On Reliability Structures with Two Common Failure Criteria Under Age-Based Maintenance Policy

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Abstract

In this work we study reliability structures with two common failure criteria under specific age-based preventive maintenance models. The aforementioned systems, which consist of n independent components, fail upon the occurrence of two different scenarios. The theoretical framework for delivering the expected cost rate of such structures is presented in detail, while a variety of numerical outcomes for different choices of the design parameters are also provided and discussed.

Keywords: reliability systems with two failure criteria; preventive maintenance policy; Samaniego's signature; cost rate.

I. Introduction

In the area of Reliability Engineering, an enthralling quest calls for the design of appropriate structures, which are related to real-life applications or existing devices and contrivances. A particular group of reliability models, which seems to reel in the scientists during the last decades, is the family of consecutive-type systems with two stopping rules. Due to the abundance of their applications in Engineering and Statistical Modelling, the aforementioned structures comprise an engrossing scope of research activity.

The general framework of constructing a consecutive-type system with two common stopping rules requires n linearly or circularly ordered units. The resulting system fails, whenever a pre-specified condition (out of two ones) is satisfied or both of them are met. In such a framework, several structures have been already introduced in the literature. For example, the (n, f, k) structure proposed in [1], fails if, and only if, there exist at least f failed units or at least k consecutive failed units. Several reliability characteristics of the (n, f, k) systems are studied in detail in [2] or [3]. Among others, the $<n, f, k>$ structure (see, e.g. [4] or [5]), the constrained (k, d)–out–of–n: F system (see, e.g. [6] or [7]) are well-known consecutive-type reliability systems with two failure criteria. For a detailed and up-to-date survey on the consecutive-type systems, we refer to the detailed reviews offered [8] or [9] and the well-documented monographs devised by [10] or [11]. Some recent advances on the topic can be found in the works [12], [13] or [14]. A survey of reliability approaches in various fields of Engineering and Physical Sciences is also provided in [15].

In the present work, we study the (n, f, k) and $<n, f, k>$ structures under specific age-based preventive maintenance policy. In Section 2, the general framework for obtaining the expected cost rate of the underlying reliability systems is presented in detail. In Section 3, extensive numerical experimentation is accomplished in order to shed light on the behavior of the cost rate of the aforementioned reliability structures under different choices for their design parameters. Finally, the Discussion section summarizes the contribution of the present manuscript, while some interesting conclusions based on previous sections are also highlighted.

II. The theoretical framework for delivering the expected cost rate of (*n,f,k*) and <*n,f,k*> structures

Let us first consider the (n, f, k) structure consisting of *n* independent and identically distributed (*i.i.d.*) components $X_1, X_2, ..., X_n$ ordered in a line ($f > k$). The particular system fails if, and only if, there exist at least *f* failed components or at least *k* consecutive failed components (see, e.g. [1]). We next denote by *T* the lifetime of the (n, f, k) system with distribution *F*, while the lifetimes $T_1, T_2, ..., T_n$ of its components share a common exponential distribution *G* with parameter *λ*, namely

$$G(t) = P(T_i \le t) = 1 - e^{-\lambda t}, \; i = 1, 2, .., n. \tag{1}$$

If we denote by $T_{1:n}, T_{2:n}, ..., T_{n:n}$ the corresponding ordered lifetimes of the components, the signature of the system is defined as the probability vector $(s_1(n), s_2(n), ..., s_n(n))$ with

$$s_i(n) = P(T = T_{i:n}), i = 1,2,..., n. \tag{2}$$

Following the so-called age-based maintenance policy, the system is replaced either at a pre-specified time t_0 with cost c_0 or at its failure with cost c_1 (whichever comes first). It is common to assume that $c_0 < c_1$. Denoting by *c* the corresponding replacement cost of a single component, we next provide the so-called expected cost rate of the (n, f, k) structure with exponentially distributed components, namely its expected cost per time unit. The proposed procedure is based on the signature vector of the underlying system.

