THE NEW LENGTH BIASED QUASI LINDLEY DISTRIBUTION AND ITS APPLICATIONS

N. W. Andure (Yawale)[1] and R. B. Ade[2]

[1] Department of Statistics,Government Vidarbha Institute of Science and Humanities, Amravati, Maharashtra, India, neetayawale@gmail.com

[2]Department of Statistics,Government Vidarbha Institute of Science and Humanities, Amravati, Maharashtra, India, rajeshwarb.sc@gmail.com

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Abstract

In this paper, Length biased Quasi Lindley (LBQL) distribution is proposed. The different properties of the proposed distribution are derived and discussed. The parameters of the proposed distribution are estimated by using method of maximum likelihood estimation and also the Fisher's Information matrix is obtained. The performance of the proposed distribution is studied using real-life data sets.

Keywords: Length Biased Distribution, Quasi Lindley Distribution, Reliability Analysis, Maximum Likelihood Estimation, Likelihood Ratio test.

I. Introduction

The Quasi Lindley (QL) distribution was introduced by Shanker and Mishra (2013). The QL distribution has two parameters α and θ. The Quasi-Lindley distribution reduces to one parameter Lindley distribution if $\alpha=\theta$. If $\alpha=0$, it reduces to the gamma distribution with parameter (2, θ). The probability density function of QL distribution is a mixture of exponential (θ) and gamma (2, θ). The Probability density function of Quasi Lindley distribution (QLD) with parameters α and θ is given by

$$f(x;\alpha,\theta)=\frac{\theta}{1+\alpha}(\alpha+\theta x)e^{-\theta x} \qquad ;x>0,\theta>0,\alpha>-1 \qquad (1)$$

and the cumulative distribution function of the two parameter Quasi Lindley distribution is given by

$$F(x;\alpha,\theta)=1-\left(\frac{1+\alpha+\theta x}{1+\alpha}\right)e^{-\theta x} \quad ;x>0,\theta>0,\alpha>-1 \qquad (2)$$

II. Length Biased Quasi Lindley Distribution

Length biased distribution is a particular case of weighted distributions that were first introduced by Fisher (1934) to model the ascertainment bias. These weighted distributions were later developed by C R Rao (1965) in a unifying manner. Weighted distributions arise when the observations generated from a stochastic process are not given equal chances of being recorded and moderately, they are recorded in accordance to some weight function. When the weight function depends only on the length of units of interest, the resulting distribution is called as length biased. Length biased concept was firstly given by Cox (1969) and Zelen (1974). The study of weighted distributions helps us to deal with model description and data interpretation problems. In the study of distribution theory, weighted distributions are useful because it provides a new understanding of existing standard probability distributions and also it provides methods for extending existing standard probability distributions for modelling lifetime data due to introduction of additional parameter in the model which creates flexibility in their nature. Much work has been done to characterize the relations between original distributions and their length biased versions.

various researchers have reviewed and studied different weighted distribution and found its applications in different fields such as reliability, biomedicine, ecology, and branching processes [refer Lappi et al. (1987), Mir et al. (2013), Mudassir et al. (2015), Shenbagaraja et al. (2019)].

Definition: The non-negative random variable X is said to have weighted distribution, if the probability density function of weighted random variable X_w is given by

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))} \quad , \quad x > 0$$

Where $w(x)$ be a non - negative weight function and

$$E\big(w(x)\big) = \int w(x)f(x)dx < \infty$$

For different weighted models, different choices of the weight function can be done. When $w(x) = x^c$,the resulting distribution is termed as weighted distribution. In this Paper, the Length biased version of Quasi Lindley distribution is studied, here choice of c = 1 is done as a weight, in order to get the Length biased Quasi Lindley distribution and the probability density function of Length biased Quasi Lindley distribution (LBQLD) is given by

$$f_L(x;\alpha,\theta) = \frac{xf(x,\alpha,\theta)}{E(x)} \tag{3}$$

Where $E(x) = \int_0^\infty x\, f(x;\alpha,\theta)dx$

$$E(x) = \frac{(\alpha+2)}{\theta(\alpha+1)} \tag{4}$$

After substituting from equation (1) and (4) in equation (3), the probability density function of Length biased Quasi Lindley distribution is obtained as

$$f_L(x;\alpha,\theta) = \frac{x\,\theta^2}{(\alpha+2)}(\alpha+\theta x)e^{-\theta x} \qquad ; x > 0, \theta > 0, \alpha > -2 \tag{5}$$

and the cumulative distribution function (cdf) of LBQL distribution is obtained as

