29}$$

and the n^{th} order statistic $X_{n:n}$ has the quantile function

$$\begin{aligned} Q_{(n)}(u) &= Q(u^{\frac{1}{n}}) \\ &= \alpha u^{\frac{1}{\beta n}} + \frac{1}{\lambda}\log\left(\frac{1-Pu^{\frac{1}{n}}}{1-u^{\frac{1}{n}}}\right) \end{aligned} \tag{30}$$

Order statistics have more applications in quantile based reliability analysis as compared to distribution function based reliability analysis.

7. Reliability Properties

Reliabilty properties have an important role in real life situations. Some most relevent quantile based functions used in reliabilty analysis are hazard quantile function, mean residual quantile function, etc.

7.1. Hazard Quantile Function

One of the basic concepts employed for modeling and analysis of lifetime data is the hazard rate. In a quantile setup, Nair and Sankaran (2009) defined the hazard quantile function, which is equivalent to the hazard rate. The hazard quantile function H(u) is defined as

$$H(u) = h(Q(u)) = [(1-u)q(u)]^{-1} \tag{31}$$

Thus, H(u) can be interpreted as the conditional probability of failure of a unit in the next small interval of time given the survival of the unit until 100(1-α)% point of the distribution. Note that H(u) uniquely determines the distribution using the identity,

$$Q(u) = \int_0^u \frac{dp}{(1-p)H(p)} \tag{32}$$

Since the PEG model is the sum of quantile functions of power and exponential geometric quantile functions, (4) and (32) give

$$\frac{1}{H(u)} = \frac{1}{H_1(u)} + \frac{1}{H_2(u)} \tag{33}$$

where H(u) is the hazard quantile function of the PEG model, $H_1(u)$ is the hazard quantile function of Power distribution and $H_2(u)$ is the hazard quantile function of exponential geometric distribution. From (33), the PEG model has hazard quantile function proportional to the harmonic average of the hazard quantile functions of Power and exponential geometric quantile functions. For the class of distributions (8), we have

$$H(u) = \frac{\beta\lambda(1-Pu)}{\alpha\lambda(1-u)(1-Pu)u^{\frac{1}{\beta}-1} + \beta(1-P)} \tag{34}$$

7.1.1 Behavior of Hazard Quantile Function

The shape of the hazard quantile function can explain the behavior of hazard quantile function. It express increasing hazard rate (IHR), decreasing hazard rate (DHR), bathtub shape (BT), upside down bathtub shape (UBT) and constant rate at different values of parameters.

The different shapes of hazard quantile function for various values of parameters are summarized in Table 1 and plots given in Figure (4).

Table 1: *Behavior of hazard quantile function for different regions of parameters.*

No.	Parameter region	Shape of hazard quantile function
1	$\alpha > 1, \beta > 1, P = 0, \lambda > 0$	IHR
2	$\alpha = 1, 0 < \beta < 1, 0 < P < 1, \lambda > 0$	DHR
3	$\alpha = 1, \beta = 1, 0 < P < 1, \lambda > 0$	UBT
4	$\alpha > 1, 0 < \beta < 1, P = 1, \lambda > 0$	BT
5	$\alpha = 0, \beta > 1, P = 0, \lambda > 0$	Constant
6	$\alpha > 1, \beta > 1, 0 < P < 1, \lambda > 0$	UBT
7	$\alpha = 1, \beta > 1, P = 0, \lambda > 0$	IHR
8	$0 < \alpha < 1, 0 < \beta < 1, 0 < P < 1, \lambda > 0$	DHR
9	$\alpha > 1, \beta > 1, P = 1, \lambda > 0$	IHR

