MATHEMATICAL ANALYSIS OF A VACCINATION MODEL WITH IMMIGRATION AND GENERALIZED SATURATED INCIDENCE RATE FUNCTION

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Abstract

In the paper , we propose a vaccine-dependent mathematical model for the treatment of diseases at the population level. Determine equilibrium points : disease-free and endemic and basic reproduction number R_0 .We formulate theorems on stability and establish the proof of the theorems by Ruth-Hurwitz criteria..In addition, numerical simulations of the model is carried out to show the efficacy of the vaccine . Moreover, graphically it is clearly seen the effectiveness of vaccine for SIR epidemic model with vaccination and without vaccination.

Keywords: Basic reproduction number, Equilibrium, Vaccine, Stability, Ruth-Hurwitz Criteria

I. Introduction

Mathematical Modeling is the art of capturing natural phenomenon of real life in the form of mathematical equations. It is a method of simulating real life situations with mathematical equations to forecast their future behavior. Mathematical modeling uses tools such as decision theory, queuing theory and linear programming and requires large amounts of number crunching. In 2013, Agrawal Ankit and Saxena G. studied an SIR epidemic model with generalized saturated incidence rate function[1]. They have analyzed stability of the disease free equilibrium and the endemic equilibrium with the help of a non linear incidence rate. In1993, Derrick W.R and team formulated a general SIRS disease transmission model under assumptions that the size of the population varies, the incidence rate is nonlinear and the recovered (removed) class may also be directly re-infected[2]. Vaccination is the administration of antigenic material (a vaccine) to stimulate an individual's immune system to develop adaptive immunity to a pathogen. Vaccines can prevent or ameliorate morbidity from infection. Gumel A.B., Moghadas S.M. [3] have proposed a new deterministic model for the dynamics of an infectious disease in the presence of a preventive (prophylactic) vaccine and effective therapeutic treatment in 2003. Many models for the spread of infectious diseases in populations have been analyzed mathematically and applied to specific diseases by Hethcote H.W. in 2000 [4]. In 2013, Jasmine D. E.C. and Henry Amrithraj proposed an epidemic model with non-monotonic incidence rate under a limited resource for treatment to understand the effect of the capacity of the treatment [5,6,7]. In 2014 , Jasmine D.E.C., Henry Amirtharaj a modified SIR epidemic model with generalized saturated incidence rate is incorporated on account of the effect of limited treatment resources on the control of epidemic disease [8].

II. Model Formulation

In the paper , considered a population of size $N(t)$,which is divided into disjoint classes $S(t), I(t), V(t)$ and $R(t)$ which denote the number of susceptible , infected, vaccinated and recovered individuals respectively at time t with the saturated incidence rate function $\frac{\lambda SI}{\rho + \alpha_1 I + \alpha_2 I^2}$.

The flow diagram of the model [**figure 1**] is given below :

