Inverse Weibull-Rayleigh Distribution Characterisation with Applications Related Cancer Data

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Abstract

The current study establishes a new three parameter Rayleigh distribution that is based on the inverse Weibull-G family and is an extension of the Rayleigh distribution. The formulation is known as the inverse Weibull-Rayleigh distribution (IWRD). The distinct structural properties of the formulated distribution including moments, moment generating function, order statistics, quantile function, and Renyi entropy have been discussed. In addition expressions for survival function, hazard rate function and reverse hazard rate function are obtained explicitly. The behaviour of probability density function (p.d.f) and cumulative distribution function (c.d.f) are illustrated through different graphs. The estimation of the formulated distribution parameters are performed by maximum likelihood estimation method. A simulation analysis has been carried out to evaluate and compare the effectiveness of estimators in terms of their bias, variance and mean square error (MSE). Eventually, the usefulness of the formulated distribution is illustrated by means of real data sets which are related distinct areas of science.

Keywords: *Inverse Weibull-G family, Rayleigh distribution, moments, Renyi entropy, simulation, maximum likelihood estimation.*

Mathematics classification: *60E05, 62FXX, 62F10, 62G05*

I. Introduction

There is a plethora of univariate distributions in the statistics literature. However, statisticians have found it difficult to find an effective distribution for analysing or modelling complicated real-life data sets. To resolve such challenges, new probability distributions must be formed or fundamental type must be modified. Over recent times, researchers have investigated a plethora of new methods and approaches, and by employing these approaches, generalization or extensions can be accomplished from baseline distributions. The main objective for these modifications is to enhance the accuracy or flexibility of distributions while assessing more complicated real-life data sets.

Waloddi Weibull, a Swedish mathematician, introduced the Weibull distribution in 1951. Because it may be used to analyse real life data with monotone failure rates, this distribution is considered versatile for data sets with bathtub shapes or unimodal. The Weibull distribution, on the other hand, may not necessarily give a best fit for data sets with a bathtub shape or failure rates that are unimodal.

Let X be a random variable follows the Weibull distribution with parameter β and θ. Then its probability density function (pdf) is defined as

$$\psi(x,\beta,\theta)=\beta\theta^{\beta}x^{\beta-1}e^{-\theta^{\beta}x^{\beta}} \quad ; x>0,\beta,\theta>0$$

The inverse of the Weibull distribution is obtained by applying the transformation $T=\frac{1}{X}$.

Thus the probability density function (pdf) of inverse Weibull distribution takes following form.

$$h(t,\beta,\theta)=\beta\theta^{\beta}t^{-\beta-1}e^{-\theta^{\beta}t^{-\beta}} \quad ; t>0,\beta,\theta>0 \tag{1}$$

The inverse Weibull distribution is a subclass of the generalised extreme value distribution, which was previously researched by B.V. Gnedenko (1941) and Frechet (1927). In this paper, we develop the inverse Weibull-Rayleigh distribution, which is an extension of the Rayleigh distribution. Rayleigh distributions have a broad array of applications in research to simulate real life data, including reliability analysis, engineering, communication theory, medical science, and applied statistics. Rayleigh distribution has been expanded by researchers to make it more comprehensive and efficient for assessing more diverse factual data, for instance, due to its immense variety of applications. Weibull-Rayleigh distribution by Faton Merovci [11], odd generalized exponential Rayleigh distribution by Albert Luguterah [2], Topp-Leone Rayleigh distribution by Fatoki olayode [12], new generalisation of Rayleigh distribution by A.A Bhat et al [8].The probability density function (pdf) of Rayleigh distribution with scale parameter α is defined by

$$g(y,\alpha)=\alpha y e^{-\frac{\alpha}{2}y^{2}} \quad ; y>0,\alpha>0 \tag{2}$$

The associated cumulative distribution function (cdf) is given by

$$G(y,\alpha)=1-e^{-\frac{\alpha}{2}y^{2}} \quad ; y>0,\alpha>0 \tag{3}$$

In recent past years researcher have focussed to explore new generators from continuous standard distributions. As a result, the obtained distribution enhances the effectiveness and flexibility of data modelling. Some generated families of distribution are as follows: beta-G family of distribution explored by Eugene et al [10], kumaraswamy-G family by Cordeiro et al [9], transformed-transformer(T-X) by Alzaatrh et al [1], Weibull-G by Bourguignon et al [5], Lindley-G by Frank Gomes-silva et al [12], Topp-Leone odd log-logistic family of distributions by Brito et al [7], inverse Weibull-G by Amal S. Hassan et al [3], among others.

II. T-X Transformation

T-X family of distributions defined by Alzaatreh et al [1] is given by

$$F(y)=\int_{0}^{W[G(y)]} r(t)dt \tag{4}$$

Where $r(t)$ be the probability density function of a random variable T and $W[G(y)]$ be a function of cumulative density function of random variable Y.

Suppose $G(y,\eta)$ denotes the baseline cumulative distribution function, which depends on parameter vector η. Now using T-X approach, the cumulative distribution function $F(y)$ of inverse Weibull generator (IWG) can be derived by replacing $r(t)$ in equation (4) with (1) and $W[G(y)]=\frac{G(y,\eta)}{\overline{G}(y,\eta)}$, where $\overline{G}(y,\eta)=1-G(y,\eta)$ which follows

$$F(y,\beta,\theta,\eta)=\int_0^{\frac{G(y,\eta)}{\overline{G}(y,\eta)}}\beta\theta^{\beta}t^{-\beta-1}e^{-\theta^{\beta}t^{-\beta}}dt$$

$$=e^{-\theta^{\beta}\left(\frac{G(y,\eta)}{\overline{G}(y,\eta)}\right)^{-\beta}}\ ; y>0,\beta,\theta>0 \tag{5}$$

The corresponding pdf of (5) becomes

$$f(y,\beta,\theta,\eta)=\beta\theta^{\beta}g(y,\eta)\frac{[G(y,\eta)]^{-\beta-1}}{[\overline{G}(y,\eta)]^{-\beta+1}}e^{-\theta^{\beta}\left(\frac{G(y,\eta)}{\overline{G}(y,n)}\right)^{-\beta}}\ ; y>0,\beta,\theta>0 \tag{6}$$

