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A Two-State Retrial Queueing Model with Feedback and Balking

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Abstract

Present paper discusses a two-state retrial queueing model with feedback and balking. If a customer on arrival finds the server free it is served immediately. Else either it joins the retrial orbit as a secondary customer or balks from the system due to impatience. Primary and secondary arrivals both follow Poisson process. If the customer feels unsatisfied after service, it may join the orbit as a feedback customer. Service times follow Exponential distribution. The transient state probabilities for exact number of arrivals and departures when the server is busy or idle are obtained by solving difference-differential equations. Numerical solution is obtained and presented graphically.

Keywords: Arrivals, Departures, Queueing, Retrial, Feedback, Balking.

I. Introduction

Apart from classical queueing systems there exists a new class of queueing systems that is referred to as retrial queueing systems. In recent years, a lot of work has been done in this direction. Here if a customer on arrival finds the server free, it is served immediately. Else it joins the orbit (virtual queue) and retries for service from the orbit after a random amount of time (as shown in Figure 1). Some of the real life phenomena where these systems are successfully used are telecommunication systems, computer network systems, telephone switching systems. For detailed overview and main results [1], [2], [3] and [4] could be referred.

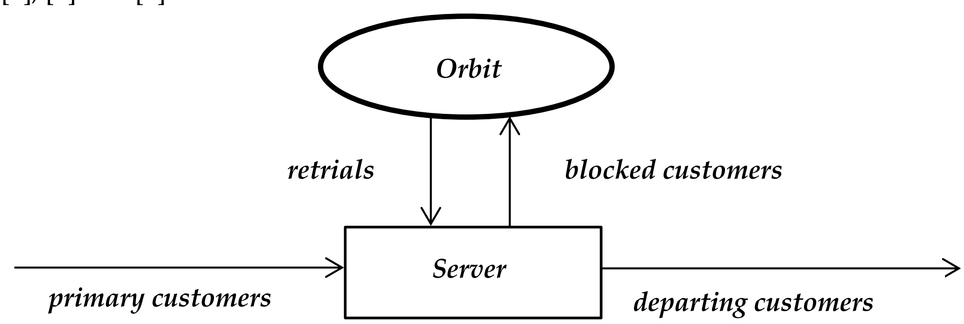

Figure 1: *Basic Structure of a Retrial Queueing System*

If on encountering a busy server, a customer leaves the system forever instead of joining the orbit (due to impatience), it is known as balking. Impatience can be commonly observed in many queueing systems dealing with which could lead to profits. [5] analyzed `Retrial queueing system with balking, optional service and vacation' where the steady state distributions of server state and number of jobs in the orbit are obtained.

If a customer feels unsatisfied after service, it may join the orbit as a feedback customer in order to obtain a satisfied service. This feature of feedback has also been widely discussed in retrial queueing theory. `A single server feedback retrial queue with collisions' was analyzed by [6]. [7] worked on `Modified vacation policy for M/G/1 retrial queue with balking and feedback' where some important measures were obtained. [8] published `Performance evaluation of two Markovian retrial queueing model with balking and feedback' in which the joint distribution of server state and retrial queue was derived.

[9] worked on `Some new results for the M/M/1 queue' where solution is obtained for the probability that exactly `i' number of arrivals, `$j$' number of services occur over a time interval $t$. In standard queueing models, total number of units in the system is considered whereas in this approach the exact number of arrivals and departures are considered. `A Single Server Retrial Queue with Impatient Customers' was studied by [10] considering the number of arrivals and departures from the orbit. [11] worked on `A two-state multiserver retrial queueing model with balking'. [12] analyzed `A Two-State Retrial Queueing Model with Feedback having Two Identical Parallel Servers' where the transient state probabilities were obtained.

The novelty of the work in the present paper is that here the solution of two-state model considering balking on the basis of immediate need and providing feedback facility to unsatisfied customers is obtained.

The present paper is categorized into various sections as under:

Section II gives the model description along with the difference-differential equations governing the system. The transient state probabilities are evaluated in section III. In section IV various performance measures are obtained. Numerical and graphical solutions are illustrated in section V. In section VI, the busy period probabilities are presented numerically and graphically. Finally, the paper is concluded in section VII which is followed by the references at the end.

II. Model Description

We consider a two-state retrial queueing model with feedback and balking. The fresh customers follow a Poisson process. On encounter with a busy server, the customer may join the orbit in order to retry for service else it balks from the system due to impatience. Service times follow Exponential distribution. The secondary customers repeatedly request for service from the orbit following a Poisson process. Also, an unsatisfied customer may join the orbit as a feedback customer in order to receive a satisfied service.

- Primary arrivals follow Poisson process with parameter λ.
- On encountering a busy server, arriving customer either joins the retrial orbit with probability β or leaves the system without joining i.e., balk from the system with parameter 1-β.
- Secondary arrivals follow Poisson process with parameter θ.
- Service times follow Exponential distribution with parameter μ.
- After receiving service, the customer joins the orbit with probability γ (when unsatisfied) and departs from the system with probability 1-γ.

The input flow of primary calls, intervals between repetitions, service times are statistically independent.

Laplace Transformation of $\bar{f}(s)$ of $f(t)$ is given by:

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt; \quad Re(s) > 0$$

The Laplace inverse of

$\frac{Q(p)}{P(p)} = \sum_{k=1}^{n}\sum_{l=1}^{m_k} \frac{t^{m_k-l}e^{a_k t}}{(m_k-l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}}\left(\frac{Q(p)}{P(p)}\right) (p-a_k)^{m_k}$ ∀ p= a_k, $a_i \neq a_k$ for i ≠ k

where, P(p)= $(p-a_1)^{m_1}$ $(p-a_2)^{m_2}$ ……….. $(p-a_n)^{m_n}$

Q(p) is a polynomial of degree < m_1+m_2+m_3 +…………m_n-1.