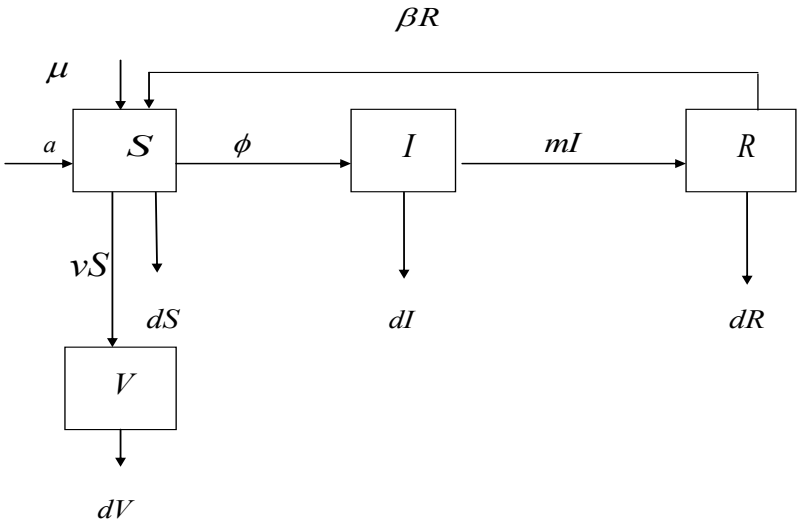

Figure 1: Transfer diagram for a vaccinated model

where the symbols stand for

S	Number of susceptible individuals
I	Number of infected individuals
R	Number of recovered individuals
V	Vaccinated population
a	Recruitment rate of population
d	Natural death rate
m	Natural Recovery rate of infective
β	Rate at which recovered individuals lose immunity and return to susceptible
μ	Increase of susceptible at constant rate
v	Rate at which susceptible population is vaccinated
$\phi = \frac{\lambda SI}{\rho + \alpha_1 I + \alpha I^2}$	Transmission rate(non linear incidence rate function)

λ	proportionality constant
$\rho \geq 1$	a positive constant
α	a positive parameter
λSI	infection force of the disease

All parameters assumed here are greater than or equal to zero .

III. Derivation of model

The differential equations corresponding to figure 1 are

$$\begin{aligned}
\frac{dS}{dt} &= a + \mu + \beta R - \frac{\lambda SI}{\rho + \alpha_1 I + \alpha_2 I^2} - (d+v)S \\
\frac{dI}{dt} &= \frac{\lambda SI}{\rho + \alpha_1 I + \alpha_2 I^2} - (d+m)I \\
\frac{dR}{dt} &= mI - (d+\beta)R \\
\frac{dV}{dt} &= vS - dV
\end{aligned} \tag{1}$$

Because of the biological meaning of the components $(S(t), I(t), V(t), R(t))$, We have focused on the model in the first octant of R^3 that is
$S(t) \geq 0, I(t) \geq 0, R(t) \geq 0, V(t) \geq 0$ and
$N(t) = S(t) + I(t) + R(t) + V(t)$.

IV. Equilibrium Points

a) <u>Disease-freee equilibrium</u> E_0

At disease-free equilibrium state,

$$\begin{aligned}
& a + \mu + \beta R - \frac{\lambda SI}{\rho + \alpha_1 I + \alpha_2 I^2} - (d+v)S = 0 \\
& \frac{\lambda SI}{\rho + \alpha_1 I + \alpha_2 I^2} - (d+m)I = 0 \\
& mI - (d+\beta)R = 0 \\
& vS - dV = 0
\end{aligned} \tag{2}$$

Assume that $I = 0$ then on solving all equations of system (2) ,we have disease-free equilibrium points such that

$$E_0 = \left(\frac{a+\mu}{d+v}, \ 0, \ 0, \frac{v(a+\mu)}{d(d+v)} \right).$$

b) <u>Endemic equlibrium</u> E^*

Assume that $I \neq 0$ then system (1) becomes

$$\begin{aligned} &a+\mu+\beta R^*-\frac{\lambda S^* I^*}{\rho+\alpha_1 I^*+\alpha_2 I^{*2}}-(d+v)S^*=0 \\ &\frac{\lambda S^* I^*}{\rho+\alpha_1 I^*+\alpha_2 I^{*2}}-(d+m)I^*=0 \\ &mI^*-(d+\beta)R^*=0 \\ &vS^*-dV^*=0 \end{aligned} \quad . \qquad (3)$$

On solving all equations of system (3) ,we have the endemic equilibrium points such that

$$S^*=\frac{(d+m)(\rho+\alpha_1 I^*+\alpha_2 I^{*2})}{\lambda},$$

$$I^*=\frac{\left[\frac{\beta m}{d+\beta}-(d+m)-\frac{\alpha_1(d+v)(d+m)}{\lambda}\right]}{2\left[\frac{\alpha_2(d+v)(d+m)}{\lambda}\right]} \pm \frac{\sqrt{\left[\frac{\beta m}{d+\beta}-(d+m)-\frac{\alpha_1(d+v)(d+m)}{\lambda}\right]^2+\frac{4\alpha_2(d+v)^2(d+m)^2\rho(R_0-1)}{\lambda^2}}}{2\left[\frac{\alpha_2(d+v)(d+m)}{\lambda}\right]},$$