Proposition 1. Let us consider the (n, f, k) structure consisting of *n* independent components sharing a common Exponential distribution with mean *λ*. The expected cost rate of the underlying system under the age-based preventive maintenance policy is given by

$$ECR(t_0) = \frac{c_0 + (c_1 - c_0)\sum_{i=k}^{n} s_i(n)\left(1 - \sum_{j=n-i+1}^{n}\binom{n}{j} e^{-\lambda j t_0}(1 - e^{-\lambda t_0})^{n-j}\right) + c\sum_{i=k}^{n}\left(\sum_{m=1}^{i} s_m(n)\right)\left(1 - \sum_{j=n-i+1}^{n}\binom{n}{j} e^{-\lambda j t_0}(1 - e^{-\lambda t_0})^{n-j}\right)}{\sum_{i=k}^{n} s_i(n)\int_0^{t_0}\sum_{j=n-i+1}^{n}\binom{n}{j} e^{-\lambda j v}(1 - e^{-\lambda v})^{n-j}\,dv} \tag{3}$$

Proof. For a coherent system consisting of *n* components with lifetime *T* and signature $p_i(n), i = 1, 2, ..., n$, the expected cost rate for the age-based preventive maintenance policy is determined via the following (see [16])

$$ECR(t_0) = \frac{c_0 + (c_1 - c_0)\sum_{i=1}^{n} s_i(n) P(T_{i:n} \le t_0) + c\sum_{i=1}^{n}\left(\sum_{m=1}^{i} s_m(n)\right) P(T_{i:n} \le t_0)}{\sum_{i=1}^{n} s_i(n)\int_0^{t_0} P(T_{i:n} > v)\,dv}. \tag{4}$$

Lifetimes $T_1, T_2, ..., T_n$ of the components are random variables with exponential distribution with parameter *λ*, namely $T_i \sim G, i = 1, 2, ..., n$, where *G* is defined in (1). Consequently, the following holds true (see, e.g. [17])

$$P(T_{i:n} > t) = \sum_{j=n-i+1}^{n} \binom{n}{j} e^{-\lambda j t}(1-e^{-\lambda t})^{n-j} \tag{5}$$

Since the first (k-1) coordinates of the signature vector of the (n, f, k) structure with n independent components ($f > k$) are equal to zero (see, e.g. [3]), the desired result is readily obtained by the aid of equations (4) and (5).

□

We next consider the $< n, f, k >$ structure with *i.i.d.* linearly ordered units $X_1, X_2, ..., X_n$ ($f > k$). The particular structure has been introduced in [4] and involves two common stopping rules. More precisely, the $< n, f, k >$ structure consists of n components and fails if, and only if, there exist at least f failed components and at least k consecutive failed components. Denoting by T^* the lifetime of the $< n, f, k >$ system with distribution F and by $T_1^*, T_2^*, ..., T_n^*$ the lifetimes of its components with Exponential distribution G as defined in (1), we next determine its expected cost rate under preventive maintenance strategy. More precisely, if we assume that the signature vector of the $< n, f, k >$ system is given by $(s_1^*(n), s_2^*(n), ..., s_n^*(n))$, the following proposition sheds light on the aforementioned issue.

Proposition 2. Let us consider the $< n, f, k >$ structure consisting of n independent components sharing a common Exponential distribution with mean λ. The expected cost rate of the underlying system under the age-based preventive maintenance policy is given by

$$ECR^*(t_0) = \frac{c_0 + (c_1 - c_0)\sum_{i=f}^{n} s_i^*(n)\left(1 - \sum_{j=n-i+1}^{n}\binom{n}{j} e^{-\lambda j t_0}(1-e^{-\lambda t_0})^{n-j}\right) + c\sum_{i=f}^{n}\left(\sum_{m=1}^{i} s_m^*(n)\right)\left(1 - \sum_{j=n-i+1}^{n}\binom{n}{j} e^{-\lambda j t_0}(1-e^{-\lambda t_0})^{n-j}\right)}{\sum_{i=f}^{n} s_i^*(n)\int_0^{t_0}\sum_{j=n-i+1}^{n}\binom{n}{j} e^{-\lambda j v}(1-e^{-\lambda v})^{n-j}\,dv} \tag{6}$$

Proof. We first take into account that the first (f-1) coordinates of the signature vector of the $< n, f, k >$ structure with n independent components ($f > k$) are equal to zero (see, e.g. [5]). Following a parallel argumentation as the one presented in the proof of Proposition 1, the outcome is readily deduced.

