A novel approach for constructing distributions with an example of the Rayleigh distribution

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Abstract

In this paper, we describe a novel technique for creating distributions based on logarithmic functions, which we referred the Log Exponentiated Transformation (LET). The LET technique is then applied to Rayleigh distributions, resulting in a new distribution known as the Log Exponentiated Rayleigh distribution (LERD). Several distributional properties of the formulated distribution have been discussed. The expressions for ageing properties have been derived and discussed explicitly. The behaviour of the pdf, cdf and hazard rate function has been illustrated through different graphs. The parameters are estimated through the technique of MLE. A simulation analysis was conducted to measure the effectiveness of all estimators. Eventually the versatility and the efficacy of the formulated distribution have been examined through real life data set.

Keywords: Log Exponentiated Transformation, Rayleigh distribution, Moments, reliability measures, maximum likelihood function.

Mathematics subject classification: 60-XX, 62-XX, 11-KXX.

I. Introduction

The adoption of an efficient statistical model is critical in a variety of practical analyses. This is especially inconvenient for specific data studies, because the typically employed distributional models are inadequate for producing a plausible fit. Several approaches, such as the generation of families of adaptable distributions, have been presented in recent times. Most of them attempt to increase the effectiveness of a baseline distribution by utilising diverse mathematical expansion approaches. As a result, the related models may incorporate some extra characteristics that provide sufficient flexibility to examine real-life data in many areas of study, such as reliability, survival analysis, computer science, finance, biological research, medicine, and so on. Academics have recently been concerned with developing new techniques for creating new families of distributions so that real data can be adequately analysed and explored. Among them are Marshall and Olkin [9], Eugene et al.[4] , Mudholkar et al. [11], Nadrajah and Kotz [12], Alzaatreh et al. [2], Mahdavi and Kundu [9], Ijaz et al. [8], Anwar Hassan et al.[3]. Based on the argumentation stated above, we suggest a novel family of distributions that adds versatility to the provided family and entitles it Log Exponentiated Transformation (LET). We give a thorough explanation of its fundamental mathematical characteristics, and subsequently employ the Rayleigh distribution as an application.

II. Log Exponentiated Transformation (LET)

This section demonstrates a novel generating family of probability distributions termed as log Exponentiated transformation, abbreviated as LET. If X is a continuous random variable, then the cumulative distribution function (cdf) of the log Exponentiated transformation is described as

$$F(x;\zeta,\theta)=1-\log\left(e+\bar{e}(G(x;\zeta))^{\theta}\right) \quad ;x\in\Re,\theta,\zeta>0 \tag{1}$$

Where $G(x;\zeta)$ denotes the cdf of baseline distribution and $\frac{d(G(x;\zeta))}{dx}=g(x;\zeta)$.

The associated probability density function (pdf) is described as

$$f(x;\zeta,\theta)=\frac{(e-1)\theta g(x;\zeta)(G(x;\zeta))^{\theta-1}}{e+\bar{e}(G(x;\zeta))^{\theta}} \quad ;x\in\Re,\theta,\zeta>0 \tag{2}$$

The survival function $s(x;\zeta,\theta)$, hazard rate function $h(x;\zeta,\theta)$ and cumulative hazard rate function $H(x;\zeta,\theta)$ are stated as respectively

$$s(x;\zeta,\theta)=\log\left(e+\bar{e}(G(x;\zeta))^{\theta}\right)$$

$$h(x;\zeta,\theta)=\frac{(e-1)\theta g(x;\zeta)(G(x;\zeta))^{\theta-1}}{\left(e+\bar{e}G^{\theta}(x;\zeta)\right)\left(\log\left(e+\bar{e}(G(x;\zeta))^{\theta}\right)\right)}$$

$$H(x;\zeta,\theta)=-\log\left(\log\left(e+\bar{e}(G(x;\zeta))^{\theta}\right)\right)$$

III. Mixture Form

This section provides an expression for the mixture form of the probability density function. Equation (2) can be written as

$$\begin{aligned}f(x;\zeta,\theta)&=(e-1)\theta g(x;\zeta)G^{\theta-1}(x;\zeta)\left(e+\bar{e}(G(x;\zeta))^{\theta}\right)^{-1}\\&=\frac{(e-1)}{e}\theta g(x;\zeta)G^{\theta-1}(x;\zeta)\left(1+\frac{\bar{e}}{e}(G(x;\zeta))^{\theta}\right)^{-1}\end{aligned} \tag{3}$$

We know that $(1+z)^{-1}=\sum_{p=0}^{\infty}(-1)^{p}z^{p} \quad ; \; |z|<1$, using it in equation (3), we have

$$f(x;\zeta,\theta)=\frac{(e-1)}{e}\theta\, g(x;\zeta)(G(x;\zeta))^{\theta-1}\sum_{p=0}^{\infty}(-1)^{p}\left(\frac{\bar{e}}{e}\right)^{p}(G(x;\zeta))^{p\theta}$$