7.2. Mean Residual Quantile function

Mean residual function is a well - known measure that has been widely used for modeling lifetime data in reliability and survival analysis. For a nonnegative random variable X, the mean residual life function is defined as,

$$m(x) = \frac{1}{1 - F(x)} \int_x^\infty (1 - F(t))dt \tag{35}$$

In quantile based reliability analysis, the mean residual life function is known as mean residual quantile function, which is the quantile version of the mean residual function (35), defined by Nair and Sankaran (2009), has the expression,

$$M(u) = \frac{1}{1 - u} \int_u^1 (Q(p) - Q(u))dp \tag{36}$$

For the PEG model, M(u) has the form,

$$\begin{aligned} M(u) = \frac{1}{1-u} \frac{\alpha\beta(1 - u^{\frac{1}{\beta}+1})}{1+\beta} + \frac{1-P}{P(1-u)}(1 - log(1-P)) - \frac{1-Pu}{P(1-u)}(1 - log(1-Pu)) \\ - (1 - log(1-u)) - \left(\alpha u^{\frac{1}{\beta}} + \frac{1}{\lambda} log(\frac{1-Pu}{1-u})\right) \end{aligned} \tag{37}$$

It is well known that increasing (decreasing) failure rate implies decreasing (increasing) mean residual life (see Lai and Xie 2006). The aging behavior of PEG model based on mean residual quantile function can be defined from Table (2)