□

III. Numerical Results

In the present section, we run through extensive numerical experimentation in order to study the cost behavior of the (n, f, k) and $< n, f, k >$ structures with exponentially distributed components. Based on the theoretical results proved previously, we compute the expected cost rate of the aforementioned reliability systems for several choices of their design parameters under preventive maintenance policy.

Table 1 displays several numerical results referring to the *ECR* function defined earlier for the (n, f, k) structure with exponentially distributed components under a pre-specified preventive maintenance policy with constants c, c_0, c_1.

Table 1: *The ECR-values of the (n,f,k) system with exponentially distributed components under preventive maintenance policy (c=0.1, c0=1, c1=3).*

n	**(f,k)**	**t0**	**λ=1**	**λ=0.8**	**λ=0.6**	**λ=0.4**	**λ=0.2**
5	(3,2)	0.01	90.6672	92.5294	94.3938	96.2603	98.1291

		0.02	40.6946	42.5469	44.4036	46.2647	48.1302
		0.03	24.0554	25.8978	27.7468	29.6024	31.4646
		0.04	15.7497	17.5821	19.4233	21.2734	23.1324
		0.05	10.7774	12.5997	14.4332	16.2778	18.3340
		0.06	7.4719	9.2842	11.1098	12.9489	14.8012
		0.07	5.1191	6.9210	8.7389	10.5723	12.4213
		0.08	3.3615	5.1532	6.9631	8.7910	10.6367
5	(4,2)	0.01	90.7073	92.5615	94.4178	96.2764	98.1371
		0.02	40.7343	42.5788	44.4276	46.2807	48.1382
		0.03	24.0944	25.9293	27.7707	29.6184	31.4726
		0.04	15.7876	17.6131	19.447	21.2894	23.1404
		0.05	10.814	12.6301	14.4566	16.2937	18.1415
		0.06	7.5069	9.3136	11.1329	12.9647	14.8092
		0.07	5.1521	6.9495	8.7614	10.588	12.4293
		0.08	3.3924	5.1805	6.9852	8.8065	10.6447
6	(3,2)	0.01	87.7976	90.2437	92.687	95.1275	97.5652
		0.02	37.7633	40.2215	42.6744	45.1218	47.5638
		0.03	21.0636	23.5333	25.9954	28.4496	30.8957
		0.04	12.6984	15.179	17.6499	20.1107	22.5609
		0.05	7.66759	10.1587	12.6381	15.1053	17.5595
		0.06	4.30444	6.8056	9.29326	11.7666	14.2248
		0.07	1.89456	4.40538	6.901	9.38041	11.8425
		0.08	0.08076	2.60088	5.1042	7.58951	10.0554
6	(4,2)	0.01	87.898	90.3241	92.7473	95.1677	97.5853
		0.02	37.8624	40.3013	42.7345	45.162	47.5838
		0.03	21.16	23.6119	26.0551	28.4897	30.9158
		0.04	12.7908	15.2556	17.7089	20.1507	22.581
		0.05	7.7548	10.2327	12.696	15.145	17.5796
		0.06	4.38558	6.8763	9.34977	11.8059	14.2448
		0.07	1.9686	4.4723	6.95589	9.41925	11.8625
		0.08	0.14711	2.66363	5.15724	7.62781	10.0753
7	(3,2)	0.01	90.6864	92.5489	94.4115	96.2743	98.1371
		0.02	40.6889	42.5502	44.412	46.2744	48.1371
		0.03	24.0266	25.8857	27.7463	29.608	31.4705
		0.04	15.6993	17.5555	19.4143	21.2751	23.1372
		0.05	10.7069	12.5594	14.416	16.2756	18.1373
		0.06	7.38259	9.23085	11.0847	12.9429	14.804
		0.07	5.01207	6.85542	8.70612	10.5626	12.4231
		0.08	3.2381	5.07595	6.92314	8.7777	10.6375

Based on Table 1, we next deduce some concluding remarks. More precisely, the *ECR* function of the (n, f, k) structure with exponentially distributed components (with mean λ), under the preventive maintenance policy with cost constants c, c_0, c_1 seems to:

- increases as λ decreases (for pre-fixed values of $t_0, n, f, k, c, c_0, c_1$)
- decreases as t_0 increases (for pre-fixed values of $\lambda, n, f, k, c, c_0, c_1$)
- decreases as n increases (for pre-fixed values of $t_0, \lambda, f, k, c, c_0, c_1$)
- increases as f increases (for pre-fixed values of $t_0, \lambda, n, k, c, c_0, c_1$)

We next investigate the impact of parameters c, c_0, c_1 on the *ECR*-behavior of the resulting (n, f, k)

structure. Table 2 provides several numerical results referring to the *ECR* function of the (n, f, k) structure with exponentially distributed components for pre-specified design parameters *n*, *f*, *k*.