The Laplace inverse of $\bar{N}^{a,b,c}_{n_1,n_2,n_3}$(s) = $\frac{1}{(s+a)^{n_1}\,(s+b)^{n_2}\,(s+c)^{n_3}}$ is

$$N^{a,b,c}_{n_1,n_2,n_3}(t) = \sum_{l=1}^{n_3}\sum_{m=1}^{l} \frac{e^{-at}\, t^{n_3-l}\,(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_2+g_2)\right)}{(n_3-l)!(m-1)!\,(b-a)^{n_2+m-1}\,(c-a)^{n_1+l-m}}$$
$$+\sum_{l=1}^{n_2}\sum_{m=1}^{l} \frac{e^{-bt}\, t^{n_2-l}\,(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_2-l)!(m-1)!\,(a-b)^{n_3+m-1}\,(c-b)^{n_1+l-m}}$$
$$+\sum_{l=1}^{n_1}\sum_{m=1}^{l} \frac{e^{-ct}\, t^{n_1-l}\,(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(n_2+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_1-l)!(m-1)!\,(a-c)^{n_3+m-1}\,(b-c)^{n_2+l-m}}$$

If L^{-1}{f(s)} = F(t) and L^{-1}{g(s)} = G(t), then

L^{-1}{f(s) g(s)} = $\int_0^t F(u)G(t-u)du$ = F * G,

F * G is called the convolution of F and G.

Two-Dimensional State Model

Definitions:

$P_{i,j,0}(t)$=Probability that there are exactly *i* number of arrivals, *j* number of departures from the system by time *t* and server is idle.

$P_{i,j,1}(t)$=Probability that there are exactly *i* number of arrivals, *j* number of departures from the system by time *t* and server is busy.

$P_{i,j}(t)$ =Probability that there are exactly *i* number of arrivals, *j* number of departures from the system by time *t*.

$$P_{i,j}(t) = P_{i,j,0}(t) + P_{i,j,1}(t)\ \forall i,j;\ \ i \geq j$$

$$P_{i,j,0}(t) = 0; i < j \quad P_{i,j,1}(t) = 0; i \leq j$$

Initially

$$P_{0,0,0}(0) = 1;\ P_{i,j,0}(0) = 0\ i \geq j;\ i,j \neq 0$$
$$P_{i,j,1}(0) = 0;\ \forall\, i,j$$

The Difference-Differential Equations Governing the System are:

$$\frac{d}{dt}P_{i,j,0}(t) = -(\lambda + (i-j)\theta)P_{i,j,0}(t) + \mu(1-\gamma)P_{i,j-1,1}(t) + \mu\gamma P_{i,j,1}(t);\quad i \geq j \geq 0 \tag{1}$$

$$\frac{d}{dt}P_{1,0,1}(t) = -(\lambda\beta + \mu)P_{1,0,1}(t) + \lambda P_{0,0,0}(t) + \theta P_{1,0,0}(t); \tag{2}$$

$$\frac{d}{dt}P_{i,j,1}(t) = -(\lambda\beta + \mu)P_{i,j,1}(t) + \lambda P_{i-1,j,0}(t) + \lambda\beta\left(1-\delta_{i-1,j}\right)P_{i-1,j,1}(t) + (i-j)\theta P_{i,j,0}(t);\quad i > 1, i > j \geq 0 \tag{3}$$

where

$$\delta_{i-1,j} = \begin{cases}1; & i-1 = j\\ 0; & otherwise\end{cases}$$

Using Laplace Transform $\bar{f}(s)$ of $f(t)$ given by:

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt;\qquad Re(s) > 0$$

and using initial condition in equations (1) to (3), we have:

$$(s+\lambda)\bar{P}_{0,0,0}(s) = \bar{P}(0)$$
$$(s+\lambda+(i-j)\theta)\bar{P}_{i,j,0}(s) = \mu(1-\gamma)\bar{P}_{i,j-1,1}(s) + \mu\gamma\bar{P}_{i,j,1}(s);\qquad i \geq j \geq 0 \tag{4}$$

$$(s+\lambda\beta+\mu)\bar{P}_{1,0,1}(s)=\lambda\bar{P}_{0,0,0}(s)+\theta\bar{P}_{1,0,0}(s) \tag{5}$$

$$(s+\lambda\beta+\mu)\bar{P}_{i,j,1}(s)=\lambda\bar{P}_{i-1,j,0}(s)+\lambda\beta\bar{P}_{i-1,j,1}(s)+(i-j)\theta\bar{P}_{i,j,0}(s);\quad i>1, i>j\geq 0 \tag{6}$$

III. Solution of the Problem

Solving equations (4) to (6) recursively, we have

$$\bar{P}_{0,0,0}(s)=\frac{1}{s+\lambda} \tag{7}$$

$$\bar{P}_{1,0,1}(s)=\frac{\lambda}{s+\lambda\beta+\mu}\left(\frac{1}{s+\lambda}\right)+\frac{\theta}{s+\lambda\beta+\mu}\bar{P}_{1,0,0}(s) \tag{8}$$

$$\bar{P}_{1,1,0}(s)=\frac{\mu(1-\gamma)}{s+\lambda}\left[\frac{\lambda}{s+\lambda\beta+\mu}\bar{P}_{0,0,0}(s)+\frac{\theta}{s+\lambda\beta+\mu}\bar{P}_{1,0,0}(s)\right] \tag{9}$$

$$\bar{P}_{i,0,0}(s)=\frac{\mu\gamma}{s+\lambda+i\theta}\bar{P}_{i,0,1}(s);\qquad i\geq 1 \tag{10}$$

$$\bar{P}_{i,1,0}(s)=\frac{\mu(1-\gamma)}{s+\lambda+(i-1)\theta}\bar{P}_{i,0,1}(s)+\frac{\mu\gamma}{s+\lambda+(i-1)\theta}\bar{P}_{i,1,1}(s);\qquad i\geq 2 \tag{11}$$

$$\bar{P}_{i,i-1,0}(s)=\frac{\mu(1-\gamma)}{s+\lambda+\theta}\bar{P}_{i,i-2,1}(s)+\frac{\mu\gamma}{s+\lambda+\theta}\bar{P}_{i,i-1,1}(s);\qquad i\geq 3 \tag{12}$$