After simplification, we obtain the mixture form of pdf as

$$f(x;\zeta,\theta)=\sum_{p=0}^{\infty}(-1)^{p+1}\left(\frac{\bar{e}}{e}\right)^{p+1}\theta g(x;\zeta)(G(x;\zeta))^{(p+1)\theta-1} \tag{4}$$

IV. Log Exponentiated Rayleigh Distribution with properties

The Rayleigh distribution, named after the Lord Rayleigh, is a continuous probability distribution. Due to its wide range of applications, researchers have extended Rayleigh distribution for instance Exponentiated Rayleigh distribution by Voda [13], Weibull-Rayleigh distribution by Faton Merovci et al.[5], transmuted generalized Rayleigh distribution by Faton Merovci [6], Topp-Leone Rayleigh distribution with application by Fatoki O [7] and inverse Weibull Rayleigh distribution by Aijaz et al. [1]. The probability density function (pdf) of Rayleigh distribution with scale parameter α is defined by

$$g(x;\alpha)=\alpha x e^{-\frac{\alpha}{2}x^{2}} \quad ;x>0,\alpha>0 \tag{5}$$

The related cumulative distribution function (cdf) is given by

$$G(x;\alpha)=1-e^{-\frac{\alpha}{2}x^2} \quad ;x>0,\alpha>0 \tag{6}$$

The cumulative distribution function (cdf) of the formulated distribution can be obtained by substituting the value of equation (6) in equation (1), which follows

$$F(x;\alpha,\theta)=1-\log\left(e+\bar{e}\left(1-e^{-\frac{\alpha}{2}x^2}\right)^{\theta}\right) \quad ;x>0,\alpha,\theta>0 \tag{7}$$

The related probability density function is stated as

$$f(x;\alpha,\theta)=\frac{\alpha\theta(e-1)xe^{-\frac{\alpha}{2}x^2}\left(1-e^{-\frac{\alpha}{2}x^2}\right)^{\theta-1}}{e+\bar{e}\left(1-e^{-\frac{\alpha}{2}x^2}\right)^{\theta}} \quad ;x>0,\alpha,\theta>0 \tag{8}$$

Equation (8) may be stated in mixture form by substituting equations (5) and (6) in equation (4).

$$f(x;\alpha,\theta)=\sum_{p=0}^{\infty}(-1)^{p+1}\left(\frac{\bar{e}}{e}\right)^{p+1}\alpha\theta xe^{-\frac{\alpha}{2}x^2}\left(1-e^{-\frac{\alpha}{2}x^2}\right)^{(p+1)\theta-1} \tag{9}$$

Since $(1-z)^{b-1}=\sum_{q=0}^{\infty}(-1)^q\binom{b-1}{q}z^q \;\; ; \;\; |z|<1$, using it in equation (9), we have

$$f(x;\alpha,\theta)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q}\binom{(p+1)\theta-1}{q}\left(\frac{\bar{e}}{e}\right)^{p+1}\alpha\theta xe^{-\frac{\alpha(q+1)}{2}x^2}$$

$$=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}\,\alpha xe^{-\frac{\alpha(q+1)}{2}x^2} \tag{10}$$

Where

$$\delta_{p,q}=(-1)^{p+q}\binom{(p+1)\theta-1}{q}\left(\frac{\bar{e}}{e}\right)^{p+1}\theta$$

Figures (1.1), (1.2), (1.3), and (1.4) depict several probable pdf and cdf layouts of LERD for distinct parameter selections.