Table 2: *The ECR-values of the (n,f,k) system with exponentially distributed units (λ=0.5) under pre-specified design (n=5, f=4, k=2, t_0=0.02).*

	Parameter c_1			
c_0	***c_1=2***	***c_1=3***	***c_1=4***	***c_1=5***
0.5	21.3479 20.6849 20.022	19.3529 18.6899 18.027	17.3579 16.6949 16.032	15.3629 14.6999 14.037
0.75	34.3482 33.6853 33.0224	32.3532 31.6903 31.0274	30.3582 29.6953 29.0324	28.3632 27.7003 27.0374
1	47.3486 46.6857 46.0228	45.3536 44.6907 44.0278	43.3586 42.6957 42.0328	41.3636 40.7007 40.0378
1.25	60.349 59.6861 59.0232	58.354 57.6911 57.0282	56.359 55.6961 55.0332	54.364 53.7011 53.0382
1.5	73.3494 72.6865 72.0235	71.3544 70.6915 70.0285	69.3594 68.6965 68.0335	67.3644 66.7015 66.0386
1.75	86.3498 85.6869 85.0239	84.3548 83.6919 83.0289	82.3598 81.3598 81.0339	80.3648 79.7019 79.0389

Each cell contains the *ECR*-values for *c*=0.1(upper entry), *c*=0.2 (middle entry), *c*=0.3 (lower entry)

Based on Table 2, we deduce that the *ECR* function of the (n, f, k) structure with exponentially distributed components, under a pre-specified design, seems to:

- decreases as c_1 increases (for pre-fixed values of remaining parameters)
- increases as c_0 increases (for pre-fixed values of remaining parameters)
- decreases as *c* increases (for pre-fixed values of remaining parameters).

On the other hand, it is of some interest to shed light on the *ECR* behavior of the $< n, f, k >$ structure with *i.i.d.* components. Table 3 displays several numerical results referring to the *ECR* function defined earlier for the $< n, f, k >$ structure with exponentially distributed components under a pre-specified preventive maintenance policy with constants *c*, c_0, c_1.

Table 3: *The ECR-values of the <n,f,k> system with exponentially distributed components under preventive maintenance policy (c=0.1, c_0=1, c_1=3).*

n	***(f,k)***	***t_0***	***λ=1***	***λ=0.8***	***λ=0.6***	***λ=0.4***	***λ=0.2***
5	(3,2)	0.01	90.7744	92.6296	94.4799	96.3252	98.1653
		0.02	40.7146	42.5902	44.4572	46.3148	48.1626
		0.03	23.9937	25.8871	27.769	29.6381	31.4933
		0.04	15.6114	17.5201	19.4153	21.2951	23.1574
		0.05	10.5674	12.4891	14.3962	16.2858	18.1549
		0.06	7.19479	9.12734	11.0449	12.9436	14.8191
		0.07	4.77896	6.72036	8.64706	10.554	12.4357
5	(4,2)	0.01	91.135	92.9183	94.6964	96.4695	98.2374
		0.02	41.0719	42.8775	44.6733	46.4591	48.2348
		0.03	24.3443	26.1712	27.984	29.7822	31.5655
		0.04	15.9522	17.7992	19.6284	21.4388	23.2296