The survival $S(y)$ and hazard rate function $h(y)$ are respectively given by

$$S(y)=1-F(y,\beta,\theta,\eta)=1-e^{-\theta^{\beta}\left(\frac{G(y,\eta)}{\overline{G}(y,n)}\right)^{-\beta}}$$

$$h(y)=\frac{\beta\theta^{\beta}g(y,\eta)\frac{[G(y,\eta)]^{-\beta-1}}{[\overline{G}(y,\eta)]^{-\beta+1}}e^{-\theta^{\beta}\left(\frac{G(y,\eta)}{\overline{G}(y,n)}\right)^{-\beta}}}{1-e^{-\theta^{\beta}\left(\frac{G(y,\eta)}{\overline{G}(y,n)}\right)^{-\beta}}}$$

III. Useful Expansion

Applying Taylor series expansion to the exponential function of the pdf in equation (6) we have

$$e^{-\theta^{\beta}\left(\frac{G(y,\eta)}{\overline{G}(y,n)}\right)^{-\beta}}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\theta^{i\beta}}{i!}\left[\frac{G(y,\eta)}{\overline{G}(y,\eta)}\right]^{-\beta i} \tag{7}$$

Substitute equation (7) in (6), we have

$$f(y,\beta,\theta,\eta)=\beta g(y,\eta)\sum_{i=0}^{\infty}\frac{(-1)^{i}\theta^{-\beta(i+1)}}{i!}\frac{(G(y,\eta))^{-\beta(i+1)-1}}{(\overline{G}(y,\eta))^{-\beta(i+1)+1}} \tag{8}$$

Since $\beta>0$ and $|z|<1$, using generalised binomial theorem, we have

$$(1-z)^{\beta-1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}z^{j}$$

$$\left(\overline{G}(y,\eta)\right)^{\beta(i+1)-1}=\left(1-G(y,\eta)\right)^{\beta(i+1)-1}=\sum_{j=0}^{\infty}(-1)^j\binom{\beta(i+1)-1}{j}\left(G(y,\eta)\right)^j \tag{9}$$

Using equation (9) in equation (8), we have

$$f(y,\beta,\theta,\eta)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{i!}\beta\theta^{\beta(i+1)}g(y,\eta)\binom{\beta(i+1)-1}{j}\left(G(y,\eta)\right)^{j-\beta(i+1)-1}$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{i,j}\,g(y,\eta)\left(G(y,\eta)\right)^{j-\beta(i+1)-1} \tag{10}$$

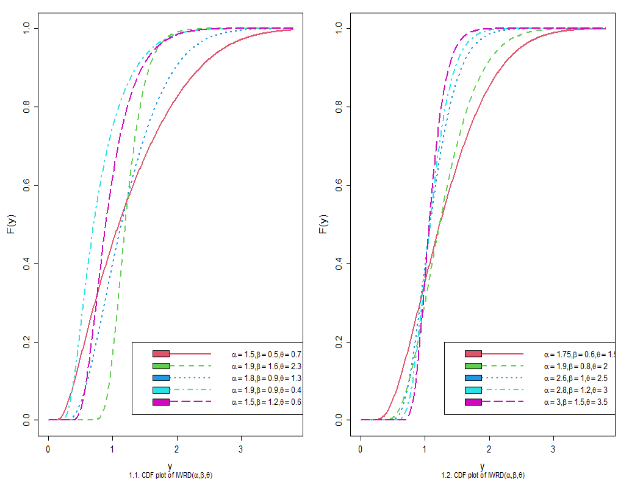
Where $\delta_{i,j}=\dfrac{(-1)^{i+j}\beta\theta^{\beta(i+1)}}{i!}\dbinom{\beta(i+1)-1}{j}$

The paper is framed as. In section 2, we derive the cumulative distribution function (cdf), probability density function (pdf). In section 3, we study the reliability measures, survival function, hazard rate function and reverse hazard rate function. In section 4, different statistical properties are studied including, moments, moment generating function, quantile function and random number generation. In section 5, Renyi entropy is discussed. In section 6, order statistics is expressed, in section 7, the estimation of parameters are performed by maximum likelihood estimation. In section 8, simulation study is performed. Finally in section 9 the efficiency of the established distribution is examined through data sets.

IV. The Inverse Weibull-Rayleigh Distribution

In this section we explore the inverse Weibull-Rayleigh distribution and studied its different statistical properties. Using equation (3) in equation (5), we obtain the cumulative distribution function (cdf) of the proposed distribution which follows

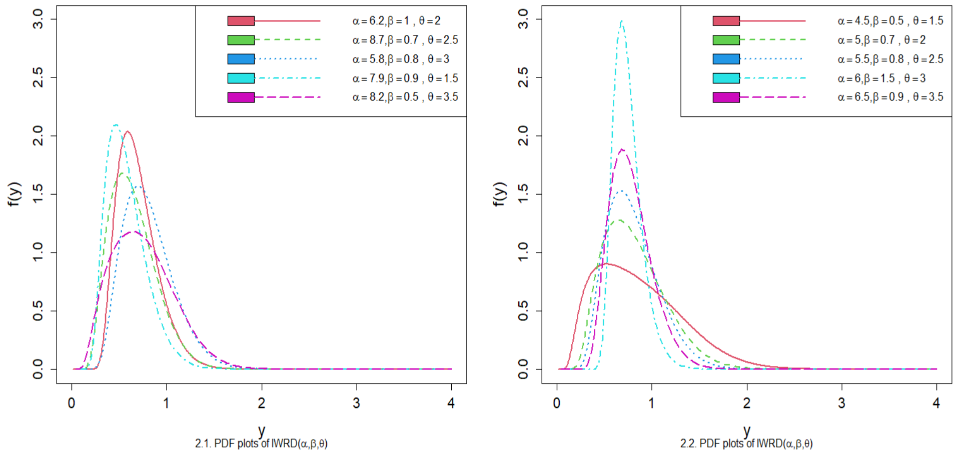
$$F(y,\alpha,\beta,\theta)=e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta}},\quad y>0,\alpha,\beta,\theta>0 \tag{11}$$



Figures (1.1) and (1.2) illustrates some of possible shapes of the cdf of IWRD for different values α,β and θ

The associated probability density function of inverse Weibull-Rayleigh distribution is given by

$$f(y,\alpha,\beta,\theta)=\alpha\beta\theta^{\beta} y e^{\frac{\alpha}{2}y^2}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta-1} e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta}}, y>0,\alpha,\beta,\theta>0 \tag{12}$$



2.1. PDF plots of IWRD(α,β,θ) 2.2. PDF plots of IWRD(α,β,θ)

Figures (2.1) and (2.2) illustrates some of possible shapes of the pdf of IWRD for different values α,β and θ