$$\bar{P}_{i,0,1}(s)=\frac{\lambda}{s+\lambda\beta+\mu}\bar{P}_{i-1,0,0}(s)+\frac{\lambda\beta}{s+\lambda\beta+\mu}\bar{P}_{i-1,0,1}(s)+\frac{i\theta}{s+\lambda\beta+\mu}\bar{P}_{i,0,0}(s);\; i\geq 2 \tag{13}$$

$$\bar{P}_{i,i-1,1}(s)=\frac{\lambda}{s+\lambda\beta+\mu}\bar{P}_{i-1,i-1,0}(s)+\frac{\theta}{s+\lambda\beta+\mu}\bar{P}_{i,i-1,0}(s);\qquad i\geq 2 \tag{14}$$

$$\bar{P}_{i,i,0}(s)=\frac{\mu(1-\gamma)}{s+\lambda}\left[\frac{\lambda}{s+\lambda\beta+\mu}\bar{P}_{i-1,i-1,0}(s)+\frac{\theta}{s+\lambda\beta+\mu}\bar{P}_{i,i-1,0}(s)\right];\qquad i\geq 2 \tag{15}$$

$$\bar{P}_{i,1,1}(s)=\frac{\lambda}{s+\lambda\beta+\mu}\bar{P}_{i-1,1,0}(s)+\frac{\lambda\beta}{s+\lambda\beta+\mu}\bar{P}_{i-1,1,1}(s)+\frac{(i-1)\theta}{s+\lambda\beta+\mu}\bar{P}_{i,1,0}(s);\; i\geq 3 \tag{16}$$

$$\bar{P}_{i,j,1}(s)=\sum_{k=1}^{i-j}\left(\frac{1}{s+\lambda\beta+\mu}\right)^{i-j-k}\lambda^{\psi'_k}\,(\lambda\beta)^{(i-j-k-1)\psi'_k}\,\eta'_k(s)\,\bar{P}_{j+k,j,0}(s) + \left(\frac{\lambda\beta}{s+\lambda\beta+\mu}\right)^{i-j-1}\bar{P}_{j+1,j,1}(s);\qquad i\geq j+2, j\geq 1 \tag{17}$$

where

$$\eta'_k(s)=\begin{cases}1; & k=1\\ 1+\dfrac{k\theta\beta}{s+\lambda\beta+\mu}; & k=2\ to\ i-j-1\\ \dfrac{k\theta}{s+\lambda\beta+\mu}; & k=i-j\end{cases}$$

$$\psi'_k=\begin{cases}1; & k=1\ to\ i-j\\ 0; & k=i-j+1\end{cases}$$

$$\bar{P}_{i,j,0}(s) = \frac{\mu(1-\gamma)}{s+\lambda+(i-j)\theta}\left\{\left[\sum_{k=1}^{i-j+1}\left(\frac{1}{s+\lambda\beta+\mu}\right)^{i-j-k+1}\lambda^{\psi'_k}(\lambda\beta)^{(i-j-k)\psi'_k}\eta'_k(s)\,\bar{P}_{j+k-1,j-1,0}(s)\right]\right.$$
$$\left.+\left(\frac{\lambda\beta}{s+\lambda\beta+\mu}\right)^{i-j}\bar{P}_{j,j-1,1}(s)\right\}$$
$$+\frac{\mu\gamma}{s+\lambda+(i-j)\theta}\left[\sum_{k=1}^{i-j+1}\left(\frac{1}{s+\lambda\beta+\mu}\right)^{i-j-k+1}\lambda^{\psi'_k}(\lambda\beta)^{(i-j-k)\psi'_k}\phi'_k(s)\,\bar{P}_{j+k-1,j,0}(s)\right]; \quad i>j>1 \quad (18)$$

where

$$\eta'_k(s) = \begin{cases} 1; & k=1 \\ 1+\dfrac{k\theta\beta}{s+\lambda\beta+\mu}; & k=2\ to\ i-j \\ \dfrac{k\theta}{s+\lambda\beta+\mu}; & k=i-j+1 \end{cases}$$

$$\phi'_k(s) = \begin{cases} 1; & k=1 \\ 1+\dfrac{(k-1)\theta\beta}{s+\lambda\beta+\mu}; & k=2\ to\ i-j \\ \dfrac{(k-1)\theta}{s+\lambda\beta+\mu}; & k=i-j+1 \end{cases}$$

$$\psi'_k = \begin{cases} 1; & k=1\ to\ i-j \\ 0; & k=i-j+1 \end{cases}$$

Taking Inverse Laplace of equations (7) to (18), we get the time dependent probabilities as:

$$P_{0,0,0}(t) = e^{-\lambda t} \quad (19)$$

$$P_{1,0,1}(t) = \lambda e^{-\lambda t}\left\{\frac{1}{\frac{\mu}{\beta}}-\frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\}+\theta e^{-(\lambda\beta+\mu)t} * P_{1,0,0}(t) \quad (20)$$

$$P_{1,1,0}(t) = \mu(1-\gamma)\lambda e^{-\lambda t}\left\{\frac{1}{\frac{\mu}{\beta}}-\frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\} * P_{0,0,0}(t)+\mu(1-\gamma)\theta e^{-\lambda t}\left\{\frac{1}{\frac{\mu}{\beta}}-\frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\} * P_{1,0,0}(t) \quad (21)$$

$$P_{i,0,0}(t) = \mu\gamma e^{-(\lambda+i\theta)t} * P_{i,0,1}(t); \quad i \ge 1 \quad (22)$$

$$P_{i,1,0}(t) = \mu(1-\gamma)e^{-(\lambda+(i-1)\theta)t} * P_{i,0,1}(t)+\mu\gamma e^{-(\lambda+(i-1)\theta)t} * P_{i,1,1}(t); \quad i \ge 2 \quad (23)$$

$$P_{i,i-1,0}(t) = \mu(1-\gamma)e^{-(\lambda+\theta)t} * P_{i,i-2,1}(t)+\mu\gamma e^{-(\lambda+\theta)t} * P_{i,i-1,1}(t); \quad i \ge 3 \quad (24)$$