$$F_L(x) = \int_0^x f_L(t;\alpha,\theta)dt$$

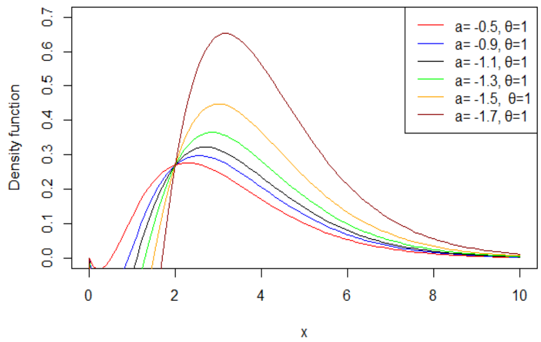
$$F_L(x) = \int_0^x \frac{t\,\theta^2}{(\alpha+2)}(\alpha+\theta t)e^{-\theta t}dt$$

$$F_L(x) = \frac{\theta^2}{(\alpha+2)}\int_0^x t(\alpha+\theta t)e^{-\theta t}dt$$

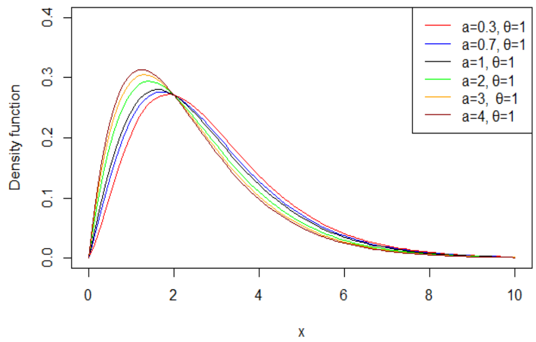
after simplification, the cumulative distribution function of Length biased Quasi Lindley distribution is

$$F_L(x) = \frac{\alpha\gamma(2,\theta x)+\gamma(3,\theta x)}{(\alpha+2)} \quad ; x>0, \theta>0, \alpha>-2 \tag{6}$$

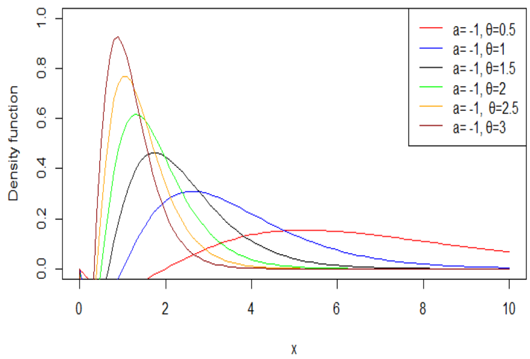
The graph of the probability density function and cumulative distribution function of length biased quasi-Lindley distribution (LBQLD) for different values of parameters, are shown in Figure 1 and 2.



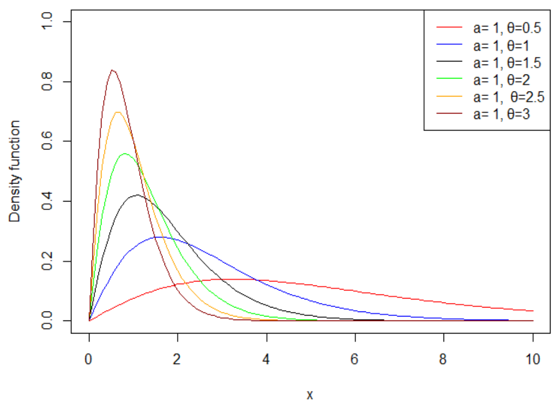
For $-2<\alpha<0$ and $\theta=1$



For $\alpha>0$ and $\theta=1$



For $-2<\alpha<0$ and $\theta>0$



For $\alpha>0$ and $\theta>0$

Figure 1: *Pdf plot of LBQLD for the different values of* α *and* θ.

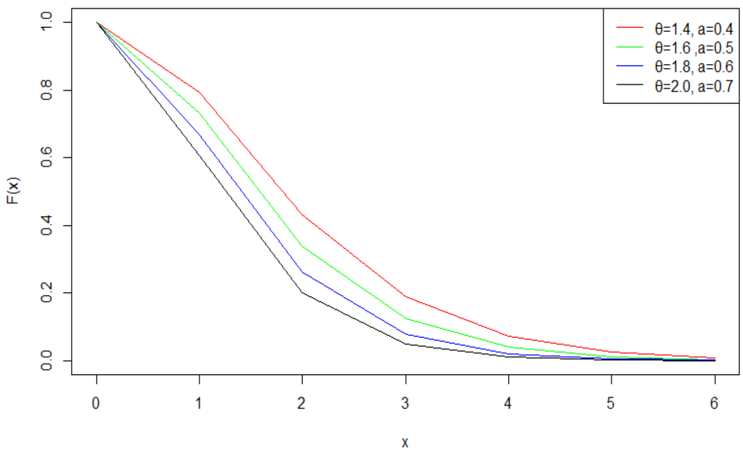


Figure 2: *cdf plot of LBQLD for the different values of α and θ.*

III. Reliability Analysis

In this section, the survival function, hazard rate, reverse hazard rate function and mill's ratio are discussed for the Length biased Quasi Lindley distribution.