$$R^*=\frac{mI^*}{d+\beta} \quad \text{and} \quad V^*=\frac{v(d+m)(\rho+\alpha_1 I^*+\alpha_2 I^{*2})}{\lambda d} \quad \text{respectively}$$

and

basic reproduction number is given by

$$R_0=\frac{\lambda(a+\mu)}{(d+m)\rho(d+v)}$$

Jacobian matrix of system (2) at disease -free equilibrium is given by

$$J=\begin{pmatrix} -(d+v) & \frac{-\lambda(a+\mu)}{\rho(d+v)} & \beta & 0 \\ 0 & \frac{\lambda(a+\mu)}{\rho(d+v)}-(d+m) & 0 & 0 \\ 0 & m & -(d+\beta) & 0 \\ v & 0 & 0 & -d \end{pmatrix}.$$

and

$$J - zI = \begin{pmatrix} -(d+v)-z & \frac{-\lambda(a+\mu)}{\rho(d+v)} & \beta & 0 \\ 0 & \frac{\lambda(a+\mu)}{\rho(d+v)} - (d+m) - z & 0 & 0 \\ 0 & m & -(d+\beta)-z & 0 \\ v & 0 & 0 & -d-z \end{pmatrix}.$$

The characteristic equation will be

$$|J - zI| = 0 \ .$$

$$\Rightarrow \ -(d+z)\left[-(d+v+z)\left\{-(d+\beta+z)\left(\frac{\lambda(a+\mu)}{\rho(d+v)} - (z+d+m)\right)\right\}\right] = 0$$

or $z = -d \ , \ z = -(d+v) \ , \ z = -(d+\beta)$ and

$$z = \frac{\lambda(a+\mu) - (d+m)\rho(d+v)}{\rho(d+v)}.$$

For the system (2) to be locally asymptotically stable all $z < 0$.

So, if we consider , $\frac{\lambda(a+\mu)}{(d+m)\rho(d+v)} < 1$. Then $R_0 < 1$.

Where $R_0 = \frac{\lambda(a+\mu)}{(d+m)\rho(d+v)}$ is called basic reproduction number .

Therefore , the system (2) is locally asymptotically stable if $R_0 < 1$.

And if $z > 0$, then

$$\frac{\lambda(a+\mu) - (d+m)(d+v)\rho}{(d+v)\rho} > 0$$

$$\frac{\lambda(a+\mu)}{(d+m)(d+v)\rho} > 1$$

Or, $R_0 > 1$.

This implies that the system (2) is globally asymptotically stable if $R_0 > 1$.

V.Mathematical Analysis

Lemma 5.1: The plane $S + I + R + V = \frac{a+\mu}{d}$ is a manifold of system (1) which is attracting in the first octant.

From the lemma , we have

$S + I + R + V = \frac{a+\mu}{d}$ which implies $S = \frac{a+\mu}{d} - I - R - V$

Therefore system (1) becomes,

$$\frac{dI}{dt} = \frac{\lambda\left[\frac{a+\mu}{d} - I - R - V\right]I}{\rho + \alpha_1 I + \alpha_2 I^2} - (d+m)I \cong P(I,R,V)$$
$$\frac{dR}{dt} = mI - (d+\beta)R \cong Q(I,R) \quad (4)$$
$$\frac{dV}{dt} = \nu\left[\frac{a+\mu}{d} - I - R - V\right] - dV \cong T(I,R,V)$$

Theorem 5.2 : System (4) does not have non-trivial periodic orbit if $\alpha_1(3d+\beta+m) > 0$.