V. Reliability Measures

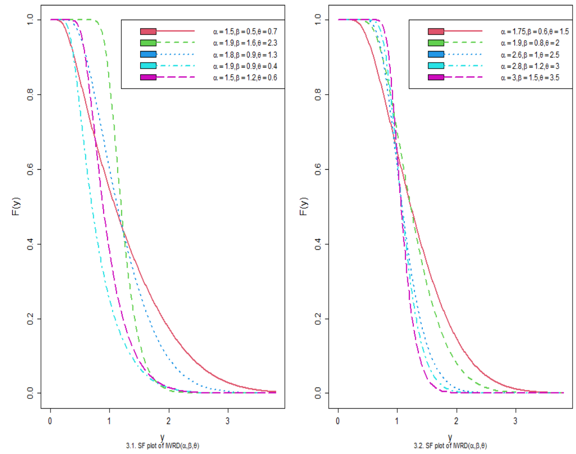
Suppose Y be a continuous random variable with cdf $F(y), y\geq 0$. Then its reliability function which is also called survival function is defined as

$$S(y)=p_r(Y>y)=\int_y^{\infty} f(y)dy=1-F(y)$$

The survival function of inverse Weibull-Rayleigh distribution is given as

$$S(y,\alpha,\beta,\theta)=1-F(y,\alpha,\beta,\theta)$$

$$=1-e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta}} \tag{13}$$



3.1. SF plot of IWRD(α,β,θ) 3.2. SF plot of IWRD(α,β,θ)

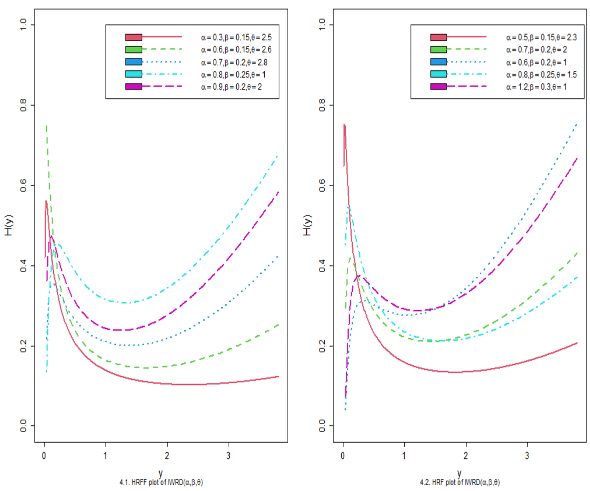
Figures (3.1) and (3.2) illustrates some of possible shapes of the survival function of IWRD for different values α, β and θ

The hazard rate function of inverse Weibull-Rayleigh distribution is given as

$$H(y,\alpha,\beta,\theta)=\frac{f(y,\alpha,\beta,\theta)}{S(y,\alpha,\beta,\theta)} \tag{14}$$

Substituting equations (12) and (13) in equation (14), we have

$$H(y,\alpha,\beta,\theta)=\frac{\alpha\beta\theta^{\beta} y e^{\frac{\alpha}{2}y^2}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta-1} e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta}}}{1-e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta}}}$$



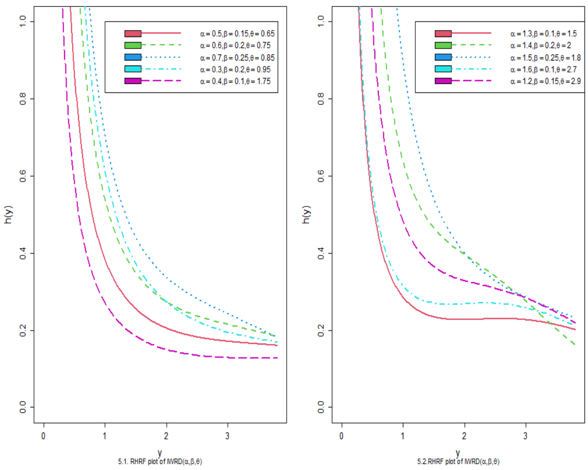
4.1. HRFF plot of IWRD(α,β,θ) 4.2. HRF plot of IWRD(α,β,θ)

Figures (4.1) and (4.2) illustrates some of possible shapes of the hazard rate function of IWRD for different values α, β and θ

Reverse hazard rate function of inverse Weibull-Rayleigh distribution is given as

$$h(y,\alpha,\beta,\theta)=\frac{f(y,\alpha,\beta,\theta)}{F(y,\alpha,\beta,\theta)}$$

$$=\alpha\beta\theta^{\beta}ye^{\frac{\alpha}{2}y^2}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta-1}$$



Figures (5.1) and (5.2) illustrates some of possible shapes of the reverse hazard rate function of IWRD for different values α,β and θ

VI. Structural properties of inverse Weibull-Rayleigh distribution

Theorem 4.1:- Suppose y denotes a random variable follows IWRD with p.d.f $f(y,\alpha,\beta,\theta)$. Then the r^{th} moment of inverse Weibull-Rayleigh distribution is given by

$$\mu_r' = \sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty}\delta_{i,j}(-1)^k\binom{j-\beta(i+1)-1}{k}\frac{2^{\frac{r}{2}}\alpha\Gamma\left(\frac{r}{2}+1\right)}{[\alpha(k+1)]^{\frac{r}{2}+1}}$$

Proof:- Let Y denotes a random variable follows inverse Weibull-Rayleigh distribution. Then r^{th} moment denoted by μ_r' is given as

$$\mu_r' = E(Y^r)=\int_0^{\infty} y^r f(y,\alpha,\beta,\theta)dy$$

Substituting equations (1) and (2) in equation (10), we get

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{i,j}\alpha\int_0^{\infty} y^{r+1} e^{-\frac{\alpha}{2}y^2}\left(1-e^{-\frac{\alpha}{2}y^2}\right)^{j-\beta(i+1)-1} dy$$

We know the formulae of generalized binomial expansion, which follows

$$(1-a)^{p-1} = \sum_{j=0}^{\infty}(-1)^j \binom{p-1}{j} a^j$$

Now applying the above formulae, we get

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{i,j}\alpha\int_0^{\infty} y^{r+1} e^{-\frac{\alpha}{2}y^2}\sum_{k=0}^{\infty}(-1)^k\binom{j-\beta(i+1)-1}{k} e^{-\frac{k\alpha}{2}y^2} dy$$