		0.05	10.8956	12.7617	14.6066	16.4288	18.227
		0.06	7.50793	9.39205	11.2519	13.0856	14.8911
		0.07	5.07495	6.97589	8.85013	10.6949	12.5076
6	(3,2)	0.01	87.9497	90.3833	92.8053	95.2155	97.6138
		0.02	37.8079	40.2909	42.7524	45.1916	47.6078
		0.03	21.0074	23.5361	26.0347	28.5016	30.9351
		0.04	12.5478	15.1186	17.6521	20.1455	22.5958
		0.05	7.4285	10.0382	12.6045	15.1232	17.5899
		0.06	3.9823	6.6281	9.2251	11.7681	14.2508
		0.07	1.4949	4.1738	6.7997	9.36582	11.8641
6	(4,2)	0.01	88.3513	90.7047	93.0463	95.3761	97.694
		0.02	38.2043	40.6103	42.993	45.3524	47.6881
		0.03	21.3927	23.8502	26.2735	28.662	31.0154
		0.04	12.9166	15.4245	17.8876	20.3051	22.6762
		0.05	7.77636	10.3333	12.8356	15.2817	17.6703
		0.06	4.3054	6.91004	9.45072	11.9251	14.331
		0.07	1.78962	4.44048	7.01874	9.52098	11.9442

Based on Table 3, we next deduce some concluding remarks. More precisely, the *ECR* function of the $<n, f, k>$ structure with exponentially distributed components (with mean λ), under the preventive maintenance policy with cost constants c, c_0, c_1 seems to:

- increases as λ decreases (for pre-fixed values of $t_0, n, f, k, c, c_0, c_1$)
- decreases as t_0 increases (for pre-fixed values of $\lambda, n, f, k, c, c_0, c_1$)
- decreases as n increases (for pre-fixed values of $t_0, \lambda, f, k, c, c_0, c_1$)
- increases as f increases (for pre-fixed values of $t_0, \lambda, n, k, c, c_0, c_1$).

We next investigate the impact of parameters c, c_0, c_1 on the *ECR*-behavior of the resulting $<n, f, k>$ structure. Table 4 provides several numerical results referring to the *ECR* function of the $<n, f, k>$ structure with exponentially distributed components for pre-specified design parameters n, f, k.

Table 4: *The ECR-values of the <n,f,k> system with exponentially distributed components (λ=0.5) under pre-specified design (n=6, f=4, k=2, t_0=0.02).*

	Parameter c_1			
c_0	***c_1=2***	***c_1=3***	***c_1=4***	***c_1=5***
0.5	21.5749	19.5601	17.5453	15.5305
	21.1719	19.1571	17.1424	15.1276
	20.769	18.7542	16.7394	14.7246
0.75	34.5786	32.5638	30.549	28.5342
	34.1756	32.1608	30.1461	28.1313
	33.7727	31.7579	29.7431	27.7283
1	47.5823	45.5675	43.5527	41.5379
	47.1793	45.1645	43.1497	41.135
	46.7764	44.7616	42.7468	40.732
1.25	60.586	58.5712	56.5564	54.5416
	60.183	58.1682	56.1534	54.1387
	59.78	57.7653	55.7505	53.7357
1.5	73.5897	71.5749	69.5601	67.5453

	73.1867 72.7837	71.1719 70.769	697542 68.7542	67.1424 66.7394
1.75	86.5933 86.1904 85.7874	84.5786 84.1756 83.7727	82.5638 82.1608 81.7579	80.549 80.1461 79.7431

Each cell contains the *ECR*-values for *c*=0.1(upper entry), *c*=0.2 (middle entry), *c*=0.3 (lower entry)

Based on Table 4, we deduce that the *ECR* function of the $<n,f,k>$ structure with exponentially distributed components, under a pre-specified design, seems to:

- decreases as c_1 increases (for pre-fixed values of remaining parameters)
- increases as c_0 increases (for pre-fixed values of remaining parameters)
- decreases as *c* increases (for pre-fixed values of remaining parameters).

IV. Discussion

In the present paper, we focus on reliability structures with two common failure criteria. More precisely, the so-called (n,f,k) and $<n,f,k>$ structures are considered, while the corresponding expected cost rate under preventive maintenance policy is determined. The numerical results are produced by the aid of the theoretical framework given in Section 2 of the present manuscript. The influence of the design parameters of (n,f,k) and $<n,f,k>$ structures on their cost behavior is studied and some concluding remarks are also provided. All design parameters seem to influence the cost behavior of the underlying structures. Finally, a parallel reliability study of different consecutive-type structures is among future plans for the authors.

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