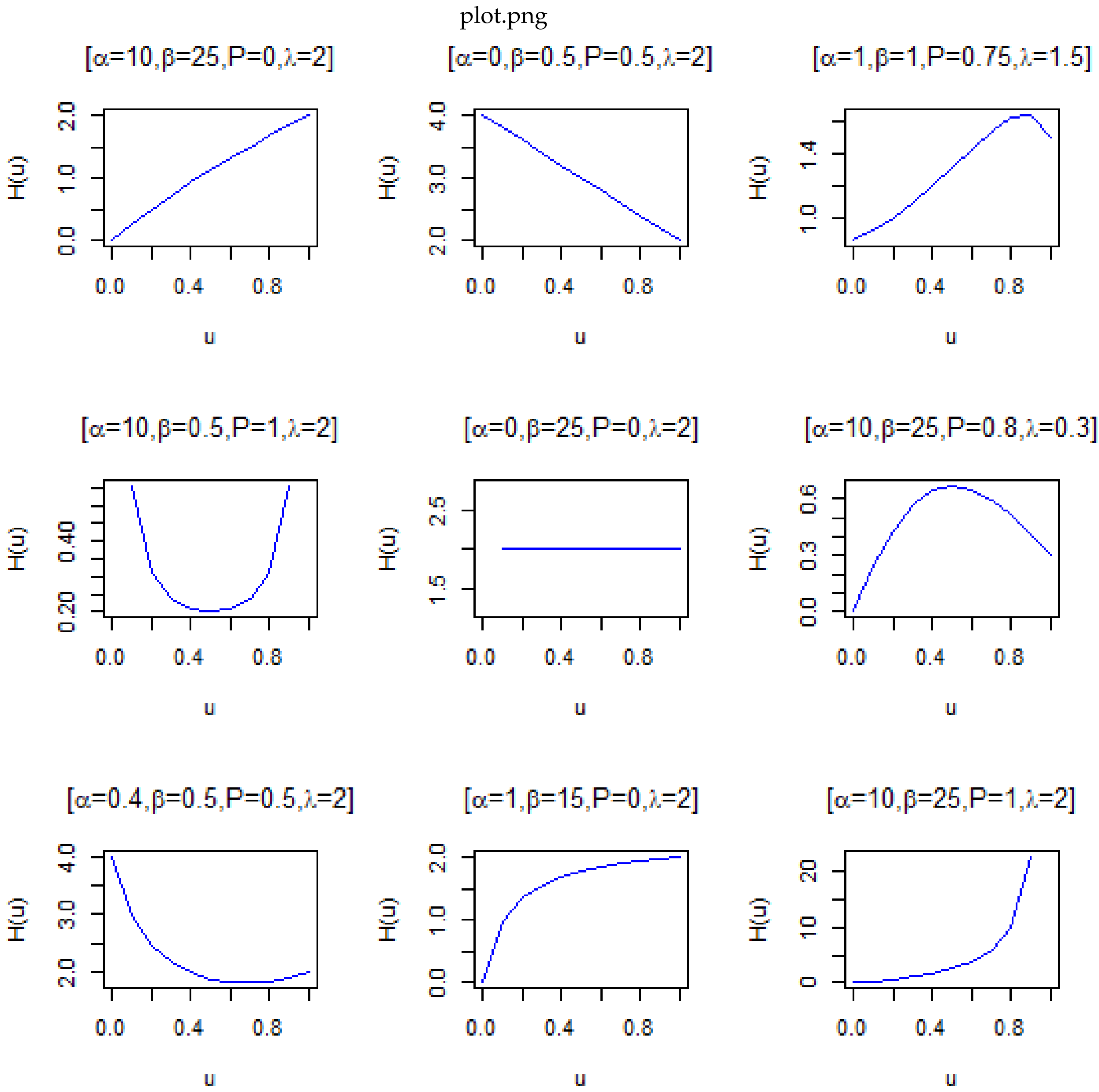

Figure 4: *Behavior of Hazard quantile function.*

7.3. Percentile Residual Quantile Function

Lillo (2005) found that Q(u) can be uniquely determined from the knowledge of $P_k(u)$, where $P_k(u)$ is the kth percentile residual quantile function. We have,

$$\begin{aligned} P_k(u) &= Q(1-(1-k)(1-u)) - Q(u) \\ &= \alpha(1-(1-k)(1-u))^{\frac{1}{\beta}} + \frac{1}{\lambda}log\left(\frac{1-P(1-(1-k)(1-u))}{1-(1-(1-k)(1-u))}\right) - \alpha u^{\frac{1}{\beta}} + \frac{1}{\lambda}log\left(\frac{1-Pu}{1-u}\right) \end{aligned} \tag{38}$$

7.4. Reversed Hazard Quantile Function

Reversed hazard quantile function is,

$$\begin{aligned}\Lambda &= (uq(u))^{-1} \\ &= \frac{\beta\lambda(1-u-Pu-Pu^2)}{u\beta(1-P)+\lambda\alpha u^{\frac{1}{\beta}}(1-u-Pu-Pu^2)}\end{aligned} \tag{39}$$

7.5. Reversed Mean Residual Quantile Function

Reversed mean residual quantile function is represented as R(u). k^{th} reversed mean residual quantile function can be obtained by using the given formula.

$$\begin{aligned}R(u) &= \frac{1}{u}\int_0^u (Q(u)-Q(k))dk \\ &= \alpha u^{\frac{1}{\beta}} + \frac{1}{\lambda}log\left(\frac{1-Pu}{1-u}\right) - \frac{\alpha\beta u^{\frac{1}{\beta}}}{1+\beta} - \frac{1}{\lambda u}\left(\frac{1-Pu}{P}(1-log(1-Pu)) - (1-u)(1-log(1-u))\right)\end{aligned} \tag{40}$$

7.6. Total Time on Test Transform (TTT)

Total time on test transform is represented as T(u), and it is calculated by using

$$T(u) = \int_0^u (1-k)q(k)dk$$

Also there is a relationship between total time on test transform and reversed mean residual quantile function (sankaran (2009)) and it is given by

$$T(u) = Q(u) - uR(u)$$

Using this relation, we can obtain the TTT of PEG model(8) and is given by

$$\begin{aligned}T(u) = (1-u)\left(\alpha u^{\frac{1}{\beta}} + \frac{1}{\lambda}log\left(\frac{1-Pu}{1-u}\right)\right) + \frac{\alpha u^{\frac{1}{\beta}+1}}{1+\beta} \\ + \frac{1}{\lambda}\left((1-Pu)(1-log(1-Pu)) - (1-u)(1-log(1-u))\right)\end{aligned} \tag{41}$$

8. Simulation Study

A simulation study is conducted to examine the performance of PEG quantile function for different sizes n=25, 50, 100, 500 using R package. Here simulate 1000 samples for the parameter values α=5, β=2, λ=3, P=0.87 and for α=0.8, β=0.05, λ=2, P=0.91, the maximum likelihood estimates for α, β, λ and P were determined for each sample, allowing the calculus of mean estimates.
We also evaluate the absolute bias and mean square erro (MSE) defined by,

$$AbsoluteBias = \frac{1}{N}|\sum_{i=1}^{N}(\hat{\epsilon}-\epsilon)|$$

and

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}-\epsilon)^2$$

From the table. we can see that, as the sample size increases, the mean square error decreases for all selected parameter values. Also the bias caused by the estimates decreases as the sample size increases. Thus the estimates tends to the true parameter values with increasing sample size.