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\delta_{i,j}\alpha\binom{j-\beta(i+1)-1}{k}\int_0^{\infty} y^{r+1} e^{-(k+1)\frac{\alpha}{2}y^2} dy$$

Making substitution $\frac{\alpha}{2}(k+1)y^2 = z$ so that $ydy = \frac{1}{\alpha(k+1)}dz$

$$\mu_r' = \sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\delta_{i,j}\alpha\binom{j-\beta(i+1)-1}{k}\frac{2^{\frac{r}{2}}}{[\alpha(k+1)]^{\frac{r}{2}+1}}\int_0^{\infty} z^{\left(\frac{r}{2}+1\right)-1} e^{-z} dz$$

$$\mu_r' = \sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\delta_{i,j}\binom{j-\beta(i+1)-1}{k}\frac{2^{\frac{r}{2}}\alpha\Gamma\left(\frac{r}{2}+1\right)}{[\alpha(k+1)]^{\frac{r}{2}+1}}$$

Theorem 4.2:- Suppose y denotes a random variable follows IWRD with pdf $f(y,\alpha,\beta,\theta)$. Then the moment generating function of inverse Weibull-Rayleigh distribution is given by

$$M_Y(t) = \sum_{r=0}^{\infty}\sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(it)^r}{r!}\delta_{i,j}(-1)^k\binom{j-\beta(i+1)-1}{k}\frac{2^{\frac{r}{2}}\alpha\Gamma\left(\frac{r}{2}+1\right)}{[\alpha(k+1)]^{\frac{r}{2}+1}}$$

Proof:- Let Y be a random variable follows inverse Weibull-Rayleigh distribution. Then the moment generating function of the distribution denoted by $M_Y(t)$ is given

$$M_Y(t) = E\left(e^{ty}\right) = \int_0^{\infty} e^{ty} f(y,\alpha,\beta,\theta)dy$$

Using Taylor's series

$$= \int_0^\infty \left(1 + ty + \frac{(ty)^2}{2!} + \frac{(ty)^3}{3!} + \ldots \right) f(y, \alpha, \beta, \theta) dy$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\infty y^r f(y, \alpha, \beta, \theta) dy$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} E(Y^r)$$

$$M_Y(t) = \sum_{r=0}^{\infty} \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^r}{r!} \delta_{i,j} (-1)^k \binom{j - \beta(i+1) - 1}{k} \frac{2^{\frac{r}{2}} \alpha \Gamma\left(\frac{r}{2} + 1\right)}{[\alpha(k+1)]^{\frac{r}{2}+1}}$$

The characteristics function of the IWRD denoted as $\phi_Y(t)$ can be obtained by replacing $t = it, i = \sqrt{-1}$ is given by

$$\phi_Y(t) = \sum_{r=0}^{\infty} \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(it)^r}{r!} \delta_{i,j} (-1)^k \binom{j - \beta(i+1) - 1}{k} \frac{2^{\frac{r}{2}} \alpha \Gamma\left(\frac{r}{2} + 1\right)}{[\alpha(k+1)]^{\frac{r}{2}+1}}$$

VII. Quantile function of inverse Weibull-Rayleigh distribution

The quantile function of random variable Y, where $Y \sim IWRD(\alpha, \beta, \theta)$, can be obtained by inverting equation (11), we have

$$Q(u) = F^{-1}(u) = \left[\frac{2}{\alpha} \log\left\{ \left(\frac{-1}{\theta^\beta} \log u \right)^{\frac{-1}{\beta}} + 1 \right\} \right]^{\frac{1}{2}}$$

In particular, the median of the distribution can be obtained by setting $u = 0.5$

$$M = \left[\frac{2}{\alpha} \log\left\{ \left(\frac{-1}{\theta^\beta} \log(0.5) \right)^{\frac{-1}{\beta}} + 1 \right\} \right]^{\frac{1}{2}}$$

VIII. Random number generation of inverse Weibull-Rayleigh distribution

Suppose y denotes a random variable with cdf given in equation (11). The random number of inverse Weibull-Rayleigh distribution can be generated as

$$F(y) = u \Rightarrow y = F^{-1}(u)$$

So that

$$y=\left[\frac{2}{\alpha}\log\left\{\left(\frac{-1}{\theta^{\beta}}\log u\right)^{\frac{-1}{\beta}}+1\right\}\right]^{\frac{1}{2}}$$

Where u is the uniform random variable defined in an open interval $(0,1)$.

IX. Renyi entropy of inverse Weibull-Rayleigh distribution

If Y is a continuous random variable having probability density function $f(y,\alpha,\beta,\theta)$. Then Renyi entropy is defined as

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\int_0^{\infty} f^{\gamma}(y)dy\right\}, \text{ where } \gamma>0 \text{ and } \gamma\neq 1$$

Using equation (6), we have

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\int_0^{\infty}\left[\beta\theta^{\beta}g(y,\eta)\frac{(G(y,\eta))^{-(\beta+1)}}{(\overline{G}(y,\eta))^{(-\beta+1)}}e^{-\theta^{\beta}\left[\frac{G(y,\eta)}{\overline{G}(y,\eta)}\right]^{-\beta}}\right]^{\gamma}dy\right\}$$

$$=\frac{1}{1-\gamma}\log\left\{\int_0^{\infty}\beta^{\gamma}\theta^{\beta\gamma}(g(y,\eta))^{\gamma}\frac{(G(y,\eta))^{-(\beta+1)\gamma}}{(\overline{G}(y,\eta))^{(-\beta+1)\gamma}}e^{-\gamma\theta^{\beta}\left[\frac{G(y,\eta)}{\overline{G}(y,\eta)}\right]^{-\beta}}dy\right\} \quad (15)$$

Now using the power series expansion for exponential function, we have

$$e^{-\gamma\theta^{\beta}\left[\frac{G(y,\eta)}{\overline{G}(y,\eta)}\right]^{-\beta}}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\gamma^{i}\theta^{\beta i}}{i!}\left[\frac{G(y,\eta)}{\overline{G}(y,\eta)}\right]^{-\beta i} \quad (16)$$

Substituting equation (16) into (17), we obtain

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\sum_{i=0}^{\infty}\frac{(-1)^{i}\beta^{\gamma}\gamma^{i}\theta^{\beta(\gamma+i)}}{i!}\int_0^{\infty}(g(y,\eta))^{\gamma}(G(y,\eta))^{-(\beta+1)\gamma-\beta i}(1-G(y,\eta))^{(\beta-1)\gamma+\beta i}dy\right\}$$

$$=\frac{1}{1-\gamma}\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\beta^{\gamma}\gamma^{i}\theta^{\beta(\gamma+i)}}{i!}\binom{\beta(\gamma+i)-\gamma}{j}\int_0^{\infty}(g(y,\eta))^{\gamma}(G(y,\eta))^{j-\beta(\gamma+i)-\gamma}dy\right\}$$