The survival function is also known as reliability function and the Survival function of Length biased Quasi Lindley distribution is defined as

$$S(x) = 1 - F_L(x)$$

$$S(x) = 1 - \left(\frac{\alpha\gamma(2, \theta x) + \gamma(3, \theta x)}{(\alpha + 2)}\right)$$

The hazard function is also known as hazard rate or instantaneous failure rate or force of mortality and the hazard function of Length biased Quasi Lindley distribution is given by

$$h(x) = \frac{f_L(x; \alpha, \theta)}{S(x)}$$

$$h(x) = \frac{x\theta^2(\alpha + \theta x)e^{-\theta x}}{(\alpha + 2) - \left(\alpha\gamma(2, \theta x) + \gamma(3, \theta x)\right)}$$

where $(\alpha + 2) - \left(\alpha\gamma(2, \theta x) + \gamma(3, \theta x)\right) > 0$

The reverse hazard function of Length biased Quasi Lindley distribution is given by

$$h_r(x) = \frac{f_L(x; \alpha, \theta)}{F_L(x)}$$

$$h_r(x) = \frac{x\theta^2(\alpha + \theta x)e^{-\theta x}}{\left(\alpha\gamma(2, \theta x) + \gamma(3, \theta x)\right)}$$

The Mills ratio of Length biased Quasi Lindley distribution is given by,

$$Mills\ ratio = \frac{1}{h_r(x)}$$

$$Mills\ ratio = \frac{\left(\alpha\gamma(2,\theta x)+\gamma(3,\theta x)\right)}{x\theta^2(\alpha+\theta x)e^{-\theta x}}$$

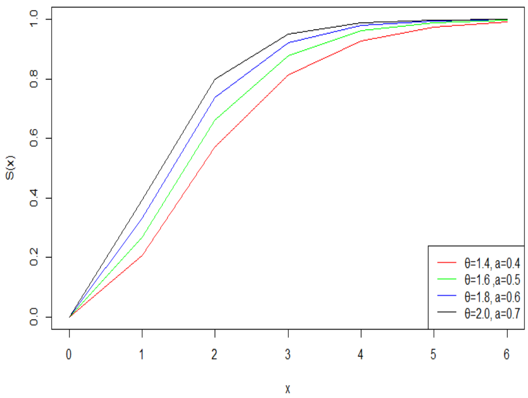


Fig.3: *Graph of survival function of LBQLD.*

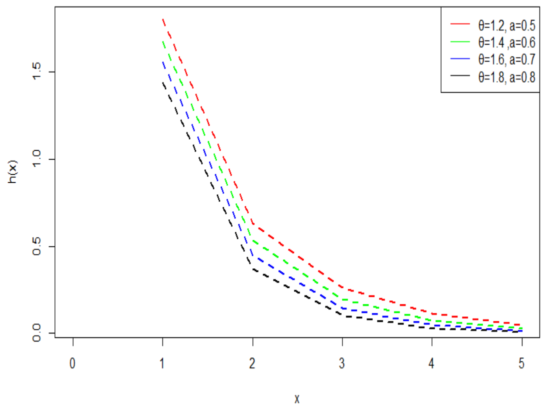


Fig.4: *Graph of Hazard function of LBQLD.*

Figure (4) shows the behavior of hazard function. For different choices of α and θ it shows decreasing failure rate.

IV. Statistical Properties

In this section, the statistical properties of Length biased Quasi Lindley distribution are discussed.

I. Moments

Let X denotes the random variable of LBQL distribution with parameters θ and α, then the rth order moment of LBQL distribution is defined as