Proof: Consider,

$$I(t) > 0, R(t) > 0, V(t) > 0 \ .$$

and consider the Dulac function,

$$D(I,R,V) = \phi^{-1} = \frac{\rho + \alpha_1 I + \alpha_2 I^2}{\lambda SI}$$

i.e. $$D(I,R,V) = \frac{\rho + \alpha_1 I + \alpha_2 I^2}{\lambda\left[\frac{a+\mu}{d} - I - R - V\right]I} \ .$$

Then ,

$$\frac{\partial(DP)}{\partial I} + \frac{\partial(DQ)}{\partial R} + \frac{\partial(DT)}{\partial V}$$

$$= \frac{-\alpha_1(3d+\beta+m)}{\lambda\left(\frac{a+\mu}{d} - I - R - V\right)} - \frac{2\alpha_2 I(d+m)}{\lambda\left(\frac{a+\mu}{d} - I - R - V\right)} - \frac{\left(\rho + \alpha_2 I^2\right)(2d+\beta)}{\lambda I\left(\frac{a+\mu}{d} - I - R - V\right)}$$
$$\frac{-d\left(\rho + \alpha_1 I + \alpha_2 I^2\right)}{\lambda\left(\frac{a+\mu}{d} - I - R - V\right)^2} - \frac{(dR + dV + \beta R)\left(\rho + \alpha_1 I + \alpha_2 I^2\right)}{\lambda I\left(\frac{a+\mu}{d} - I - R - V\right)^2}$$

.

$$\frac{\partial(DP)}{\partial I} + \frac{\partial(DQ)}{\partial R} + \frac{\partial(DT)}{\partial V} < 0 \quad if \quad \alpha_1(3d+\beta+m) > 0 \ .$$

where,

$$\frac{\partial(DP)}{\partial I} = \frac{-(d+m)(\alpha_1 + 2\alpha_2 I)}{\lambda\left[\frac{a+\mu}{d} - I - R - V\right]} - \frac{(d+m)(\rho + \alpha_1 I + \alpha_2 I^2)}{\lambda\left[\frac{a+\mu}{d} - I - R - V\right]^2} \ .$$

$$\frac{\partial(DQ)}{\partial R} = \frac{m(\rho + \alpha_1 I + \alpha_2 I^2)}{\lambda\left[\frac{a+\mu}{d} - I - R - V\right]^2} - \frac{(d+\beta)(\rho + \alpha_1 I + \alpha_2 I^2)}{\lambda I\left[\frac{a+\mu}{d} - I - R - V\right]} - \frac{(d+\beta)R(\rho + \alpha_1 I + \alpha_2 I^2)}{\lambda I\left[\frac{a+\mu}{d} - I - R - V\right]^2} \ .$$

and

$$\frac{\partial(DT)}{\partial V} = \frac{-d(\rho + \alpha_1 I + \alpha_2 I^2)}{\lambda I\left[\frac{a+\mu}{d} - I - R - V\right]} - \frac{dV(\rho + \alpha_1 I + \alpha_2 I^2)}{\lambda I\left[\frac{a+\mu}{d} - I - R - V\right]^2} \ .$$

This completes the proof .

Now rescaling (4) by

$$x=\frac{\lambda}{d+\beta}I,\ y=\frac{\lambda}{d+\beta}R,\ z=\frac{\lambda}{d+\beta}V\ ,\ \tau=(d+\beta)t$$

.

Then

$$\frac{dx}{d\tau}=\frac{dx}{dI}.\frac{dI}{dt}.\frac{dt}{d\tau}\ ,\ \frac{dy}{d\tau}=\frac{dy}{dR}.\frac{dR}{dt}.\frac{dt}{d\tau}\ \text{ and }\ \frac{dz}{d\tau}=\frac{dz}{dV}.\frac{dV}{dt}.\frac{dt}{d\tau}$$

Therefore,

$$\frac{dx}{d\tau}=\frac{px\left[A-x-y-z\right]}{\left[1+qx\right]}-Tx\ ,\ \frac{dy}{d\tau}=sx-y$$ and

$$\frac{dz}{d\tau}=g(A-x-y-z)-hz$$

.