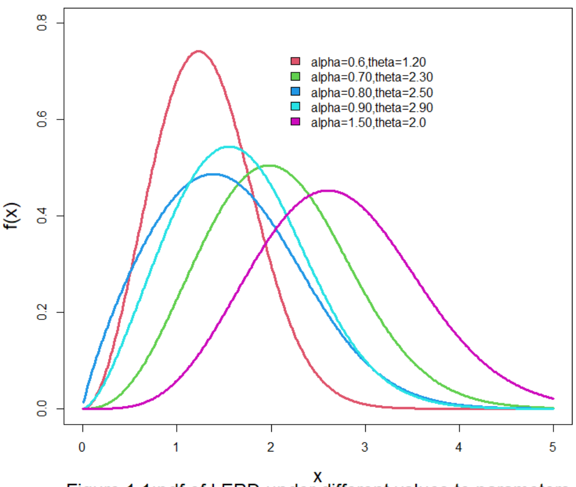


Figure 1.1:pdf of LERD under different values to parameters

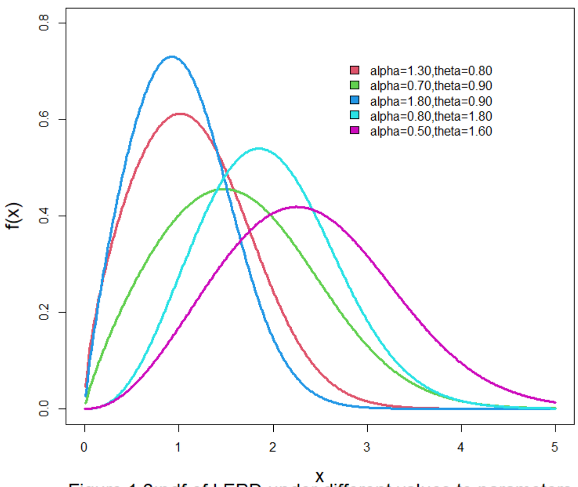


Figure 1.2:pdf of LERD under different values to parameters

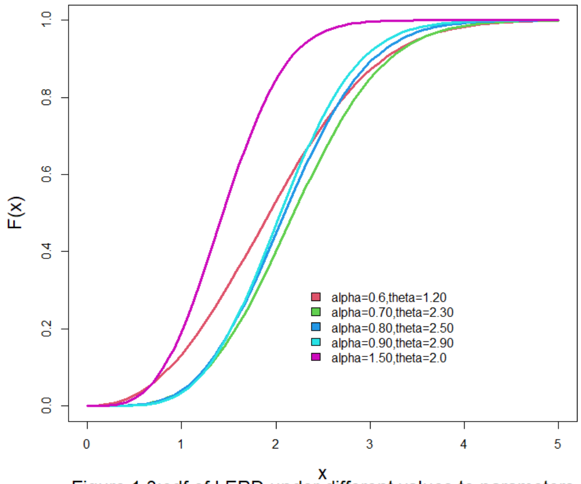


Figure 1.3:cdf of LERD under different values to parameters

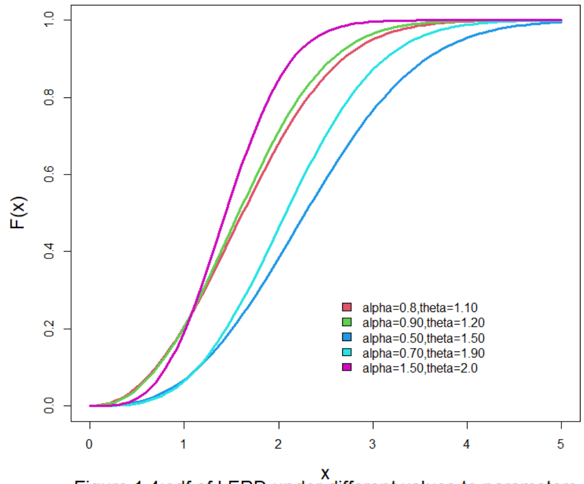


Figure 1.4:cdf of LERD under different values to parameters

V. Mathematical Properties of LER Distribution

I. Moments of LER Distribution

Let suppose X denotes random variable follows LERD. Then r^{th} moment denoted by μ_r is given as

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x;\alpha,\theta)dx$$

Using equation (10), we have

$$\mu_r' = \int_0^\infty x^r \sum_{p=0}^\infty \sum_{q=0}^\infty \delta_{p,q}\, \alpha x e^{-\frac{\alpha(q+1)}{2}x^2} dx$$

$$= \sum_{p=0}^\infty \sum_{q=0}^\infty \delta_{p,q}\, \alpha \int_0^\infty x^{r+1} e^{-\frac{\alpha(q+1)}{2}x^2} dx$$

Making substitution $\frac{\alpha}{2}(q+1)x^2 = z$, so that $0 < z < \infty$, we have

$$\mu_r' = \sum_{p=0}^\infty \sum_{q=0}^\infty \delta_{p,q} \left(\frac{2}{\alpha(q+1)}\right)^{\frac{r}{2}} \frac{1}{(q+1)} \int_0^\infty z^{\frac{r}{2}} e^{-z} dz$$

After solving the integral, we get

$$\mu_r' = \sum_{p=0}^\infty \sum_{q=0}^\infty \delta_{p,q} \frac{\alpha}{2} \left(\frac{2}{\alpha(q+1)}\right)^{\frac{r}{2}+1} \Gamma\left(\frac{r}{2}+1\right)$$

II. Moment Generating Function of LER Distribution

Let X be a random variable follows LERD. Then the moment generating function of the distribution denoted by $M_X(t)$ is given

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x;\alpha,\theta)dx$$

Using Taylor's series

$$= \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots\right) f(x;\alpha,\theta)dx$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} \int_0^\infty x^r f(x;\alpha,\theta)dx$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} E(X^r)$$

$$M_X(t) = \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{r=0}^\infty \frac{t^r}{r!} \delta_{p,q} \frac{\alpha}{2} \left(\frac{2}{\alpha(q+1)}\right)^{\frac{r}{2}+1} \Gamma\left(\frac{r}{2}+1\right)$$

III. Quantile Function of LER Distribution

The quantile function of any distribution may be described as follows:

$$Q(u) = X_q = F^{-1}(u)$$

Where $Q(u)$ denotes the quantile function of $F(x)$ for $u \in (0,1)$.