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_L(x;\alpha,\theta)dx$$

$$= \int_0^\infty x^{r+1}\frac{\theta^2}{(\alpha+2)}(\alpha+\theta x)e^{-\theta x}dx$$

$$= \frac{\theta^2}{(\alpha+2)}\int_0^\infty x^{r+1}(\alpha+\theta x)e^{-\theta x}dx$$

$$= \frac{\theta^2}{(\alpha+2)}\left(\alpha\int_0^\infty x^{r+2-1}e^{-\theta x}dx + \theta\int_0^\infty x^{r+3-1}e^{-\theta x}dx\right)$$

after simplification,

$$E(X^r) = \mu_r' = \frac{\theta^2}{(\alpha+2)}\left(\frac{\alpha\Gamma(r+2)+\Gamma(r+3)}{\theta^{r+2}}\right) \quad (7)$$

putting r = 1 in equation (7), the mean of LBQL distribution is given by

$$\mu_1' = E(X) = \frac{2(\alpha+3)}{\theta(\alpha+2)}$$

After putting r = 2,3 and 4 in equation (7), the second, third and fourth raw moments of Length biased Quasi Lindley distribution are obtained as,

$$\mu_2' = E(X^2) = \frac{6(\alpha+4)}{\theta^2(\alpha+2)}$$

$$\mu_3' = E(X^3) = \frac{24(\alpha+5)}{\theta^3(\alpha+2)}$$

$$\mu_4' = E(X^4) = \frac{120(\alpha+6)}{\theta^4(\alpha+2)}$$

Therefore,

$$Variance = \sigma^2 = \frac{2(\alpha^2+6\alpha+6)}{\theta^2(\alpha+2)^2}$$

$$S.D. = \sigma = \frac{\sqrt{2(\alpha^2+6\alpha+6)}}{\theta(\alpha+2)}$$

$$C.V = \frac{\sigma}{\mu} = \frac{\sqrt{2(\alpha^2+6\alpha+6)}}{2(\alpha+3)}$$

$$C.D.(\gamma) = \frac{\sigma^2}{\mu} = \frac{(\alpha^2+6\alpha+6)}{\theta(\alpha+2)(\alpha+3)}$$

II. Moment generating Function and Characteristic Function LBQLD

Let X follows LBQL distribution, then the moment generating function (MGF) of X is,

$$M_X(t) = \int_0^\infty e^{tx} f_L(x;\alpha,\theta)dx$$

$$= \int_0^\infty \left(1 + (tx) + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots\right) f_L(x,\alpha,\theta)dx$$

$$= \int_0^\infty \sum_{r=0}^\infty \frac{(tx)^r}{r!} f_L(x,\alpha,\theta)dx$$

$$= \sum_{r=0}^\infty \frac{(t)^r}{r!} \int_0^\infty x^r f_L(x,\alpha,\theta)dx$$

$$M_X(t) = \sum_{r=0}^\infty \frac{(t)^r}{r!} E(x^r) \tag{8}$$

Substituting value of $E(x^r)$ from equation (7) in equation (8),

$$M_X(t)=\sum_{r=0}^{\infty}\frac{(t)^r}{r!}\left\{\frac{\theta^2}{(\alpha+2)}\left(\frac{\alpha\Gamma(r+2)+\Gamma(r+3)}{\theta^{r+2}}\right)\right\}$$

Similarly, the characteristic function of LBQL distribution can be obtained as,

$\varphi_X(t)=M_X(it)$

$$=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left\{\frac{\theta^2}{(\alpha+2)}\left(\frac{\alpha\Gamma(r+2)+\Gamma(r+3)}{\theta^{r+2}}\right)\right\}$$

III. Harmonic Mean

Let X follows LBQL distribution then, the harmonic mean is obtained as

$$H=\int_0^{\infty}\frac{1}{x}f_L(x,\alpha,\theta)dx$$

$$=\int_0^{\infty}\frac{\theta^2}{(\alpha+2)}(\alpha+\theta x)e^{-\theta x}dx$$

$$=\frac{\theta^2}{(\alpha+2)}\left[\alpha\int_0^{\infty}e^{-\theta x}\,dx+\theta\int_0^{\infty}x\,e^{-\theta x}dx\right]$$

after simplification,

$$H=\frac{\theta(\alpha+1)}{(\alpha+2)}$$

V. Order Statistics for LBQL Distribution

Order statistics have central role in statistical theory. It deals with the ordered data that is necessary to take for quality control, reliability, hydrological and extreme values analysis. Suppose $X_{(1)},X_{(2)},\dots\dots\dots,X_{(n)}$ be the jth order statistic and it is denoted by $X_{(j)}$.
The probability density function of the j^{th} order statistics $X_{(j)}$ for $1\le j\le n$ is

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j}f_1(x) \tag{9}$$

Substitute the value from equation (5) and (6) in equation (9), the probability density function of jth order statistics of LBQL distribution is given as

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!(n-j)!}\left[\frac{\alpha\gamma(2,\theta x)+\gamma(3,\theta x)}{(\alpha+2)}\right]^{j-1}\times\left[1-\frac{\alpha\gamma(2,\theta x)+\gamma(3,\theta x)}{(\alpha+2)}\right]^{n-j}\frac{x\,\theta^2}{(\alpha+2)}(\alpha+\theta x)e^{-\theta x} \tag{10}$$