Where ,

$$g=\frac{v}{d+\beta}\ ,\ h=\frac{d}{d+\beta}\ s=\frac{m}{d+\beta}\ \ p=\frac{1}{\rho}\ ,A=\frac{\lambda(a+\mu)}{d(d+\beta)}\ ,T=\frac{(d+m)}{(d+\beta)}$$

$$q=\frac{(d+\beta)}{\rho\lambda}\left(\alpha_1+\frac{\alpha_2\left(d+\beta\right)x}{\lambda}\right)=\frac{(d+\beta)}{\rho\lambda}\left(\alpha_1+\alpha_2 I\right)$$

.

Thus we have new system of equations,

$$\begin{aligned}\frac{dx}{d\tau}&=\frac{px(A-x-y-z)}{1+qx}-Tx\\ \frac{dy}{d\tau}&=sx-y\\ \frac{dz}{d\tau}&=g(A-x-y-z)-hz\end{aligned}\qquad(5)$$

The trivial equilibrium (0, 0, 0) of (5) is the disease-free equilibrium and endemic equilibrium points after rescaling the system (4) is obtained as

$$x^*=\frac{h(Ap-T)-gT}{ph(s+1)+Tq(g+h)}\ ,\ y^*=sx^*\ ,\ z^*=\frac{g(A-x^*-sx^*)}{(g+h)}\ .$$

VI. Stability Analysis of Disease-free and Endemic Equilibria after Rescaling

Now the Jacobian matrix of system (5) at disease free equilibrium will be

$$J_1=\begin{pmatrix}Ap-T & 0 & 0\\ s & -1 & 0\\ -g & -g & -(g+h)\end{pmatrix}.$$

Then

$$J_1 - \xi I = \begin{pmatrix} Ap - T - \xi & 0 & 0 \\ s & -1 - \xi & 0 \\ -g & -g & -(g+h+\xi) \end{pmatrix}$$

and the characteristic equation is

$$\left|J_1 - \xi I\right| = 0$$

$$-(g+h+\xi)\left[-(1+\xi)(Ap - T - \xi)\right] = 0$$

$$\Rightarrow \quad \xi = -(g+h) < 0 \ , \quad \xi = -1 < 0 \ , \ \xi = Ap - T \ .$$

For the third eigen value three conditions arises:

1. Stable hyperbolic node if

$$T - Ap > 0$$

2.Saddle node if

$$T - Ap = 0$$

3.Hyperbolic saddle node if

$$T - Ap < 0 \ .$$

When

$$T - Ap > 0 \Rightarrow Ap - T < 0 \ .$$

So ,by Routh-Hurwitz criteria the disease-free equilibrium after rescaling is locally asymptotically stable .

Now , discussing the stability of the endemic equilibrium when

$$T - Ap < 0$$

.

Theorem 6.1: Suppose $T - Ap < 0$ then there is a unique endemic equilibrium (x^*, y^*, z^*) of model (5) which is a saddle node .

Proof: Since $T < Ap$, therefore , we neglect T and so the system (5) can

be written as

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{px(A - x - y - z)}{1 + qx} \\ \frac{dy}{d\tau} &= sx - y \\ \frac{dz}{d\tau} &= g(A - x - y - z) - hz \end{aligned} \qquad (6).$$

And

$$x^* = \frac{Ap}{p(s+1)} \ , \ y^* = sx^* \ , z^* = \frac{g\left[A - x^*(s+1)\right]}{g+h} \ . \qquad (7)$$

Jacobian matrix of system (6) at endemic points is given by,

$$M=\begin{pmatrix}\left[\dfrac{px^*(qsx^*-(Aq+1))}{(1+qx^*)^2}-\dfrac{pg\left[A-x^*(s+1)\right]}{(g+h)(1+qx^*)^2}\right] & \dfrac{-px^*(1+qx^*)}{(1+qx^*)^2} & \dfrac{-px^*(1+qx^*)}{(1+qx^*)^2}\\ s & -1 & 0\\ -g & -g & -(g+h)\end{pmatrix}$$

$$|M|=\frac{1}{(1+qx^*)^2}\left[px^*\left\{g\left[-(s+1)+Aq-qx^*(s+1)\right]+h\left[Aq+(s+1)\right]\right\}\right]+\frac{Apg}{(1+qx^*)^2}$$