Let us suppose

$$F(x) = 1 - \log\left(e + \bar{e}\left(1 - e^{-\frac{\alpha}{2}x^2}\right)\right) = u \tag{11}$$

After simplifying equation (11), we obtain quantile function of LER distribution as

$$Q(u) = X_q = \left[-\frac{2}{\lambda}\log\left(1 - \left(\frac{e^{1-u} - e}{e}\right)^{\frac{1}{\theta}}\right)\right]^{\frac{1}{2}}$$

VI. Mean Deviation From Mean and Median of LE R Distribution

The entirety of deviations is apparently a measure of amount of dispersion in a population. Let X be a random variable from LER distribution with mean μ . Then the mean deviation from mean is defined as.

$$D(\mu) = E(|X - \mu|)$$

$$= \int_0^\infty |X - \mu| f(x)dx$$

$$= 2\mu F(\mu) - 2\int_0^\mu x f(x)dx \tag{12}$$

Now

$$\int_0^\mu x f(x)dx = \sum_{p=0}^\infty \sum_{q=0}^\infty \delta_{p,q} \alpha \int_0^\mu x^2 e^{-\frac{\alpha(q+1)}{2}x^2} dx$$

Making substitution $\frac{\alpha(q+1)}{2} = z$ so that $0 < z < \frac{\alpha(q+1)}{2}\mu^2$, we have

$$\int_0^{\mu} xf(x)dx=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}\left(\frac{2}{\alpha(q+1)}\right)^{\frac{1}{2}}\left(\frac{1}{q+1}\right)\int_0^{\frac{\alpha(q+1)}{2}\mu^2} z^{\frac{1}{2}}e^{-z}dz$$

After solving the integral, we have

$$\int_0^{\mu} xf(x)dx=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}\left(\frac{2}{\alpha(q+1)}\right)^{\frac{1}{2}}\left(\frac{1}{q+1}\right)\gamma\left(\frac{3}{2},\frac{\alpha(q+1)}{2}\mu^2\right) \tag{13}$$

Substituting value of equation (7) and (13) in equation (12), we get

$$D(\mu)=2\mu\left(1-\log\left(e+\bar{e}\left(1-e^{-\frac{\alpha}{2}\mu^2}\right)\right)\right)-2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}\left(\frac{2}{\alpha(q+1)}\right)^{\frac{1}{2}}\left(\frac{1}{q+1}\right)\gamma\left(\frac{3}{2},\frac{\alpha(q+1)}{2}\mu^2\right)$$

Let X be a random variable from LER distribution with median M. Then the mean deviation from median is defined as.

$$D(M)=E(|X-M|)=\int_0^{\infty}|X-M|f(x)dx=\mu-2\int_0^{M}xf(x)dx \tag{14}$$

Now

$$\int_0^{M} xf(x)dx=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}\alpha\int_0^{M}x^2e^{-\frac{\alpha(q+1)}{2}x^2}dx$$

Making substitution $\frac{\alpha(q+1)}{2}=z$ so that $0<z<\frac{\alpha(q+1)}{2}M^2$, we have

$$\int_0^{M} xf(x)dx=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}\left(\frac{2}{\alpha(q+1)}\right)^{\frac{1}{2}}\left(\frac{1}{q+1}\right)\int_0^{\frac{\alpha(q+1)}{2}M^2} z^{\frac{1}{2}}e^{-z}dz$$

After solving the integral, we have

$$\int_0^{M} xf(x)dx=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}\left(\frac{2}{\alpha(q+1)}\right)^{\frac{1}{2}}\left(\frac{1}{q+1}\right)\gamma\left(\frac{3}{2},\frac{\alpha(q+1)}{2}M^2\right) \tag{15}$$

Substituting value of equation (15) in equation (14), we get

$$D(M)=\mu-2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}\left(\frac{2}{\alpha(q+1)}\right)^{\frac{1}{2}}\left(\frac{1}{q+1}\right)\gamma\left(\frac{3}{2},\frac{\alpha(q+1)}{2}M^2\right)$$

VII. Ageing Properties of LER Distribution

Suppose X be a continuous random variable with cdf $F(x), x\geq 0$. Then its reliability function which is also known survival function is stated as

$$S(x)=p_r(X>x)=\int_x^{\infty}f(x)dx=1-F(x)$$

Therefore, the survival function for LER distribution is given as

$$\begin{aligned}S(x,\alpha,\theta)&=1-F(x,\alpha,\theta)\\&=\log\left(e+\bar{e}\left(1-e^{-\frac{\alpha}{2}x^2}\right)\right)\end{aligned} \tag{16}$$