Thus, the Renyi entropy for inverse Weibull-Rayleigh distribution, is given by

$$=\frac{1}{1-\gamma}\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\alpha^{\gamma}\beta^{\gamma}\gamma^{i}\theta^{\beta(\gamma+i)}}{i!}\binom{\beta(\gamma+i)-\gamma}{j}\int_0^{\infty}y^{\gamma}e^{-\frac{\alpha\gamma}{2}y^{2}}\left(1-e^{-\frac{\alpha}{2}y^{2}}\right)^{j-\beta(\gamma+i)-\gamma}dy\right\}$$

$$=\frac{1}{1-\gamma}\log\left\{\sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{k}\alpha^{\gamma}\delta_{i,j}\binom{j-\beta(\gamma+i)-\gamma}{k}\int_0^{\infty}y^{\gamma}e^{-\frac{\alpha}{2}(\gamma+k)y^{2}}dy\right\}$$

Where

$$\delta_{i,j}=\frac{(-1)^{i+j}\beta^{\gamma}\gamma^{i}\theta^{\beta(\gamma+i)}}{i!}\binom{\beta(\gamma+i)-\gamma}{j}$$

Making the substitution $\frac{\alpha}{2}(\gamma+k)y^2=z$, we have

$$=\frac{1}{1-\gamma}\log\left\{\sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\alpha^{\gamma}\delta_{i,j}\binom{j-\beta(\gamma+i)-\gamma}{k}\frac{1}{2}\left(\frac{2}{\alpha(\gamma+k)}\right)^{\frac{\gamma+1}{2}}\int_0^{\infty}z^{\frac{\gamma+1}{2}-1}e^{-z}dz\right\}$$

After solving above integral, we obtain

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\alpha^{\gamma}\delta_{i,j}\binom{j-\beta(\gamma+i)-\gamma}{k}\frac{1}{2}\left(\frac{2}{\alpha(\gamma+k)}\right)^{\frac{\gamma+1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)\right\}$$

X. Order statistics of inverse Weibull-Rayleigh distribution

Let us suppose $Y_1,Y_2,...,Y_n$ be random samples of size n from IWRD distribution with pdf $f(y)$ and cdf $F(y)$. Then the probability density function of k^{th} order statistics is given as

$$f_{Y_{(k)}}(y)=\frac{n!}{(k-1)!(n-k)!}f(y)[F(y)]^{k-1}[1-F(y)]^{n-k} \tag{17}$$

Now using the equation (11) and (12) in (17). The probability of k^{th} order statistics of inverse Weibull-Rayleigh distribution is given as

$$f_{Y_{(k)}}(y)=\frac{n!}{(k-1)!(n-k)!}\alpha\beta\theta^{\beta}ye^{\frac{\alpha}{2}y^2}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta-1}e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta}}\left[e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta}}\right]^{k-1}\left[1-e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta}}\right]^{n-1}$$

Then, the pdf of first order statistics Y_1 inverse Weibull-Rayleigh distribution is given as

$$f_{Y_{(1)}}(y)=n\alpha\beta\theta^{\beta}ye^{\frac{\alpha}{2}y^2}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta-1}e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta}}\left[1-e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta}}\right]^{n-1}$$

Then, the pdf of nth order statistics Y_n inverse Weibull-Rayleigh distribution is given as

$$f_{Y_{(n)}}(y) = n\alpha\beta\theta^{\beta} y e^{\frac{\alpha}{2}y^2}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta-1} e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta}}\left[e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y^2}-1\right)^{-\beta}}\right]^{n-1}$$

XI. Maximum likelihood estimation and Fisher's information matrix of inverse Weibull- Rayleigh distribution

Suppose $Y_1, Y_2, ..., Y_n$ denotes random sample of size n from inverse Weibull-Rayleigh distribution then its likelihood function is given by

$$l = \prod_{i=1}^{n} f(y,\alpha,\beta,\theta)$$

$$= \prod_{i=1}^{n} \alpha\beta\theta^{\beta} y_i e^{\frac{\alpha}{2}y_i^2}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta-1} e^{-\theta^{\beta}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta}}$$

$$= \left(\alpha\beta\theta^{\beta}\right)^n \prod_{i=0}^{n} y_i \left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta-1} e^{\frac{\alpha}{2}\sum_{i=1}^{n} y_i^2} e^{-\theta^{\beta}\sum_{i=1}^{n}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta}}$$

The log likelihood function is given by

$$\log l = n\log\alpha + n\log\beta + n\beta\log\theta + \sum_{i=1}^{n}\log y_i + \frac{\alpha}{2}\sum_{i=1}^{n} y_i^2 - (\beta+1)\sum_{i=1}^{n}\log\left(e^{\frac{\alpha}{2}y_i^2}-1\right)$$
$$-\theta^{\beta}\sum_{i=1}^{n}\left(e^{-\frac{\alpha}{2}y_i^2}-1\right)^{-\beta} \qquad (18)$$

Differentiating equation (18) with respect each parameter α, β and θ, we have

$$\frac{\partial \log l}{\partial\alpha} = \frac{n}{\alpha} + \frac{1}{2}\sum_{i=1}^{n} y_i^2 - \frac{(\beta+1)}{2}\sum_{i=1}^{n}\frac{y_i^2 e^{\frac{\alpha}{2}y_i^2}}{e^{\frac{\alpha}{2}y_i^2}-1} + \frac{\beta\theta^{\beta}}{2}\sum_{i=1}^{n} y_i^2 e^{\frac{\alpha}{2}y_i^2}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta-1} \qquad (19)$$

$$\frac{\partial \log l}{\partial\beta} = \frac{n}{\beta} + n\log\theta - \sum_{i=1}^{n}\log\left(e^{\frac{\alpha}{2}y_i^2}-1\right) + \theta^{\beta}\sum_{i=1}^{n}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta}\log\left(e^{\frac{\alpha}{2}y_i^2}-1\right)$$

$$-\theta^{\beta}\log(\theta)\sum_{i=1}^{n}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta} \qquad (20)$$

$$\frac{\partial \log l}{\partial\theta} = \frac{n\beta}{\theta} - \beta\theta^{\beta-1}\sum_{i=1}^{n}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta} \qquad (21)$$

By setting equations (19), (20) and (21) to zero the MLE of parameters can be obtained. However the above equations are non-linear which cannot be expressed in closed form. So numerical techniques such as Newton-Raphson, Regula-Falsi and bisection methods must be applied to obtain MLE of parameters denoted by $\hat{\varsigma}(\hat{\alpha},\hat{\beta},\hat{\theta})$ of $\varsigma(\alpha,\beta,\theta)$.