$$P_{i,0,1}(t) = \lambda e^{-(\lambda\beta+\mu)t} * P_{i-1,0,0}(t)+\lambda\beta e^{-(\lambda\beta+\mu)t} * P_{i-1,0,1}(t)+i\theta e^{-(\lambda\beta+\mu)t} * P_{i,0,0}(t); \quad i \ge 2 \quad (25)$$

$$P_{i,i-1,1}(t) = \lambda e^{-(\lambda\beta+\mu)t} * P_{i-1,i-1,0}(t)+\theta e^{-(\lambda\beta+\mu)t} * P_{i,i-1,0}(t); \quad i \ge 2 \quad (26)$$

$$P_{i,i,0}(t) = \mu(1-\gamma)\lambda e^{-\lambda t}\left\{\frac{1}{\frac{\mu}{\beta}}-\frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\} * P_{i-1,i-1,0}(t)+\mu(1-\gamma)\theta e^{-\lambda t}\left\{\frac{1}{\frac{\mu}{\beta}}-\frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\} * P_{i.i-1,0}(t); \quad i \ge 2 \quad (27)$$

$$P_{i,1,1}(t) = \lambda e^{-(\lambda\beta+\mu)t} * P_{i-1,1,0}(t) + \lambda\beta e^{-(\lambda\beta+\mu)t} * P_{i-1,1,1}(t) + (i-1)\theta e^{-(\lambda\beta+\mu)t} * P_{i,1,0}(t); \quad i \geq 3 \quad (28)$$

$$\begin{aligned} P_{i,j,1}(t) = {} & \lambda^{i-j-1}\beta^{i-j-2}\frac{t^{i-j-2}}{(i-j-2)!}e^{-\left(\lambda+\frac{\mu}{\beta}\right)t} * P_{j+1,j,0}(t) \\ & + \sum_{k=2}^{i-j-1}\lambda^{i-j-k}\beta^{i-j-k-1}\frac{t^{i-j-k-1}}{(i-j-k-1)!}e^{-\left(\lambda+\frac{\mu}{\beta}\right)t} * P_{j+k,j,0}(t) \\ & + \sum_{k=2}^{i-j-1}(k\theta)(\lambda\beta)^{i-j-k}\frac{t^{i-j-k}}{(i-j-k)!}e^{-\left(\lambda+\frac{\mu}{\beta}\right)t} * P_{j+k,j,0}(t) + (i-j)\theta e^{-\left(\lambda+\frac{\mu}{\beta}\right)t} * P_{i,j,0}(t) \\ & + (\lambda\beta)^{i-j-1}\frac{t^{i-j-2}}{(i-j-2)!}e^{-\left(\lambda+\frac{\mu}{\beta}\right)t} * P_{j+1,j,1}(t); \quad i \geq j+2, j \geq 1 \quad (29) \end{aligned}$$

$$\begin{aligned} P_{i,j,0}(t) = {} & \mu(1-\gamma)\lambda^{i-j}\beta^{i-j-1}e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-r}}\right\} * P_{j,j-1,0}(t) \\ & + \mu(1-\gamma)\lambda e^{-(\lambda+(i-j)\theta)t}\sum_{k=2}^{i-j}(\lambda\beta)^{i-j-k}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k+1}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-k}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k-r+1}}\right\} \\ & * P_{j+k-1,j-1,0}(t) \\ & + \mu(1 \\ & - \gamma)e^{-(\lambda+(i-j)\theta)t}\sum_{k=2}^{i-j}(k\theta)(\lambda\beta)^{i-j-k+1}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k+2}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-k+1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k-r+2}}\right\} \\ & * P_{j+k-1,j-1,0}(t) + \mu(1-\gamma)(i-j+1)e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\frac{\mu}{\beta}} - \frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\} * P_{i,j-1,0}(t) \\ & + \mu(1-\gamma)(\lambda\beta)^{i-j}e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-r}}\right\} * P_{j,j-1,1}(t) \\ & + \mu\gamma\lambda^{i-j}\beta^{i-j-1}e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-r}}\right\} * P_{j,j,0}(t) \\ & + \mu\gamma\lambda e^{-(\lambda+(i-j)\theta)t}\sum_{k=2}^{i-j}(\lambda\beta)^{i-j-k}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k+1}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-k}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k-r+1}}\right\} \\ & * P_{j+k-1,j,0}(t) + \mu\gamma e^{-(\lambda+(i-j)\theta)t}\sum_{k=2}^{i-j}(k-1)\theta(\lambda\beta)^{i-j-k+1}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k+2}}\right. \\ & \left. - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-k+1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k-r+2}}\right\} * P_{j+k-1,j,0}(t) \\ & + \mu\gamma(i-j)\theta e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\frac{\mu}{\beta}} - \frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\frac{\mu}{\beta}}\right\} * P_{i,j,0}(t); \quad i > j \\ & > 1 \quad (30) \end{aligned}$$

IV. Some Performance Measures

- The Laplace transform $\bar{P}_{i.}(s)$ is given by:

$$\bar{P}_{i.}(s) = \sum_{j=0}^{i} \bar{P}_{i,j}(s) = \frac{\lambda^i}{(s+\lambda)^{i+1}}; \quad i > 0$$

and its Laplace Inverse is:

$$P_{i.}(t) = \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

which verifies the basic assumption that primary arrivals follow Poisson process.

- The probability that exactly *j* customers depart from the system by time *t* is given by:

$$\bar{P}_{.j}(t) = \sum_{i=j}^{\infty} P_{i,j}(t)$$

- Summing equations (7)-(18) over *i* and *j* we get:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left\{\bar{P}_{i,j,0}(s) + \bar{P}_{i,j,1}(s)\right\} = \frac{1}{s}$$

and hence

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left\{P_{i,j,0}(t) + P_{i,j,1}(t)\right\} = 1$$

which is a verification of our results.

- Define $Q_{n,m}(t)$ = Probability that there are exactly *n* customers in the orbit when *m (m*=0, 1*)* i.e., either the server is idle or busy at time *t*.