The hazard rate function of a random variable x is given as

$$h(x,\alpha,\theta)=\frac{f(x,\alpha,\theta)}{S(x,\alpha,\theta)} \tag{17}$$

Using equation (8) and (16) in equation (17), we have

$$h(x,\alpha,\theta)=\frac{\alpha\theta(e-1)xe^{-\frac{\alpha}{2}x^2}\left(1-e^{-\frac{\alpha}{2}x^2}\right)^{\theta-1}}{\left(e+\bar{e}\left(1-e^{-\frac{\alpha}{2}x^2}\right)^{\theta}\right)\left(\log\left(e+\bar{e}\left(1-e^{-\frac{\alpha}{2}x^2}\right)\right)\right)}$$

Figures (1.5) and (1.6) depict several probable hazard rate function layouts of LERD for distinct parameter selections.

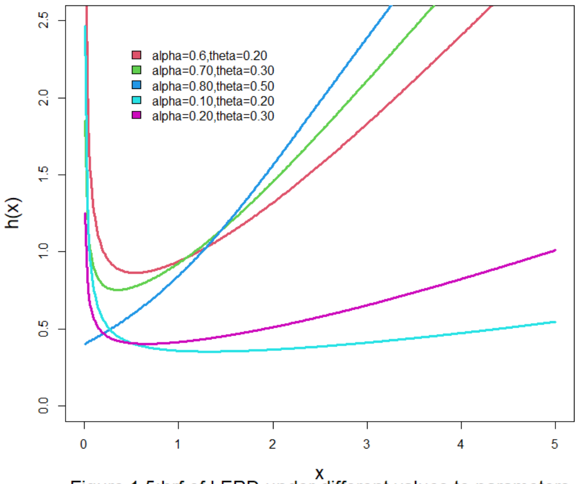


Figure 1.5:hrf of LERD under different values to parameters

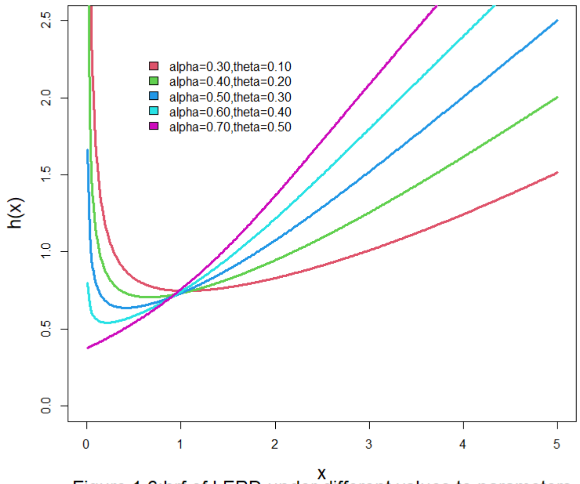


Figure 1.6:hrf of LERD under different values to parameters

The cumulative hazard rate function of a continuous random variable x is defined as

$$H(x,\alpha,\theta)=-\log\left[\bar{F}(x;\alpha,\theta)\right] \tag{18}$$

Using equation (16) in equation (18), we obtain the cumulative hazard rate function of LER distribution

$$H(x,\alpha,\theta)=-\log\left[\log\left(e+\bar{e}\left(1-e^{-\frac{\alpha}{2}x^2}\right)\right)\right]$$

VIII. Renyi Entropy of LER Distribution

If X denotes a continuous random variable having probability density function $f(x)$. Then Renyi entropy is stated as

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\int_0^\infty f^{\delta}(x)dx\right\}, \text{ where } \delta>0 \text{ and } \delta\neq 1$$

Thus, the Renyi entropy of LER distribution is given as

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\int_0^\infty\left(\frac{(e-1)\theta\, g(x;\zeta)(G(x;\zeta))^{\theta-1}}{e+\bar{e}(G(x;\zeta))^{\theta}}\right)^{\delta}dx\right\}$$

$$=\frac{1}{1-\delta}\log\left\{(e-1)^{\delta}e^{-\delta}\theta^{\delta}\int_0^\infty(g(x;\zeta))^{\delta}(G(x;\zeta))^{(\theta-1)\delta}\left(1+\frac{\bar{e}}{e}(G(x;\zeta))^{\theta}\right)^{-\delta}dx\right\} \tag{19}$$

Since $(1+z)^{-b}=\sum_{p=0}^{\infty}(-1)^p\binom{b+p-1}{p}z^p$; $|z|<1$, using it in equation (19), we have

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{(e-1)^\delta e^{-\delta}\theta^\delta\int_0^\infty (g(x;\zeta))^\delta (G(x;\zeta))^{(\theta-1)\delta}\sum_{p=0}^{\infty}(-)^p\binom{p+\delta-1}{p}\left(\frac{\bar{e}}{e}\right)^p (G(x;\zeta))^{p\theta}dx\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\sum_{p=0}^{\infty}(-1)^{p+\delta}\binom{p+\delta-1}{p}\left(\frac{\bar{e}}{e}\right)^{p+\delta}\theta^\delta\int_0^\infty (g(x;\zeta))^\delta (G(x;\zeta))^{\theta(p+\delta)-\delta}dx\right\}$$