Since the MLE of $\hat{\varsigma}$ follows asymptotically normal distribution which is given as

$$\sqrt{n}(\hat{\varsigma}-\varsigma)\to N\left(0,I^{-1}(\varsigma)\right)$$

Where $I^{-1}(\varsigma)$ is the limiting variance-covariance matrix $\hat{\varsigma}$ and $I(\varsigma)$ is a 3×3 Fisher information matrix

i,e
$$I(\varsigma)=-\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2 \log l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 \log l}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 \log l}{\partial\alpha\partial\theta}\right) \\ E\left(\frac{\partial^2 \log l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial\beta^2}\right) & E\left(\frac{\partial^2 \log l}{\partial\beta\partial\theta}\right) \\ E\left(\frac{\partial^2 \log l}{\partial\theta\partial\alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial\theta\partial\beta}\right) & E\left(\frac{\partial^2 \log l}{\partial\theta^2}\right) \end{bmatrix}$$

Where

$$\frac{\partial^2 \log l}{\partial\alpha^2}=\frac{-n}{\alpha^2}+\frac{(\beta+1)}{4}\sum_{i=1}^{n}\frac{y_i^4 e^{\frac{\alpha}{2}y_i^2}}{\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^2}$$

$$+\frac{\beta\theta^\beta}{4}\sum_{i=1}^{n} y_i^4 e^{\frac{\alpha}{2}y_i^2}\left\{\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta-1}-(\beta+1)\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta-2}\right\}$$

$$\frac{\partial^2 \log l}{\partial\beta^2}=\frac{-n}{\beta^2}-\theta^\beta\sum_{i=1}^{n}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta}\left\{\log\left(e^{\frac{\alpha}{2}y_i^2}-1\right)\right\}^2-\theta^\beta(\log\theta)^2\sum_{i=1}^{n}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta}$$

$$+2\theta^\beta\sum_{i=1}^{n}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta}\log\left(e^{\frac{\alpha}{2}y_i^2}-1\right)$$

$$\frac{\partial^2 \log l}{\partial\theta^2}=\frac{-n\beta}{\theta^2}-\beta(\beta-1)\theta^{\beta-2}\sum_{i=1}^{n}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta}$$

$$\frac{\partial^2 \log l}{\partial\alpha\partial\beta}=\frac{\partial^2 \log l}{\partial\beta\partial\alpha}=-\sum_{i=1}^{n}\frac{y_i^2 e^{\frac{\alpha}{2}y_i^2}}{\left(e^{\frac{\alpha}{2}y_i^2}-1\right)}-\frac{\beta\theta^\beta}{2}\sum_{i=1}^{n} y_i^2 e^{\frac{\alpha}{2}y_i^2}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta-1}\log\left(e^{\frac{\alpha}{2}y_i^2}-1\right)$$

$$+\theta^{\beta}(1+\beta\log\theta)\sum_{i=1}^{n} y_i^2 e^{\frac{\alpha}{2}y_i^2}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta-1}$$

$$\frac{\partial^2 \log l}{\partial\alpha\partial\theta}=\frac{\partial^2 \log l}{\partial\theta\partial\alpha}=\frac{\beta^2\theta^{\beta-1}}{2}\sum_{i=1}^{n} y_i^2 e^{\frac{\alpha}{2}y_i^2}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta-1}$$

$$\frac{\partial^2 \log l}{\partial\beta\partial\theta}=\frac{\partial^2 \log l}{\partial\theta\partial\beta}=\frac{n}{\theta}+\beta\theta^{\beta-1}\sum_{i=1}^{n}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta}\log\left(e^{\frac{\alpha}{2}y_i^2}-1\right)$$

$$-\theta^{\beta-2}\left(\theta+\beta^2-\beta\right)\sum_{i=1}^{n}\left(e^{\frac{\alpha}{2}y_i^2}-1\right)^{-\beta}$$

Hence the approximate $100(1-\psi)\%$ confidence interval for α,β and θ are respectively given by

$$\hat{\alpha}\pm Z_{\frac{\psi}{2}}\sqrt{I_{\alpha\alpha}^{-1}(\hat{\varsigma})}\ ,\hat{\beta}\pm Z_{\frac{\psi}{2}}\sqrt{I_{\beta\beta}^{-1}(\hat{\varsigma})}\ \text{And}\ \hat{\theta}\pm Z_{\frac{\psi}{2}}\sqrt{I_{\theta\theta}^{-1}(\hat{\varsigma})}$$

Where $Z_{\frac{\psi}{2}}$ denotes the ψ^{th} percentile of the standard normal distribution.

XII. Results

I. Simulation Analysis

In this section we demonstrate the simulation analysis which examines the effectiveness of the M L estimators. The inverse cdf method is employed to generate random samples of size $n = 30,50,75,100$ and 150 which is discussed in section (4). This procedure is repeated $N = 500$ times for calculation of bias, variance and MSE. Four separate combinations of parameters are selected and it is observed that bias, variance and MSE decrease significantly, when we increase sample size. The efficiency of ML estimators is therefore relatively strong, consistent in case of IWRD.

Table 1: Average bias, variance and MSEs of 5,00 simulations of IWRD for different parameters values.

II. Applications

This section is dedicated to demonstrate the effectiveness of the established distribution by taking into account real data sets taken from medical science. The established distribution is compared with power Erlang distribution (PED), Weighted Gumbel-II distribution (WG-IID), power Gompertz distribution (PGD), inverse Weibull distribution (IWD), Rayleigh distribution (RD), inverse Rayleigh distribution (IRD) and inverse Lindley distribution (ILD). It is revealed that the developed distribution offers an appropriate fit.