For idle server we represent it by probability $Q_{n,0}(t)$

$$Q_{n,0}(t) = \sum_{j=0}^{\infty} P_{j+n,j,0}(t)$$

The number of customers in the orbit, in this case are calculated with the following formula:
n = (number of arrivals − number of departures)
When the server is busy, it is represented by probability $Q_{n,1}(t)$

$$Q_{n,1}(t) = \sum_{j=0}^{\infty} P_{j+n+1,j,1}(t)$$

The number of customers in the orbit in this case is calculated by the following formula:
n = (number of arrivals − number of departures − 1)
Using above definitions in (1)-(3) and letting *γ=0*, the equations we get under statistical equilibrium are:

$$(\lambda + n\theta)Q_{n,0} = \mu Q_{n.1}; \qquad n \geq 0 \qquad (31)$$

$$(\lambda\beta + \mu)Q_{n,1} = \lambda Q_{n,0} + \lambda\beta Q_{n-1,1} + (n+1)\theta Q_{n+1,0}; \qquad n \geq 2 \qquad (32)$$

which coincides with the result (3.68) of [13].

V. Numerical Solution and Graphical Representation

Using MATLAB programming for the case *ρ=0.7*, *η=0.5*, *γ=0.5* and *1-β=0.4* the numerical solutions are generated. Some of which are given in Table 1 to Table 5. Observing the below tables for various time instants it is observed that the sum of probabilities approaches to 1.

Table 1: *At t=1*

$P_{0,0,0}$	$P_{1,0,0}$	$P_{1,1,0}$	$P_{4,4,0}$	$P_{7,1,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{6,5,1}$	Sum
0.4966	0.0623	0.0736	0.2735	0.0492	0.0165	0	0	0	0.9717

Table 2: *At t=5*

$P_{0,0,0}$	$P_{1,0,0}$	$P_{1,1,0}$	$P_{2,0,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,0,0}$	$P_{3,1,0}$	$P_{3,2,0}$	$P_{3,3,0}$
0.0302	0.0379	0.0792	0.0238	0.0699	0.0552	0.0101	0.0344	0.0409	0.0171

$P_{4,2,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{3,0,1}$	$P_{3,1,1}$	$P_{3,2,1}$	$P_{4,0,1}$	$P_{4,1,1}$	$P_{4,2,1}$
0.0172	0.0551	0.069	0.0812	0.0436	0.0795	0.039	0.0185	0.00415	0.0328

$P_{4,3,1}$	$P_{5,1,1}$	$P_{5,2,1}$	*Sum*
0.0092	0.0151	0.015	0.9154

Table 3: *At t=15*

$P_{2,2,0}$	$P_{3,3,0}$	$P_{4,2,0}$	$P_{4,3,0}$	$P_{4,4,0}$	$P_{5,3,0}$	$P_{5,4,0}$	$P_{5,5,0}$	$P_{6,4,0}$	$P_{6,5,0}$
0.0028	0.0076	0.007	0.0143	0.0122	0.0157	0.0206	0.0129	0.0128	0.0204

$P_{6,6,0}$	$P_{7,0,0}$	$P_{7,4,0}$	$P_{7,5,0}$	$P_{7,6,0}$	$P_{8,3,0}$	$P_{8,4,0}$	$P_{8,5,0}$	$P_{8,6,0}$	$P_{8,7,0}$
0.0199	0.0095	0.015	0.018	0.0137	0.0118	0.0244	0.0366	0.04	0.0302

$P_{8,8,0}$	$P_{4,2,1}$	$P_{4,3,1}$	$P_{5,2,1}$	$P_{5,3,1}$	$P_{5,4,1}$	$P_{6,2,1}$	$P_{6,3,1}$	$P_{6,4,1}$	$P_{6,5,1}$
0.0128	0.0142	0.0123	0.0196	0.0272	0.0164	0.0193	0.0333	0.0326	0.0147

$P_{7,2,1}$	$P_{7,3,1}$	$P_{7,4,1}$	$P_{7,5,1}$	$P_{7,6,1}$	$P_{8,2,1}$	$P_{8,3,1}$	$P_{8,4,1}$	$P_{8,5,1}$	$P_{8,6,1}$
0.0149	0.0293	0.0362	0.0268	0.0095	0.0187	0.0416	0.0619	0.0624	0.0404

$\mathrm{P}_{8,7,1}$	*Sum*
0.0134	0.9029

Table 4: *At t=25*

$P_{2,1,0}$	$P_{3,2,0}$	$P_{5,5,0}$	$P_{7,6,0}$	$P_{8,4,0}$	$P_{8,5,0}$	$P_{8,6,0}$	$P_{8,7,0}$	$P_{8,8,0}$	$P_{6,4,1}$
0	0.0001	0.0015	0.0065	0.0142	0.0406	0.0939	0.1804	0.2744	0.0033

$P_{7,4,1}$	$P_{8,3,1}$	$P_{8,5,1}$	$P_{8,6,1}$	$P_{8,7,1}$	Sum
0.0059	0.0141	0.0744	0.104	0.0893	0.9026

Table 5: *At t=35*

$P_{6,0,0}$	$P_{8,4,0}$	$P_{8,6,0}$	$P_{8,7,0}$	$P_{8,8,0}$	$P_{4,2,1}$	$P_{5,4,1}$	$P_{6,4,1}$	$P_{8,4,1}$	$P_{8,5,1}$
0	0.0014	0.029	0.1263	0.7119	0	0	0.0001	0.0041	0.0133

$P_{8,6,1}$	$P_{8,7,1}$	Sum
0.0349	0.0689	0.9899

Various probabilities are graphically presented against time t in Figures 2 to 5. Here, traffic intensity from primary calls is $\rho = \frac{\lambda}{\mu}$ and traffic intensity from secondary calls is $\eta = \frac{\theta}{\mu}$.