Using equation (5) and (6), we have

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\sum_{p=0}^{\infty}(-1)^{p+\delta}\binom{p+\delta-1}{p}\left(\frac{\bar{e}}{e}\right)^{p+\delta}\theta^\delta\int_0^\infty\left(\alpha x e^{-\frac{\alpha}{2}x^2}\right)^\delta\left(1-e^{-\frac{\alpha}{2}x^2}\right)^{\theta(p+\delta)-\delta}dx\right\} \tag{20}$$

Since $(1-z)^b=\sum_{q=0}^{\infty}(-1)^q\binom{b}{q}z^q$; $|z|<1$, using it in equation (20), we have

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q+\delta}\binom{p+\delta-1}{p}\binom{\theta(p+\delta)-\delta}{q}\left(\frac{\bar{e}}{e}\right)^{p+\delta}\theta^\delta\alpha^\delta\int_0^\infty x^\delta e^{-\frac{\alpha(q+\delta)}{2}x^2}dx\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\omega_{p,q}\int_0^\infty x^\delta e^{-\frac{\alpha(q+\delta)}{2}x^2}dx\right\}$$

Where

$$\omega_{p,q}=(-1)^{p+q+\delta}\binom{p+\delta-1}{p}\binom{\theta(p+\delta)-\delta}{q}\left(\frac{\bar{e}}{e}\right)^{p+\delta}\theta^\delta\alpha^\delta$$

Making substitution $\frac{\alpha(q+\delta)}{2}x^2=z$ so that $0<z<\infty$, we have

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\omega_{p,q}\frac{2^{\frac{\delta-1}{2}}}{(\alpha(q+\delta))^{\frac{\delta+1}{2}}}\int_0^\infty z^{\frac{\delta+1}{2}-1}e^{-z}dz\right\}$$

After solving the integral, we get

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\omega_{p,q}\frac{2^{\frac{\delta-1}{2}}}{(\alpha(q+\delta))^{\frac{\delta+1}{2}}}\Gamma\left(\frac{\delta+1}{2}\right)\right\}$$

IX. Maximum Likelihood Estimation of LER Distribution

Let $X_1, X_2, .., X_n$ be a random sample of size n from LERD then its likelihood function is given by

$$l=\prod_{i=1}^{n}f(y_i,\alpha,\theta)$$

$$=\prod_{i=1}^{n}\frac{\alpha\theta(e-1)x e^{-\frac{\alpha}{2}x^2}\left(1-e^{-\frac{\alpha}{2}x^2}\right)^{\theta-1}}{e+\bar{e}\left(1-e^{-\frac{\alpha}{2}x^2}\right)^{\theta}}$$

The log likelihood function is given as

$$\log l = n\log\alpha + n\log\theta + n\log(e-1) - \frac{\alpha}{2}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log(x_i)$$
$$+(\theta-1)\sum_{i=1}^{n}\log\left(1-e^{-\frac{\alpha}{2}x_i^2}\right) - \sum_{i=1}^{n}\log\left(e+\overline{e}\left(1-e^{-\frac{\alpha}{2}x_i^2}\right)^{\theta}\right) \tag{21}$$

Differentiate equation (21), partially with respect parameters, we have

$$\frac{\partial \log l}{\partial\alpha} = \frac{n}{\alpha} - \frac{1}{2}\sum_{i=1}^{n} x_i + \alpha(\theta-1)\sum_{i=1}^{n}\frac{x e^{-\frac{\alpha}{2}x_i^2}}{1-e^{-\frac{\alpha}{2}x_i^2}} - \alpha\theta\sum_{i=1}^{n}\frac{\left(1-e^{-\frac{\alpha}{2}x_i^2}\right)^{\theta-1} x e^{-\frac{\alpha}{2}x_i^2}}{e+\overline{e}\left(1-e^{-\frac{\alpha}{2}x_i^2}\right)^{\theta}} \tag{22}$$

$$\frac{\partial \log l}{\partial\alpha} = \frac{n}{\theta} + \sum_{i=1}^{n}\log\left(1-e^{-\frac{\alpha}{2}x_i^2}\right) + \overline{e}\sum_{i=1}^{n}\frac{\left(1-e^{-\frac{\alpha}{2}x_i^2}\right)^{\theta}\log\left(1-e^{-\frac{\alpha}{2}x_i^2}\right)}{e+\overline{e}\left(1-e^{-\frac{\alpha}{2}x_i^2}\right)^{\theta}} \tag{23}$$

The equations (22) and (23) are non-linear equations and hence cannot be expressed in compact form. Therefore to solve these equations explicitly for α and θ is difficult. So we can apply iterative methods such as Newton–Raphson method, secant method, Regula-falsi method etc. The MLE of the parameters denoted as $\hat{\varsigma}(\hat{\alpha}, \hat{\theta})$ of $\varsigma(\alpha,\theta)$ can be obtained by using the above methods.