To compare the versatility of the explored distribution, we consider the criteria like AIC (Akaike

Sample Size n	Parameters	$\alpha=0.4$, $\beta=0.7$, $\theta=0.8$			$\alpha=0.4$, $\beta=0.5$, $\theta=0.7$		
		Bias	variance	MSE	Bias	variance	MSE
30	α	0.10667	0.09549	0.10687	0.03396	0.04469	0.04584
	β	0.01601	0.03820	0.03846	0.03240	0.01454	0.01559
	θ	0.99840	6.86551	7.86231	0.46101	2.29695	2.50948
50	α	0.03325	0.04242	0.04353	0.00960	0.02461	0.02471
	β	0.02828	0.01952	0.02032	0.02701	0.00881	0.00954
	θ	0.29418	1.33982	1.42637	0.22597	0.96048	1.01155
75	α	0.03266	0.02648	0.02755	0.01118	0.01587	0.01599
	β	0.01992	0.01395	0.01434	0.01276	0.006403	0.00656
	θ	0.23048	0.54670	0.59982	0.14271	0.26261	0.28298
100	α	0.01522	0.01673	0.01696	0.00267	0.01220	0.01221
	β	-0.0008	0.00980	0.00980	0.01304	0.00462	0.00479
	θ	0.15424	0.34874	0.37253	0.06209	0.12997	0.13383
150	α	0.01240	0.01161	0.01177	0.00537	0.00995	0.00998
	β	0.01080	0.00754	0.00765	0.01761	0.00403	0.00434
	θ	0.09099	0.16775	0.17603	0.06589	0.13020	0.13255
		$\alpha=0.6$, $\beta=0.5$, $\theta=0.7$			$\alpha=0.6$, $\beta=0.8$, $\theta=0.9$		
30	α	0.01382	0.01359	0.01379	-0.0018	0.01282	0.01282
	β	0.00500	0.00398	0.00400	0.01203	0.00447	0.00462
	θ	0.10007	0.19036	0.20038	0.07110	0.19553	0.20058
50	α	0.01956	0.01135	0.01174	0.00314	0.01279	0.01280
	β	0.00397	0.00422	0.00423	0.00645	0.00402	0.00406
	θ	0.11939	0.15628	0.17053	0.07557	0.19296	0.19867
75	α	0.01085	0.01139	0.01151	-0.0052	0.00959	0.00962
	β	0.00533	0.00384	0.00386	0.01512	0.00440	0.00403
	θ	0.08410	0.13345	0.14052	0.03145	0.09593	0.09691
100	α	0.00531	0.01086	0.01089	0.00303	0.00873	0.00874
	β	0.00616	0.00401	0.00305	0.01137	0.00396	0.00402
	θ	0.06256	0.12807	0.13199	0.04911	0.10033	0.00275
150	α	0.01534	0.010668	0.01070	0.02240	0.01451	0.00502
	β	0.00422	0.00389	0.00301	0.00551	0.00436	0.00400
	θ	0.12016	0.17372	0.12816	0.14134	0.10026	0.00084

information criterion), CAIC (Consistent Akaike information criterion), BIC (Bayesian information criterion) and HQIC. Distribution having lesser AIC, CAIC, BIC, HQIC and KS values is considered better also having higher probability value (p-value).

$$AIC = 2k - 2\ln l \,;\, CAIC = \frac{2kn}{n-k-1} - 2\ln l \,;\, BIC = k\ln n - 2\ln l$$

And $$HQIC = 2k\ln(\ln(n)) - 2\ln l$$

The descriptive statistics of the data set 1 and 2 are presented in Table 1and 4.The estimates of the parameters are shown in Table 2 and 5 for data set 1and 2 respectively. Log-likelihood, Akaike information criteria (AIC) etc for the data set 1 and 2 are generated and presented in Table 3 and 6 respectively.

Data set 1:- The data was collected from a group of 46 patients, per years, upon the recurrence of leukemia whom received autologous marrow. The data repoted by Jhon H Kersey [14], as follows

0.0301,0.0384,0.063, 0.0849, 0.0877, 0.0959, 0.1397, 0.1616, 0.1699, 0.2137,0.2137, 0.2164, 0.2384, 0.2712, 0.274, 0.3863, 0.4384, 0.4548, 0.5918, 0.6,0.6438, 0.6849, 0.7397, 0.8575, 0.9096, 0.9644, 1.0082, 1.2822, 1.3452, 1.4,1.526, 1.7205, 1.989, 2.2438, 2.5068, 2.6466, 3.0384, 3.1726, 3.4411, 4.4219,4.4356, 4.5863, 4.6904, 4.7808, 4.9863, 5

Table 2: Descriptive statistics of data set first

Min	Q_1	Median	Mean	Q_3	Skew	Kurt.	Max
0.0301	0.221	0.798	1.517	2.441	1.036	2.655	5

Table 3: The ML Estimates and standard error of the unknown parameters

Model		IWRD	PED	WG-IID	PGD	IWD	RD	IRD	ILD
$\hat{\alpha}$		0.6737	1.2685	0.7023	0.6483	……..	0.4075	0.0343	0.4333
$\hat{\beta}$		0.2764	0.6654	0.4651	0.2006	0.7017	……..	……..	……….
$\hat{\theta}$		0.0641	1.4619	0.0010	0.7515	0.3371	…….	…….	……….
S.E	$\hat{\alpha}$	0.2252	2.6342	0.3287	0.1832	0.1154	0.0600	0.0050	0.0462
	$\hat{\beta}$	0.0399	0.6548	0.6414	0.3050	……..	……..	……..	……….
	$\hat{\theta}$	0.0491	2.3917	0.5574	0.1553	……..	……..	……..	……….

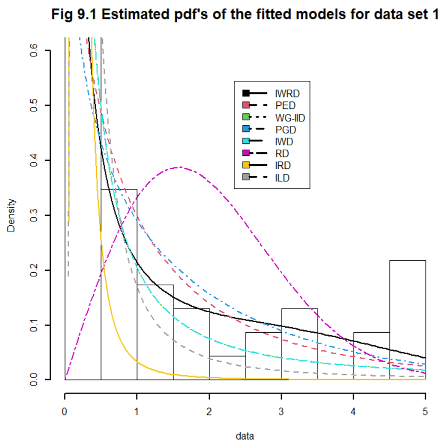
Table 4: Performance of distributions for data set first