For $j=1$ in equation (10), therefore the probability density function of first order statistics of LBQL distribution is obtained as

$$f_{X_{(1)}}(x)=n\left[1-\frac{\alpha\gamma(2,\theta x)+\gamma(3,\theta x)}{(\alpha+2)}\right]^{n-1}\times\frac{x\,\theta^2}{(\alpha+2)}(\alpha+\theta x)e^{-\theta x} \tag{11}$$

Put $j=n$ in equation (10), the probability density function of n^{th} order statistics of LBQL distribution is given by,

$$f_{X_{(n)}}(x) = n\left[\frac{\alpha\gamma(2,\theta x)+\gamma(3,\theta x)}{(\alpha+2)}\right]^{n-1} \times \frac{x\,\theta^2}{(\alpha+2)}(\alpha+\theta x)e^{-\theta x} \tag{12}$$

VI. Entropy

The concept of Entropies points out the diversity, uncertainty, or randomness of a system and the entropies have large application in several fields such as probability & statistics, physics, communication theory and economics. Entropy of a random variable X is a measure of variation of the uncertainty.

I. Renyi Entropy

The entropy termed as Renyi entropy is important in ecology and statistics as index of diversity. Renyi entropy is an extension of Shannon's entropy. Renyi (1961) give an expression of the entropy function is defined by

$$e(\delta) = \frac{1}{1-\delta}\log\left(\int_0^\infty {f_L}^{\delta}\,(x;\alpha,\theta)dx\right)$$

where $\delta > 0$ and $\delta \neq 1$

$$e(\delta) = \frac{1}{1-\delta}\log\left(\int_0^\infty\left(\frac{\theta^2}{(\alpha+2)}x(\alpha\ +\theta x)e^{-\theta x}\right)^{\delta} dx\right)$$

$$e(\delta) = \frac{1}{1-\delta}\log\left(\left(\frac{\theta^2}{(\alpha+2)}\right)^{\delta}\int_0^\infty x^{\delta}\,(\alpha+\theta x)^{\delta}e^{-\theta\delta x}dx\right) \tag{13}$$

Using binomial expansion in equation (13),

$$e(\delta) = \frac{1}{1-\delta}\log\left\{\left(\frac{\theta^2}{(\alpha+2)}\right)^{\delta}\sum_{k=0}^{\infty}\binom{\delta}{k}(\alpha)^{\delta-k}\theta^k\int_0^\infty (x)^{\delta+k}\,e^{-\theta\delta x}dx\right\}$$

$$e(\delta) = \frac{1}{1-\delta}\log\left\{\left(\frac{\theta^2}{(\alpha+2)}\right)^{\delta}\sum_{k=0}^{\infty}\binom{\delta}{k}(\alpha)^{\delta-k}\theta^k\int_0^\infty (x)^{\delta+k+1-1}e^{-\theta\delta x}\,dx\right\}$$

$$e(\delta) = \frac{1}{1-\delta}\log\left\{\left(\frac{\theta^2}{(\alpha+2)}\right)^{\delta}\sum_{k=0}^{\infty}\binom{\delta}{k}(\alpha)^{\delta-k}\theta^k\frac{\Gamma(k+\delta+1)}{(\theta\delta)^{(k+\delta+1)}}\right\}$$

II. Tsallis Entropy

Tsallis entropy was introduced by Tsallis (1988) as a basis for generalizing the standard statistical mechanics. For a continuous random variable X, Tsallis entropy is defined as follows.

$$S_\lambda = \frac{1}{1-\lambda}\left(1 - \int_0^\infty f_L{}^\lambda(x;\alpha,\theta)dx\right)$$

$$S_\lambda = \frac{1}{1-\lambda}\left(1 - \int_0^\infty \left(\frac{\theta^2}{(\alpha+2)}x(\alpha+\theta x)e^{-\theta x}\right)^\lambda dx\right)$$

$$S_\lambda = \frac{1}{1-\lambda}\left(1 - \left(\frac{\theta^2}{(\alpha+2)}\right)^\lambda \int_0^\infty x^\lambda \left((\alpha+\theta x)e^{-\theta x}\right)^\lambda dx\right)$$

$$S_\lambda = \frac{1}{1-\lambda}\left(1 - \left(\frac{\theta^2}{(\alpha+2)}\right)^\lambda \int_0^\infty x^\lambda (\alpha+\theta x)^\lambda e^{-\lambda\theta x} dx\right) \tag{14}$$