For interval estimation and hypothesis tests on the model parameters, an information matrix is required. The 2 by 2 observed matrix is

$$I(\varsigma) = \begin{bmatrix} E\left(\frac{\partial^2 \log l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 \log l}{\partial\alpha\partial\theta}\right) \\ E\left(\frac{\partial^2 \log l}{\partial\theta\partial\alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial\theta^2}\right) \end{bmatrix}$$

The elements of above information matrix can obtain by differentiating equations (22) and (23) again partially. Under standard regularity conditions when $n \to \infty$ the distribution of $\hat{\varsigma}$ can be approximated by a multivariate normal $N\left(0, I(\hat{\varsigma})^{-1}\right)$ distribution to construct approximate confidence interval for the parameters.

Hence the approximate $100(1-\psi)\%$ confidence interval for α, θ and λ are respectively given by

$$\hat{\alpha} \pm Z_{\frac{\psi}{2}}\sqrt{I^{-1}_{\alpha\alpha}(\hat{\varsigma})} \text{ and } \hat{\theta} \pm Z_{\frac{\psi}{2}}\sqrt{I^{-1}_{\theta\theta}(\hat{\varsigma})}$$

Where $Z_{\frac{\psi}{2}}$ denotes the ς^{th} percentile of the standard normal distribution.

X. Simulation Analyses

In this segment, a Monte Carlo simulation analysis was performed using R software to evaluate the consistency of the MLE's. This analysis was performed 500 times using sample sizes of n=30, 50, 150, 250,350 and 450 and various parameter combinations (0.5, 0.7) and (0.7, 0.5) created from LERD. In each case, the bias, variance, and mean square errors (MSEs) were calculated. Table 10.1 shows the simulation findings. In particular, we see that, pursuant to the theory, the MSEs and bias decrease as sample size increases.

Table 1: *Average bias, variance and MSEs of 500 simulations of LERD for different parameter combinations.*

Sample Size n	**parameters**	$\alpha = 0.5$, $\theta = 0.7$			$\alpha = 0.7$, $\theta = 0.5$		
		Bias	Variance	MSE	Bias	Variance	MSE
30	α	0.04106	0.01906	0.02075	0.06495	0.04404	0.04826
	θ	0.08050	0.05995	0.06643	0.04658	0.02140	0.02357
50	α	0.01797	0.00826	0.00858	0.03166	0.02274	0.02375
	θ	0.03925	0.02402	0.02556	0.02152	0.01128	0.01175
150	α	0.01085	0.00321	0.00333	0.01693	0.00608	0.00637
	θ	0.01481	0.00729	0.00751	0.01100	0.00305	0.00317
250	α	0.00280	0.00186	0.00187	0.00456	0.00366	0.00368
	θ	0.00702	0.00401	0.00406	0.00280	0.00169	0.00170
350	α	0.00219	0.00102	0.00102	0.00461	0.00271	0.00273
	θ	0.00175	0.00232	0.00232	0.00296	0.00123	0.00124
450	α	0.00309	0.00088	0.00089	-0.0002	0.00200	0.00200
	θ	0.00311	0.00222	0.00223	0.00188	0.00102	0.00103

XI. Data Analysis

This section assesses the effectiveness of the stated distribution using real-world data. We fitted the LER distribution to many other models for comparative purposes, including Weibull distribution (WD), Exponentiated exponential distribution (EED), Frechet distribution (FD), inverse Burr distribution (IBD), Rayleigh distribution (RD) and exponential distribution (EXD).

We will use certain measures to evaluate which of the competitive models is the strongest, including AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion) and HQIC (Hannan-Quinn Information Criterion). Such criteria can be represented mathematically by

$$AIC = 2k - 2\ln l \qquad CAIC = \frac{2kn}{n-k-1} - 2\ln l$$

$$BIC = k\ln n - 2\ln l \text{ and } HQIC = 2k\ln(\ln(n)) - 2\ln l$$

We compute Anderson-Darling (A*), Cramer-Von Misses (W*), Kolmogorov-Smirnov Statistic, and P-value in addition to the aforementioned goodness of measures. The model with the lowest value of these indicators and the greatest p-value is considered the best among the competing models.