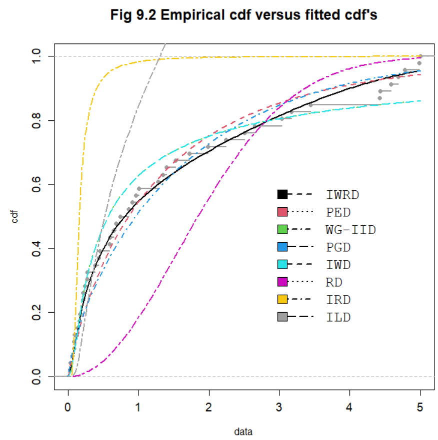
Model	IWRD	PED	WG-IID	PGD	IWD	RD	IRD	ILD
$-2\log l$	118.90	127.91	138.90	127.36	138.89	207.42	303.76	176.43
AIC	124.90	133.91	144.90	133.36	142.89	209.42	305.76	178.43
CAIC	125.47	134.48	145.47	133.93	143.17	209.51	305.85	178.52
BIC	130.39	139.40	150.38	138.84	146.55	211.25	307.59	180.25
HQIC	126.96	135.97	146.95	135.41	144.26	210.10	306.45	179.11
K-S Value	0.079	0.0980	0.1399	0.1013	0.139	0.3998	0.566	1.342
P Value	0.935	0.7686	0.3286	0.7325	0.328	8.1e-07	2.9e-13	2.2e-16

The asymptotic variance-covariance matrix of maximum likelihood estimates under IWRD for data set first is computed as

$$I^{-1}(\omega)=\begin{pmatrix} 0.0507 & -0.0053 & 0.0075 \\ -0.0053 & 0.0015 & -0.0011 \\ 0.0075 & -0.0011 & 0.0024 \end{pmatrix}$$

Therefore, the 95% confidence interval for α, β and θ are given as $(0.2322, 1.1152)$, $(0.1982, 0.3546)$ and $(-0.0321, 0.1604)$, respectively.





Data set 2:- The data set represents the survival times(in years) of a group of patients given chemotherapy treatment reported by Bekker et al.[4]. The data follows

0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033

Table 5: Descriptive statistics of data set second

Min	Q_1	Median	Mean	Q_3	Skew	Kurt.	Max
0.047	0.39	0.84	1.34	2.17	0.972	2.663	4.03

Table 6: The ML Estimates of the unknown parameters for data set second

Model		IWRD	PED	WG-IID	PGD	IWD	RD	IRD	ILD
$\hat{\alpha}$		0.9335	1.9113	0.8677	0.6504	…….	0.6026	0.1072	0.6584
$\hat{\beta}$		0.3361	0.6756	0.4977	0.1215	0.8671	…….	…….	……..
$\hat{\theta}$		0.1524	2.1173	0.0010	0.9762	0.4482	……..	…….	……...
S.E	$\hat{\alpha}$	0.3257	3.6555	0.3392	0.1719	……..	0.0898	0.0159	0.0723
	$\hat{\beta}$	0.0505	0.6238	0.5701	0.2587	0.0927	…….	…….	…………
	$\hat{\theta}$	0.1078	3.4214	0.5884	0.1949	0.0818	……..	……..	……….

Table 7: Performance of distributions for data set second

Model	IWRD	PED	WG-IID	PGD	IWD	RD	IRD	ILD
$-2\log l$	110.24	115.97	127.64	115.96	127.63	155.83	230.17	138.88
AIC	116.24	121.97	133.64	121.96	131.63	157.83	232.17	140.88
CAIC	116.82	122.56	134.23	122.54	131.92	157.92	232.26	140.97
BIC	121.66	127.39	139.06	127.38	135.25	159.63	233.98	142.69
HQIC	118.26	123.99	135.66	123.98	132.98	158.50	232.84	141.56
K-S Value	0.0660	0.9811	0.1383	0.1139	0.138	0.353	0.507	1.252
P Value	0.982	2.2e-16	0.3251	0.5638	0.325	1.5e-05	2.8e-11	2.2e-16

The asymptotic variance-covariance matrix of maximum likelihood estimates under IWRD for data set first is computed a

$$I^{-1}(\omega)=\begin{pmatrix} 0.1061 & -0.0105 & 0.0263 \\ -0.0105 & 0.0025 & -0.0034 \\ 0.0263 & -0.0034 & 0.0116 \end{pmatrix}$$

Therefore, the 95% confidence interval for α,β and θ are given as $(0.2950,1.5720)$, $(0.2370,0.4353)$ and $(-0.0590,0.3638)$, respectively.

Fig 9.3 Estimated pdf's of the fitted models for data set 2

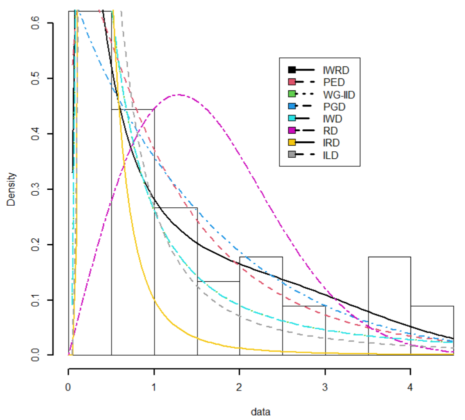
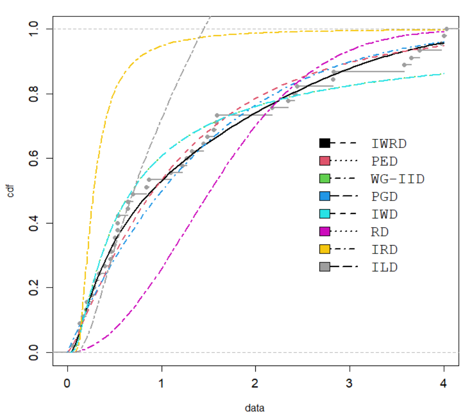


Fig 9.4 Empirical cdf versus fitted cdf's



it is evident from Table (4) and (7) that IWRD has lesser values of AIC, CAIC, BIC, HQIC and K-S statistics along with higher p-value. When it is compared with IWD, RD, IRD and ILD models. Hence we conclude that IWRD provides an adequate fit than compare ones

XIII. Discussion

This paper deals with a new generalisation of Rayleigh distribution called inverse Weibull-Rayleigh distribution. We have added extra two parameters to the Rayleigh distribution by inverse Weibull-G generator, the main purpose for such modification is that the formulated distribution become more richer and flexible in modelling datasets. Several distinct properties of formulated distribution has been studied and discussed. The model parameters of the distribution are estimated by the known method of maximum likelihood estimation. Eventually, the efficiency of the explored distribution is examined through real data sets which reveals that the formulated distribution provides an adequate model fit than competing ones.

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