Substituting the value of Ap from (7) ,

$$|M|=\frac{1}{(1+qx^*)^2}\left[px^*\left\{g\left[Aq-qx^*(s+1)\right]+h\left[Aq+(s+1)\right]\right\}\right].$$

Since $q>0$ which implies $\det(M)>0$.

Now , Trace of *M* will be

$$tr(M)=\frac{1}{(g+h)(1+qx^*)^2}\begin{bmatrix}px^*(qsx^*-(Aq+1))(g+h)-pg\left[A-x^*(s+1)\right]\\ -(g+h)(1+qx^*)^2-(1+qx^*)^2(g+h)^2\end{bmatrix}$$

Sign of *trace (M)* depends on the nature of S_1 which is given as

$$S_1=px^*(qsx^*-(Aq+1))(g+h)-pg\left[A-x^*(s+1)\right]$$

using (7) , we have

$$S_1=p\left[\frac{h(Ap-T)-gT}{\left[ph(s+1)+Tq(g+h)\right]^2}\right]\begin{Bmatrix}-qsT(g+h)^2-Aqph(g+h)-ph(s+1)\\ (g+h)-Tq(g+h)^2(Aq+1)+ghpA(s+1)\\ -gT(g+h)(s+1)\end{Bmatrix}-pgA$$

which implies $S_1<0$ since $q>0$.

$$tr(M)=S_1-1-(g+h)<0$$

Thus, by Routh-Hurwitz Criterion the endemic equilibrium points (x^*,y^*,z^*) are locally asymptotically stable.

7. Numerical Simulation and Graphical Representation

Case I: SIR epidemic model without vaccination:

$$S(0)=4, I(0)=1, R(0)=1, \alpha_1=3.1, \alpha_2=4.7, d=2.29, \beta=1.5,$$
$$\mu=2, \rho=1, a=3.1, m=0.19, \lambda=9,\ R_0=0.8980<1$$

Figure 2 shows that *S(t)* approaches to its steady state value while *I(t)* and *R(t)* approaches zero as time progresses, disease dies out.

Case II: SIR epidemic model with vaccination:

$$S(0)=4, I(0)=1, R(0)=1\ ,\ V(0)=1, \alpha_1=3.1, \alpha_2=4.7,$$
$$d=2.29, \beta=1.5, \mu=2, \rho=1, a=3.1, m=0.19, \lambda=9,\ \nu=0.5,$$
$$R_0=0.7370<1$$

Figure 3 shows that *S(t) ,V(t)* approaches to its steady state value while *I(t)* and *R(t)* approaches zero as time progresses, disease dies out.

Case III: SIR epidemic model without vaccination:

$$S(0)=4, I(0)=1, R(0)=1\ , \alpha_1=3.1, \alpha_2=4.7, d=2.29, \beta=1.5,$$
$$\mu=2, \rho=1, a=3.1, m=0.19, \lambda=1, R_0=8.08212>1$$

Figure 4 shows that *S(t)* approaches to its steady state value while *I(t)* and *R(t)* approaches zero as time progresses, disease becomes endemic.

Case IV: SIR epidemic model with vaccination:

$$S(0)=4, I(0)=1, R(0)=1,\ V(0)=1, \alpha_1=3.1, \alpha_2=4.7,$$
$$d=2.29, \beta=1.5, \mu=2, \rho=1, a=3.1, m=0.19, \lambda=1, \nu=0.5$$
$$R_0=6.63371>1$$

Figure 5 shows that *S(t) ,V(t)* approaches to its steady state value while *I(t)* and *R(t)* approaches zero as time progresses, disease becomes endemic.