Using binomial expansion in equation (14),

$$S_\lambda = \frac{1}{1-\lambda}\left\{1 - \left(\frac{\theta^2}{(\alpha+2)}\right)^\lambda \sum_{k=0}^{\infty}\binom{\lambda}{k}(\alpha)^{\lambda-k}\theta^k \int_0^\infty (x)^{\lambda+k}\, e^{-\lambda\theta x}dx\right\}$$

$$S_\lambda = \frac{1}{1-\lambda}\left\{1 - \left(\frac{\theta^2}{(\alpha+2)}\right)^\lambda \sum_{k=0}^{\infty}\binom{\lambda}{k}(\alpha)^{\lambda-k}\theta^k \int_0^\infty (x)^{\lambda+k+1-1} e^{-\lambda\theta x}\, dx\right\}$$

$$S_\lambda = \frac{1}{1-\lambda}\left\{1 - \left(\frac{\theta^2}{(\alpha+2)}\right)^\lambda \sum_{k=0}^{\infty}\binom{\lambda}{k}(\alpha)^{\lambda-k}\theta^k \frac{\Gamma(\lambda+k+1)}{(\theta\lambda)^{(\lambda+k+1)}}\right\}$$

VII. Likelihood Ratio Test

Let $x_1, x_2, x_3, \ldots\ldots, x_n$ be a random sample from the LBQL distribution. To test the hypothesis

$H_0: f(x) = f(x;\alpha,\theta)$ Against $H_1: f(x) = f_L(x;\alpha,\theta)$

In order to test whether the random sample of length n has been drawn from length biased Quasi Lindley distribution or not the following test statistics is used

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\frac{f_L(x;\alpha,\theta)}{f(x;\alpha,\theta)}$$

$$\Delta = \left(\frac{\theta(\alpha+1)}{(\alpha+2)}\right)^n \prod_{i=1}^{n} x_i$$

Reject the null hypothesis, if

$$\Delta = \left(\frac{\theta(\alpha+1)}{(\alpha+2)}\right)^n \prod_{i=1}^{n} x_i > k$$

$$\Delta = \prod_{i=1}^{n} x_i > k\left(\frac{(\alpha+2)}{\theta(\alpha+1)}\right)^n$$

$$\Delta^* = \prod_{i=1}^{n} x_i > k^* \quad \text{where } k^* = k\left(\frac{(\alpha+2)}{\theta(\alpha+1)}\right)^n > 0$$

For large sample size n, $2\log\Delta$ is distributed as chi-square distribution with 1 degree of freedom (df) and also p-value is obtained from the chi-square distribution. Thus, reject the null hypothesis, when the probability value is given by

$$P(\Delta^* > \beta^*)$$

Where $\beta^* = \prod_{i=1}^{n} x_i$ is less than specified level of significance and $\prod_{i=1}^{n} x_i$ is observed value of the statistics Δ^*.

VIII. Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves are given as

$$B(p) = \frac{1}{p\mu}\int_0^q x\, f_L(x;\alpha,\theta)dx$$

$$L(p) = pB(p) = \frac{1}{\mu}\int_0^q x\, f_L(x,\alpha,\theta)dx$$

Where $E(x) = \mu = \frac{2(\alpha+3)}{\theta(\alpha+2)}$ and $q = F^{-1}(p)$

$$B(p) = \frac{\theta(\alpha+2)}{p2(\alpha+3)}\int_0^q x^2 \frac{\theta^2}{(\alpha+2)}(\alpha+\theta x)e^{-\theta x}dx$$

$$B(p) = \frac{\theta^3}{p2(\alpha+3)}\int_0^q x^2\,(\alpha+\theta x)e^{-\theta x}dx$$

$$B(p) = \frac{\theta^3}{p2(\alpha+3)} \times \left(\alpha\int_0^q x^{3-1}e^{-\theta x}\,dx + \theta\int_0^q x^{4-1}e^{-\theta x}\,dx\right)$$

$$B(p) = \frac{\alpha\gamma(3,\theta q) + \gamma(4,\theta q)}{2(\alpha+3)p}$$

$$L(p) = pB(p) = \frac{\alpha\gamma(3,\theta q) + \gamma(4,\theta q)}{2(\alpha+3)}$$

IX. Maximum Likelihood Estimation

In this section, the maximum likelihood estimation of the parameters of Length biased Quasi Lindley (LBQL) distribution is discussed. Let $x_1, x_2, \dots, x_n$ be a random sample of size n from the LBQL distribution, then the corresponding likelihood function is given by

$$L(x;\alpha,\theta)=\prod_{i=1}^{n}\left(\frac{x_i\theta^2(\alpha+\theta x_i)e^{-\theta x_i}}{(\alpha+2)}\right)$$

$$L(x;\alpha,\theta)=\left(\frac{\theta^2}{(\alpha+2)}\right)^n\prod_{i=1}^{n}x_i\,(\alpha+\theta x_i)e^{-\theta\sum_{i=1}^{n}x_i}$$