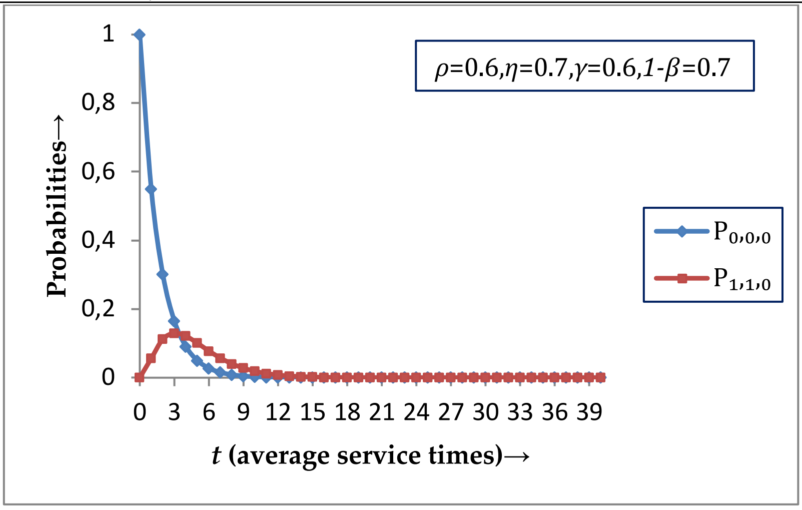

Figure 2: *Probabilities $P_{0,0,0}$ and $P_{1,1,0}$ against t (average service times)*

The probabilities $P_{0,0,0}$ and $P_{1,1,0}$ are compared in Figure 2 by plotting against time t for the case ρ=0.6, η=0.7, γ=0.6 and 1-β=0.7. It can be seen from the plot that the probability $P_{0,0,0}$ with initial value 1 at t=0 decreases rapidly whereas probability $P_{1,1,0}$ initiates with value 0 at t=0 increases in the beginning and then decreases gradually with time.

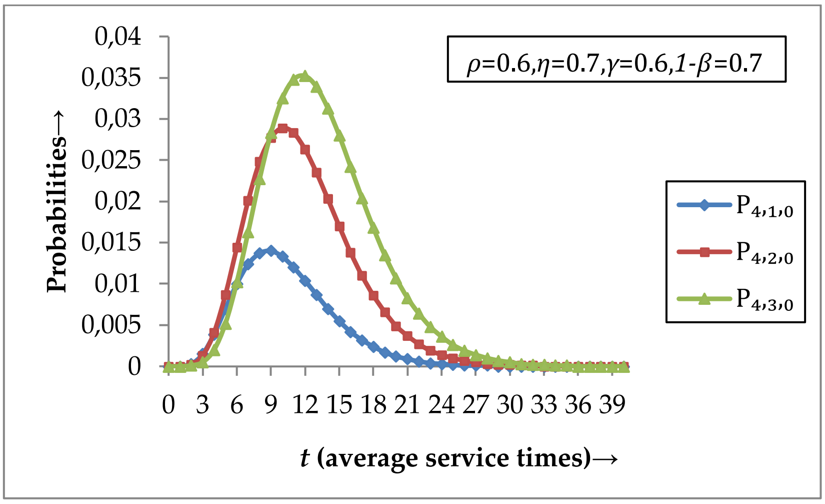

Figure 3: *Probabilities $P_{4,1,0}$, $P_{4,2,0}$ and $P_{4,3,0}$ against t (average service times)*

In Figure 3, the probabilities $P_{4,1,0}$, $P_{4,2,0}$ and $P_{4,3,0}$ are plotted against time t for the case where ρ=0.6, η=0.7, γ=0.6 and 1-β=0.7. It can be observed from the plot that the probabilities increase initially and then decrease gradually. In general, the probabilities are higher for larger number of departures.

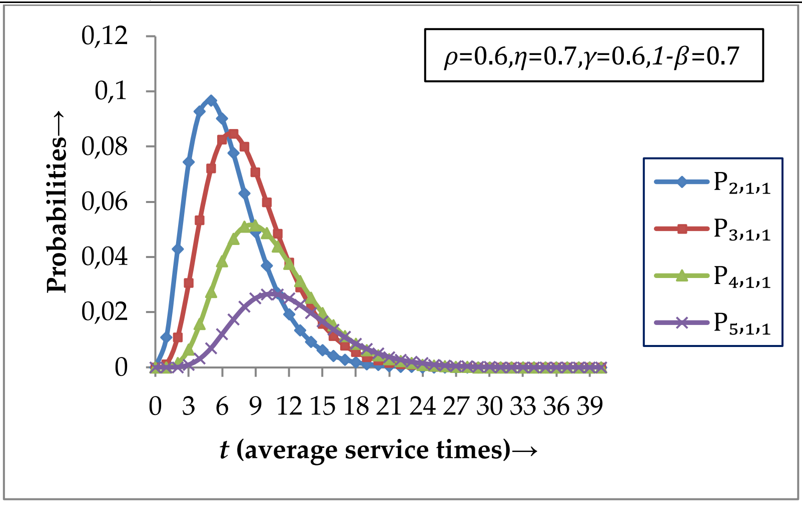

Figure 4: *Probabilities $P_{2,1,1}$, $P_{3,1,1}$, $P_{4,1,1}$ and $P_{5,1,1}$ against t (average service times)*

Figure 4 shows the comparison between the probabilities $P_{2,1,1}$, $P_{3,1,1}$, $P_{4,1,1}$ and $P_{5,1,1}$ when plotted against time t. It is interpreted from the graph that these probabilities increase initially from the value 0 at t=0. Highest achieved values of various probabilities are higher for lower i (number of arrivals). After reaching their respective peaks, probabilities start decreasing and the trend gets reversed i.e., now the probabilities take higher values for larger i (number of arrivals).