Case I

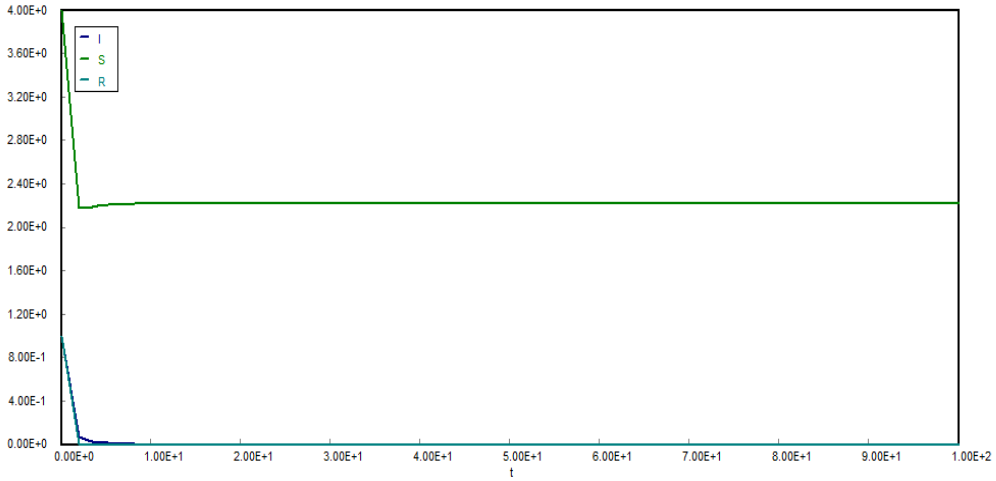

Figure 2: SIR graph without vaccination

Case II

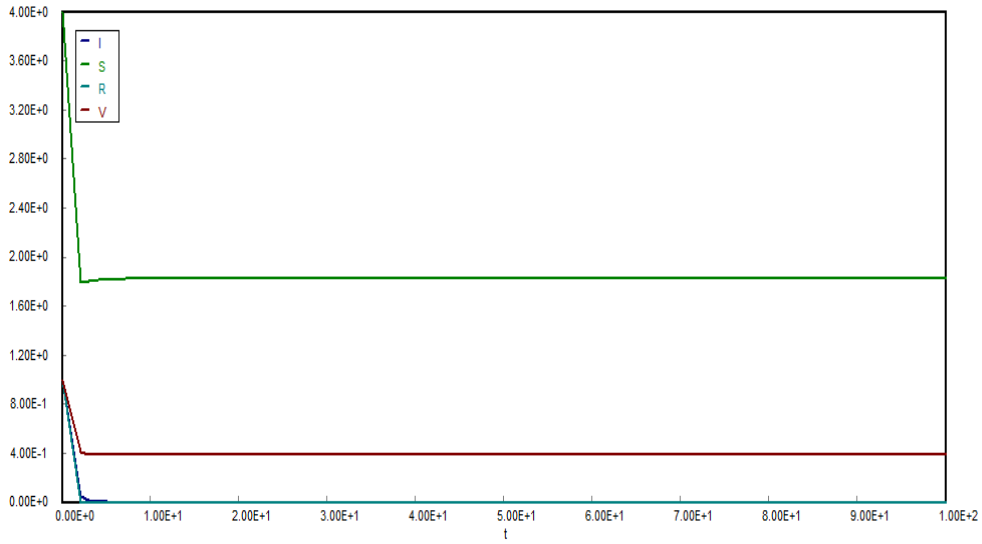

Figure 3: SIR model with vaccination.