Takin log and solving likelihood function is obtained as follows

$$\frac{\partial \log L}{\partial \alpha}=\frac{-n}{(\alpha+2)}+\sum_{i=1}^{n}\frac{1}{(\alpha+\theta x_i)}=0 \tag{15}$$

$$\frac{\partial \log L}{\partial \theta}=\frac{2n}{\theta}+\sum_{i=1}^{n}\frac{x_i}{\alpha+\theta x_i}-\sum_{i=1}^{n}x_i=0 \tag{16}$$

The MLE of the parameters cannot be obtain in close form. The exact solution of above equation for unknown parameters is not possible manually. So, we can solve above equations with the help of *R* Software using (optim function, nlminb (), nlm ()).

To obtain confidence interval we use the asymptotic normality tests. If as $\hat{\lambda}=(\hat{\alpha},\hat{\theta})$ denote the MLE of $\lambda=(\alpha,\theta)$, state the results as follows:

$$\sqrt{n}(\hat{\lambda}-\lambda)\rightarrow N(0,I^{-1}(\lambda))$$

Where $I(\lambda)$is Fisher's Information Matrix is

$$I(\lambda)=-\frac{1}{n}\begin{bmatrix}E\left(\frac{\partial^2\log l}{\partial^2\theta}\right) & E\left(\frac{\partial^2\log l}{\partial\theta\partial\alpha}\right)\\ E\left(\frac{\partial^2\log l}{\partial\alpha\partial\theta}\right) & E\left(\frac{\partial^2\log l}{\partial^2\alpha}\right)\end{bmatrix}$$

Where

$$\frac{\partial^2\log l}{\partial^2\alpha}=\frac{n}{(\alpha+2)^2}-\sum_{i=1}^{n}\frac{1}{(\alpha+\theta x_i)^2}$$

$$\frac{\partial^2\log l}{\partial^2\theta}=\frac{-2n}{\theta^2}-\sum_{i=1}^{n}\frac{{x_i}^2}{(\alpha+\theta x_i)^2}$$

$$\frac{\partial^2\log l}{\partial\theta\partial\alpha}=\frac{\partial^2 log\ l}{\partial\alpha\partial\theta}=-\sum_{i=1}^{n}\frac{x_i}{(\alpha+\theta x_i)^2}$$

Since λ being unknown, $I^{-1}(\lambda)$ is estimated by $I^{-1}(\hat{\lambda})$and this can be used to obtain asymptotic confidence intervals for α and θ.

X. Application

In this section, three real life data set are studied for the purpose of illustration to show the usefulness and flexibility of the LBQL distribution.

To compare the length biased Quasi Lindley (LBQL) distribution with QL, Power Lindley (PL), Exponential (Exp.) distributions, the criteria like Bayesian information criterion (BIC), Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (AICC), HQIC are used and parameters are estimated using ML method of estimation.

The real-life data sets are given as follows:

Data set I: The first real life data set represents the breaking stress of carbon fibres (in Gba) observed and reported by Nichols and Padgett (2006) and is executed below in table 1.

Table 1. *Data consists of breaking stress of carbon fibres (in Gba) observed by Nichols and Padgett (2006).*

Data set I								
3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47
3.11	3.56	4.42	2.41	3.19	3.22	1.69	3.28	3.09
1.87	3.15	4.90	1.57	2.67	2.93	3.22	3.39	2.81
4.20	3.33	2.55	3.31	3.31	1.25	4.38	1.84	0.39
3.68	2.48	0.85	1.61	2.79	4.70	2.03	1.89	2.88
2.82	2.05	3.65	3.75	2.43	2.95	2.97	3.39	2.96
2.35	2.55	2.59	2.03	1.61	2.12	3.15	1.08	2.56
2.85	1.80	2.53						

Data set 2: The second real life data set represent the fatigue life of some aluminum's coupons cut in specific manner (see, Birnbaum and Saunders, 1969). The dataset (after subtracting 65) is given below in table 2.

Table2. *The fatigue life of some aluminum's coupons cut in specific manner (Birnbaum and Saunders, 1969).*

Data set II								
5	25	31	32	34	35	38	39	39
40	42	43	43	43	44	44	47	47
48	49	49	49	51	54	55	55	55
56	56	56	58	59	59	59	59	59
63	63	64	64	65	65	65	66	66
66	66	67	67	67	68	69	69	69
69	71	71	72	73	73	73	74	74
76	76	77	77	77	77	77	77	79
79	80	81	83	83	84	86	86	87
90	91	92	92	92	92	93	93	94
97	98	98	99	101	101	103	105	109
139	147							

Data set 3:

This data set presented in Murthy et al. (2004) and used by some researchers. This data set present failure times for a particular windshield model including 85 observations that are classified as failure times of windshields.