Table 2: *Simulation study*

Sample Size	Parameter	Mean	Bias	MSE	Mean	Bias	MES
	α	0.68112	4.31887	22.05722	16.77764	15.97764	6904.116
25	β	0.00028	1.99971	3.99884	0.00779	0.04220	0.00278
	λ	0.24744	2.75255	8.23978	-0.97483	2.97483	40.40615
	P	0.11449	0.75550	0.67065	-2.04215	2.95215	61.96811
	α	0.85121	4.14879	21.05278	2.03723	1.23723	192.11053
50	β	0.20056	1.79943	3.59774	-1.01769	0.06769	0.02431
	λ	0.46151	2.53848	7.57733	-0.29682	2.29682	14.10438
	P	0.13915	0.73084	0.63639	-0.01065	0.9206	1.45400
	α	1.05596	3.94404	20.04939	0.01600	0.78399	0.62799
100	β	0.24022	1.75977	3.51908	0.00054	0.04972	0.00241
	λ	0.57022	2.42977	7.29000	0.03955	1.96003	3.92267
	P	0.18156	0.68874	0.60625	0.01768	0.89245	0.81158
	α	5.00104	0.00104	0.00055	0.28842	0.51200	0.40922
500	β	1.99600	0.00399	0.00797	0.01826	0.03211	0.00165
	λ	3.00001	$3.77e^{-6}$	$7.12e^{-9}$	0.72364	1.28413	2.56012
	P	0.87020	0.00020	$2.16e^{-5}$	0.32690	0.58245	0.52998

9. Data Analysis

This section explains the application of newly proposed quantile function in a real data set. There are many methods to estimate the unknown parameters of the quantile function. Method of maximum likelihood, method of L moments, method of minimum absolute deviation are some of the main mathods to estimate the parameters. In this work, method of maximum likelihood estimation procedure is used to estimate the parameters. Here the PEG function is applied to real data set reported in Zimmer et al. (1998). The data set consist of first failure times of small 20 electric carts used for internal transportation and delivery in a manufacturing company.The estimates so obtained are given by,

$$\hat{\alpha} = 11.2417, \hat{\beta} = 0.0056, \hat{\lambda} = 2.5998, \hat{P} = 0.7101 \tag{42}$$

The model adequacy is checked by using chi-squared goodness of fit test. The test result gives the p-value 0.2358. The significance level is 0.05, hence the test result indicates the adequacy of the PEG model to the data.

10. Summary and Conclusion

In this project work, a new quantile function is introduced, that is the sum of quantile functions of the power and exponential geometric (PEG) quantile functions. And also discovered that some well-known distributions are members of the PEG quantile function. Then plot different shapes of the density model for different values of parameters, also find the distributional characteristics of the model such as median, inter quartile range, skewness and kurtosis. Then derived first four L-moments and order statistics of the PEG quantile function. Important reliability properties are studied such as hazard quantile function, mean residual quantile function, etc. The different shapes of hazard quantile function for different parameter regions are plotted. Simulation study is conducted and it results that the bias and MSE are decreases as sample size increases. Finally the PEG model is applied to a real data set, and parameters are estimated by using maximum likelihood estimation procedure and model adequacy is checked by chi-square goodness of fit test. All these are done by using R software.

Several properties and extensions are possible in this PEG quantile function they are not considerd in this work, such as stochastic ordering, parameter estimation by L- moment, etc. since the parameters estimated by L moment method will be more efficient.

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