Case III

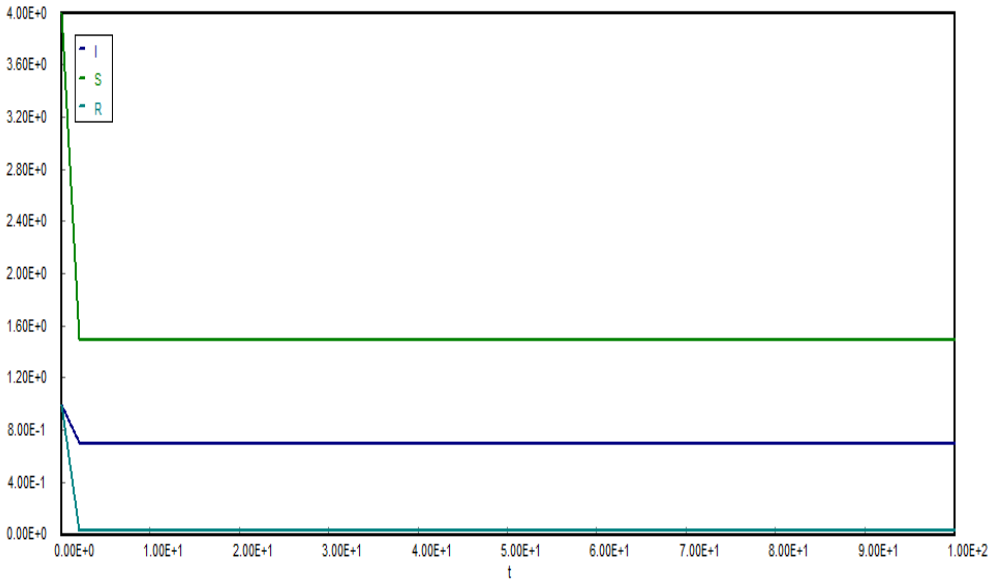

Figure 4: SIR model without vaccination.

Case IV

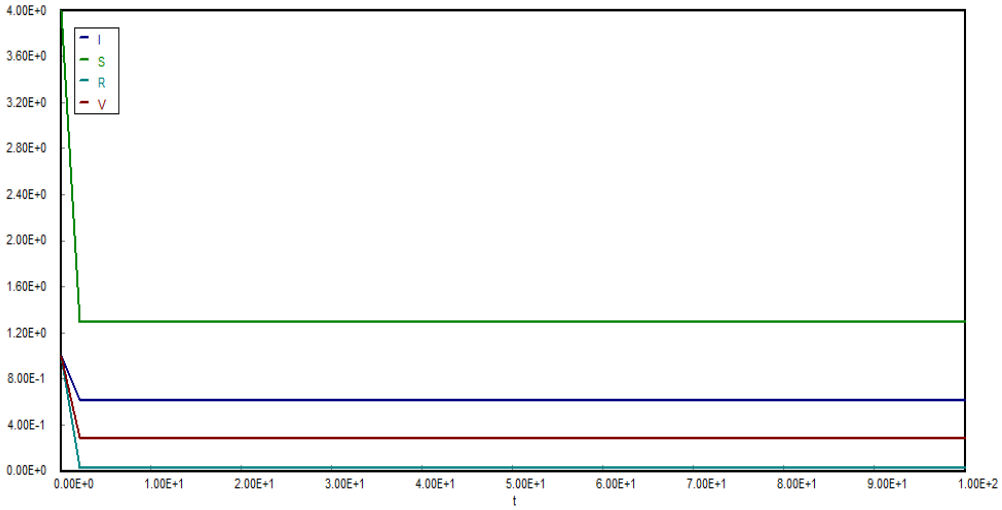

Figure 5: SIR model with vaccination.

VIII. Conclusion

In this paper, we have considered a vaccinated epidemic model with generalized incidence rate function. The global stability of the endemic equilibrium $E^* = (S^*,I^*,R^*,V^*)$ depends on the basic reproduction number. It plays an important role in controlling the disease. When reproduction number is less than or equal to one the disease free equilibrium state is globally attractive in the first octant and is globally stable, that is the disease dies out. When basic reproduction number is greater than one the endemic equilibrium state E^* exists and is globally stable in the interior. I have also plotted *SIR* and *SIR-V* graphs and compared the graphs for both reproduction number greater than one and less than one. These results and parametric conditions help to develop social consciousness about the disease among the susceptible.

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