Table3.*failure times for a particular windshield model including 85 observations that are classified as failure times of windshields.*

Data set III						
0.040	1.866	2.385	3.443	0.301	1.876	2.481
3.467	0.309	1.899	2.610	3.478	0.557	1.911
2.625	3.578	0.943	1.912	2.632	3.595	1.070
1.914	2.646	3.699	1.124	1.981	2.661	3.779
1.248	2.010	2.688	3.924	1.281	2.038	2.822
3.000	4.035	1.281	2.085	2.890	4.121	1.303
2.089	2.902	4.167	1.432	2.097	2.934	4.240
1.480	2.135	2.962	4.255	1.505	2.154	2.964
4.278	1.506	2.190	3.000	4.305	1.568	2.194
3.103	4.376	1.615	2.223	3.114	4.449	1.619
2.224	3.117	4.485	1.652	2.229	3.166	4.570
1.652	2.300	3.344	4.602	1.757	2.324	3.376
4.663						

R software is used for determining the estimation of unknown parameters and is also used for estimating the model comparison criterion values (AIC, BIC, AICC, HQIC) and -2logL. To compare the Length biased Quasi Lindley distribution with Quasi Lindley and Power Lindley, Exponential distributions, the criterion like AIC (Akaike information criterion), AICC (corrected Akaike information criterion), BIC (Bayesian information criterion) and HQIC (Hannen-Quinn information criterion) are used for comparison. The better distribution corresponds to lesser values of AIC, AICC, BIC, HQIC and -2logL.

Table 4. *Estimate and goodness of fit measures under considered distribution based on data set I.*

Distribution	M.L. E		- 2**log**L	AIC	AICC	BIC	HQIC
	$\hat{\alpha}$	$\hat{\theta}$					
LBQL Distribution	**-0.3883002**	**1.1745**	**199.7838**	**203.7838**	**203.9743**	**208.1631**	**205.5142**
QL Distribution	-0.3401128	0.9116	204.4596	208.4596	208.6501	212.8389	210.1901
Exponential Distribution	0.3900	2.36944	245.8762	249.8762	249.9397	258.2555	255.6067
PL Distribution	0.5781	1.1286	490.4955	494.4955	494.5590	498.8748	496.2260

Table 5. *Estimate and goodness of fit measures under considered distribution based on data set II.*

Distribution	M.L. E		- 2**log**L	AIC	AICC	BIC	HQIC
	$\hat{\alpha}$	$\hat{\theta}$					
LBQL Distribution	**-0.16555**	**0.04489**	**899.4582**	**903.4582**	**903.499**	**908.6884**	**905.5756**
QL Distribution	-0.14116	0.03144	982.2110	986.2110	986.2518	991.4412	988.3284
Exponential Distribution	5.000	63.83168	1041.562	1045.562	1045.603	1050.792	1047.679

PL Distribution	0.1447	1.09255	1688.346	1692.346	1692.387	1697.577	1694.463

Table 6. *Estimate and goodness of fit measures under considered distribution based on data set III.*

Distribution	M.L. E		- 2**log**L	AIC	AICC	BIC	HQIC
	$\hat{\alpha}$	$\hat{\theta}$					
LBQL Distribution	**0.1458**	**1.14416**	**276.632**	**280.632**	**280.6808**	**285.5173**	**282.5970**
Exponential Distribution	0.040	2.5626	276.7906	280.7906	280.8394	285.6759	282.7556
QL Distribution	0.00362	0.77904	289.5131	293.5131	293.5619	298.3984	295.4781
PL Distribution	0.58946	1.18093	594.0202	598.0202	598.069	602.9055	599.9852

From table (4), (5) and (6) it can be seen that the value of the statistics -2logL, AIC, BIC, AICC and HQIC of the Length biased Quasi Lindley distribution are comparatively smaller than the other distributions on a real-life data set. Therefore, the result shows that the Length biased Quasi Lindley distribution provides a significantly better fit than other models. So, it can be chosen to model the life testing data.

XI. Conclusion

In this paper, the Length biased Quasi Lindley distribution is proposed as a new extension of Quasi Lindley distribution. The newly introduced distribution is generated by using the Length biased techniques and taking the Quasi-Lindley distribution as the base distribution. The various statistical properties of the proposed distribution have been derived and discussed. Supremacy of the new distribution in real life is established with demonstration of real-life data sets and it is found from the results of data sets that the Length biased Quasi Lindley distribution performs better than the Quasi Lindley, Power Lindley and Exponential distributions.

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