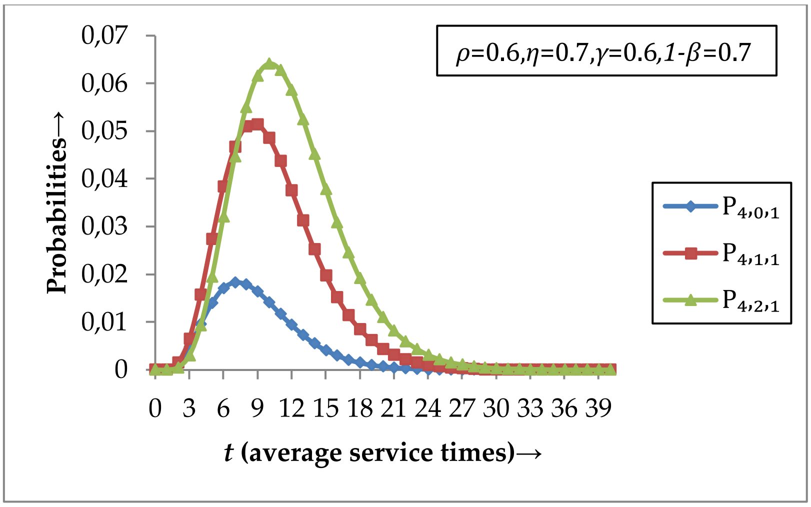

Figure 5: *Probabilities $P_{4,0,1}$, $P_{4,1,1}$ and $P_{4,2,1}$ against t (average service times)*

The probabilities $P_{4,0,1}$, $P_{4,1,1}$ and $P_{4,2,1}$ are plotted against time t in Figure 5. Beginning with value 0 at t=0, all the probabilities increase rapidly to their highest values and then decrease gradually. Also, it is observed that $P_{4,2,1}$ is higher than $P_{4,1,1}$ which is in turn greater than $P_{4,0,1}$ i.e., probabilities are higher for larger j (number of departures).

VI. Busy Period Probabilities

The probability that server is busy is given by

$$P(Server\ is\ busy) = \sum_{i>j\geq 0} P_{i,j,1}(t) \tag{33}$$

The probability that system is busy is given by

$$P(System\ is\ busy) = \sum_{i>j\geq 0} (P_{i,j,0}(t) + P_{i,j,1}(t)) \tag{34}$$

Numerical and Graphical Representation of Busy Period Probabilities:

Following the work of [14] and using MATLAB programming, the numerical results are obtained. Here the probabilities for system busy as well as for server busy are obtained which are presented in the Table 6 below for various values of ρ keeping values of η, γ and 1-β constant.

Table 6: *Probabilities of System busy and Server busy to study the effect of ρ*

t	Probability(System Busy)			Probability(Server Busy)		
	ρ=0.3	ρ=0.6	ρ=0.9	ρ=0.3	ρ=0.6	ρ=0.9
0	0	0	0	0	0	0
1	0.2231	0.3937	0.5245	0.1772	0.3152	0.4232
2	0.3565	0.5774	0.7166	0.2508	0.4159	0.5281
3	0.4455	0.6798	0.8081	0.2946	0.4676	0.5764
4	0.5085	0.7438	0.8599	0.3255	0.5019	0.608
5	0.555	0.7872	0.8927	0.3491	0.5275	0.6317
6	0.5905	0.8184	0.915	0.3678	0.5478	0.6506
7	0.6183	0.8416	0.9308	0.383	0.5645	0.6661
8	0.6405	0.8596	0.9424	0.3957	0.5784	0.6786
9	0.6586	0.8738	0.9512	0.4063	0.5903	0.6886
10	0.6735	0.8853	0.958	0.4153	0.6003	0.6959

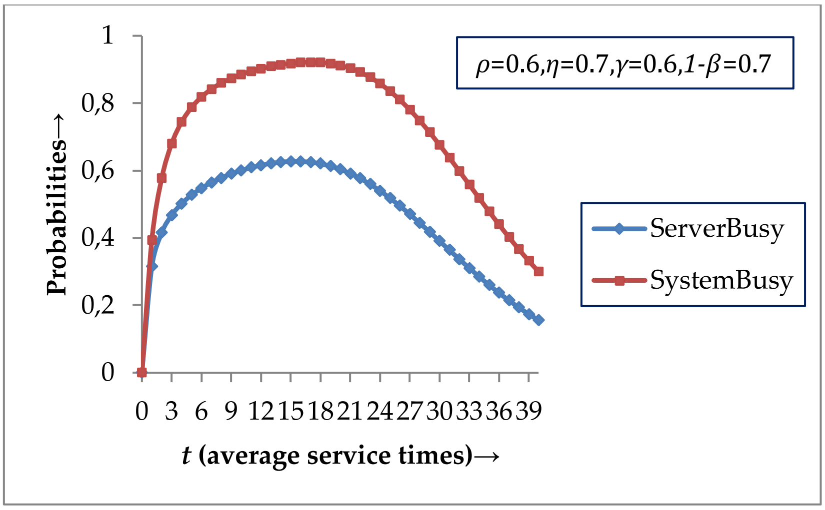

Figure 6: *Probabilities of System busy and Server busy against t (average service times)*

The probabilities for System busy and Server busy are compared in Figure 6 for the case ρ=0.6, η=0.7, γ=0.6 and 1-β=0.7. It is clearly visible that the probabilities for System busy remained higher than that of Server busy throughout, as expected. General trend shows that probabilities start increasing in the beginning, achieve some highest values and then start decreasing.