Data Set: The data set was originally reported by Bader and Priest (1982), on failure stresses (in GPa) of 65 single carbon fibres of lengths 50 mm, respectively. The data set is given as follows 1.339,1.434,1.549,1.574,1.589,1.613,1.746,1.753,1.764,1.807,1.812,1.84,1.852,1.852,1.862,1.864,1.931,1.952,1.974,2.019,2.051,2.055,2.058,2.088,2.125,2.162,2.171,2.172,2.18,2.194,2.211,2.27,2.272,2.28,2.299,2.308,2.335,2.349,2.356,2.386,2.39,2.41,2.43,2.458,2.471,2.497,2.514,2.558,2.577,2.593,2.601,2.604,2.62,2.633,2.67,2.682,2.699,2.705,2.735,2.785,3.02,3.042, 3.116, 3.174.

Table 2: *The descriptive statistics for data set*

Min	Q_1	Med.	Mean	Q_3	Kurt.	Skew.	Max
1.339	1.914	2.271	2.241	2.563	2.5270	0.0419	3.174

Table 3: *The ML Estimates for data set (standard error in parenthesis)*

Model	ML Estimates		Standard Error	
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
LERD	1.3473	12.660	0.1356	3.7652
WD	0.0059	5.8363	0.0022	0.4026
EED	2.3310	115.52	0.2045	46.011
FD	1.9940	4.9923	0.0530	0.4439
RD	0.3849	……	0.0481	……
IBD	5.0822	34.299	0.4311	9.5300
EXD	0.4462	…….	0.0557	……

Table 4: *Comparison criterion and goodness of fit statistics for data set*

Model	$-2\log l$	AIC	CAIC	BIC	HQIC
LERD	69.712	73.712	73.909	78.030	75.413
WD	70.756	74.756	74.952	79.073	76.457
EED	76.657	80.657	80.853	84.974	82.358
FD	86.443	90.443	90.642	94.761	92.144
RD	149.168	151.16	151.23	153.32	152.01
IBD	85.506	89.506	89.702	93.824	91.207
EXD	231.29	233.29	233.35	235.45	234.14

Table 5: *Other goodness of fit statistics criterion for data set*

Model	W^*	A^*	K-S value	p-value
LERD	0.04714	0.2987	0.0670	0.9357
WD	0.0590	0.3836	0.0787	0.9181
EED	0.1173	0.7114	0.1006	0.5363
FD	0.2547	1.5484	0.1221	0.2949
RD	0.0834	0.3266	0.3501	3.054e-07
IBD	0.2428	1.4748	0.1186	0.3288
EXD	0.04735	0.3986	0.4677	1.374e-12

Fig. 1.7 :Estimated pdf's of the fitted models for data set

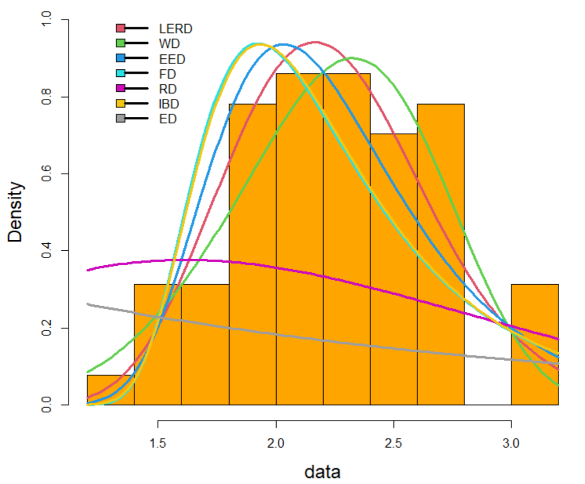
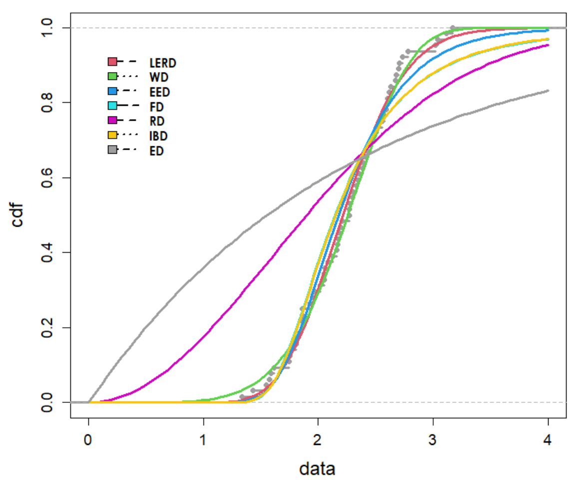


Fig 1.8 Empirical cdf versus fitted cdf's for data set



XII. Conclusions

In this study, a novel technique known as log exponentiated transformation (LET) is suggested. As an illustration, the Rayleigh distribution is employed as the baseline distribution, and a novel two-parameter log exponentiated Rayleigh distribution (LERD) which proved more flexible has been studied. Several mathematical aspects of the newly developed distribution are deduced and analysed. The MLE approach is used to acquire the parameters. From table 8.3 and 8.6 it is evident that the formulated distribution outranks than compared ones.

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