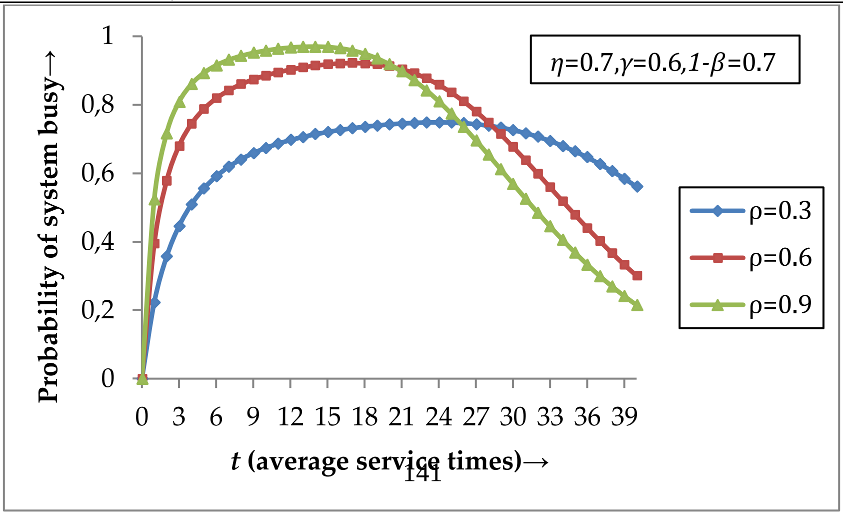

Figure 7: *Effect of ρ on System busy against t (average service times)*

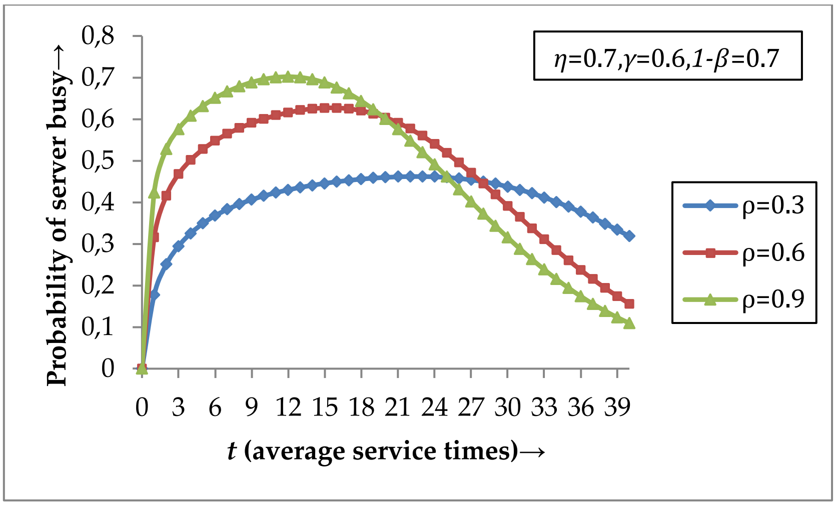

Figure 8: *Effect of ρ on Server busy against t (average service times)*

The effect of changing primary customers traffic intensity i.e., $\rho = \left(\frac{\lambda}{\mu}\right)$ on probability of system busy and probability of server busy is studied through Figure 7 and Figure 8 respectively. In both the graphs the trend followed is similar. The probabilities increases in the beginning and are higher for larger values of ϱ but the trend gets reversed for higher values of *t*.

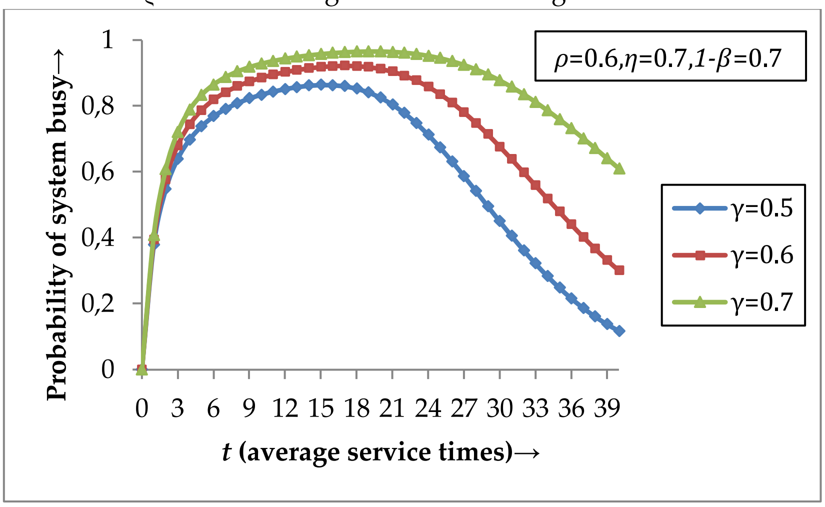

Figure 9: *Effect of γ on System busy against t (average service times)*

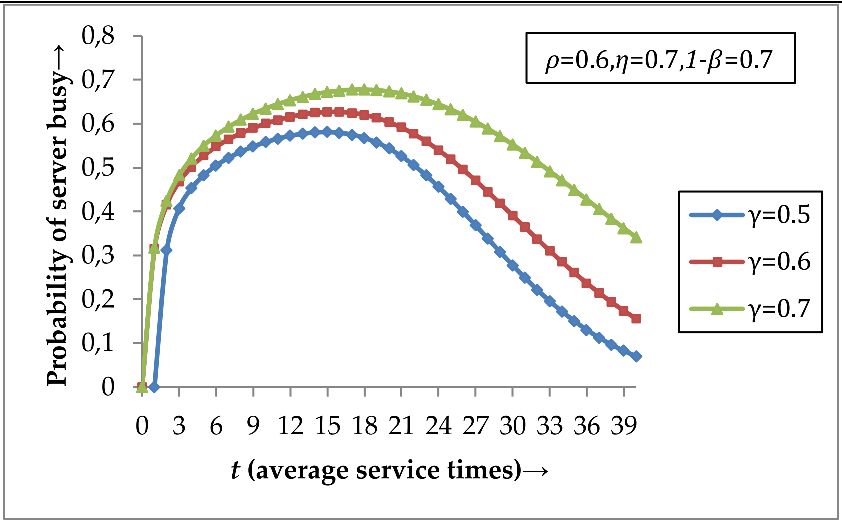

Figure 10: *Effect of γ on Server busy against t (average service times)*

The effect of change of γ (feedback factor) on probability of system busy and server busy is observed through Figures 9 and 10 respectively. It is interpreted from the plots that initially both the probabilities increase rapidly from the value 0 at t=0 and then decrease gradually for higher t. In both the cases, the probabilities are higher for larger value of γ.

VII. Conclusion

In this paper we analyzed a two-state retrial queueing system with feedback and balking. As we know balking is one aspect of impatient customers. Managing impatience leads to profit in business. The time dependent probabilities for exact number of arrivals and exact number of departures from the system are derived. Here due to dealing with two-state probabilities, results are more quantified and informative. Some performance measures are obtained in order to verify results. Numerical results are generated using MATLAB programming. Also, the graphical illustrations are provided in order to understand the effect of change of various parameters. The present model can serve as a base for the future research to model various practical situations applying the concept of balking and feedback where more than one homogeneous or heterogeneous servers would be required.

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