A New Reliability Model and Applications

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Abstract

The Lomax or Pareto Type II distribution has a wide range of applications in many areas including reliability and life testing. In this paper, we modify the Lomax distribution using KM transformation to enhance the applicability of the Lomax distribution. The distribution introduced using KM transformation is parsimonious in parameter. Substituting the cumulative distribution function (cdf) of the Lomax distribution in KM transformation provides a new modified Lomax distribution. The behavior of hazard rate function is studied graphically and also theoretically using Glacer method. Its analytical properties are derived and parameters are estimated using maximum likelihood estimation method. We consider two real data sets to show the flexibility of the proposed model. The model proposed in this paper provides a better fit to the data sets compared to other well-known distributions given in this study.

Keywords: Parsimonious model Lifetime KM transformation Lomax distribution Decreasing failure rate.

1. Introduction

The Lomax distribution has wide applications in many fields like economics, actuarial science, and so on. The Lomax distribution is also called Pareto Type II distribution. The distribution was introduced by Lomax [13] and it is a heavy-tailed distribution. It has also been useful in reliability and life testing problems in engineering and survival analysis as an alternative distribution [[9], [11]]. The Lomax distribution shows decreasing failure rate. Modified and extended versions of the Lomax distribution have been studied; examples include the weighted Lomax distribution [11], exponential Lomax distribution [7], exponentiated Lomax distribution [19], gamma Lomax distribution [5], transmuted Lomax distribution [3], Poisson Lomax distribution [2], McDonald Lomax distribution [12], Weibull Lomax distribution [21], power Lomax distribution [18], Kumaraswamy-Generalized Lomax distribution [20], Gompertz-Lomax distribution [16], and DUS-Lomax distribution [6]. Besides, estimation of the parameters of Lomax distribution under general progressive censoring has been considered by Al-Zahrani and Al-Sobhi [1].

The principal objective of the study is to introduce a modified Lomax distribution which is parsimonious in parameter and enhance the application of the Lomax distribution in reliability theory and survival analysis. We try to improve the properties of the Lomax distribution as a useful lifetime model.

We organize the paper as follows: In Section 2, we introduce a new life distribution using the Lomax distribution as the baseline distribution in the KM transformation. We then discuss the analytical characteristics of the new distribution in Section 3. In Section 4, we establish the ordering of the new distribution. In section 5, the parametric estimation for the new distribution is studied. We carry out an analysis using a real-life data set to illustrate the model's flexibility in Section 6. In section 7, we summarize the conclusions and outline our future works.

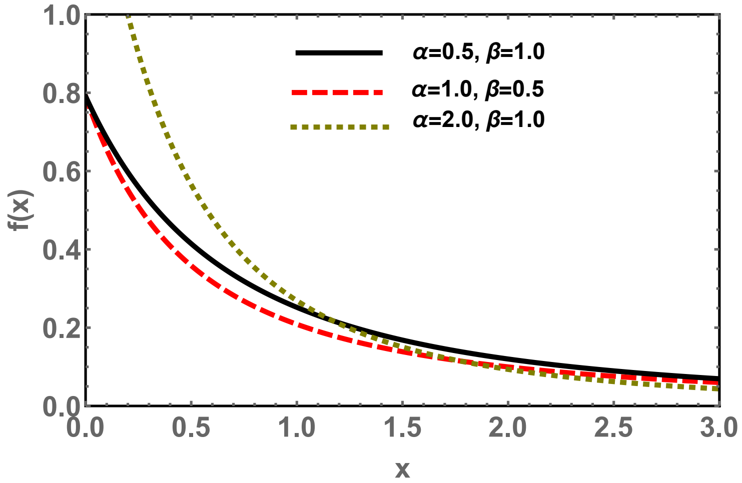

Figure 1: *Probability density plot*

2. The modified Lomax model

In this paper we have modified the Lomax distribution with cumulative distribution function (cdf)

$$G(x) = 1 - (1 + \beta x)^{-\alpha}, \quad x > 0, \ \alpha, \ \beta \ > 0, \tag{1}$$

using KM (Kavya and Manoharan) transformation introduced by Kavya and Manoharan [10]. Let X be a random variable with cdf $G(x)$ and probability density function (pdf) $g(x)$ of some baseline distribution. Then the cdf $F(x)$ of new distribution is defined as,

$$F(x) = \frac{e}{e-1}[1 - e^{-G(x)}]. \tag{2}$$

Here we introduce a new distribution by substituting the cdf of Lomax distribution (1) in (2). The cdf and pdf of the new distribution are respectively obtained as

$$F(x) = \frac{e}{e-1}[1 - e^{-(1-(1+\beta x)^{-\alpha})}], \quad x > 0, \ \alpha, \ \beta \ > 0, \tag{3}$$

$$f(x) = \frac{\alpha\beta(1+\beta x)^{-(\alpha+1)} e^{(1+\beta x)^{-\alpha}}}{e-1}, \quad x > 0, \ \alpha, \ \beta \ > 0, \tag{4}$$

(5)

The graphical representation of pdf is given in Fig. 1 for different values of parameters. In the whole paper we used the software MATHEMATICA [23] for plotting the graphs.

3. Hazard rate function of the model

The hazard function is defined as

$$h(x) = \frac{f(x)}{1 - F(x)} \tag{6}$$

The hazard function of the proposed model is obtained as

$$h(x) = \frac{\alpha\beta(1+\beta x)^{-(\alpha+1)} e^{(1+\beta x)^{-\alpha}}}{e^{(1+\beta x)^{-\alpha}} - 1} \tag{7}$$

The shape of the hazard rate function is given in Fig. 2.

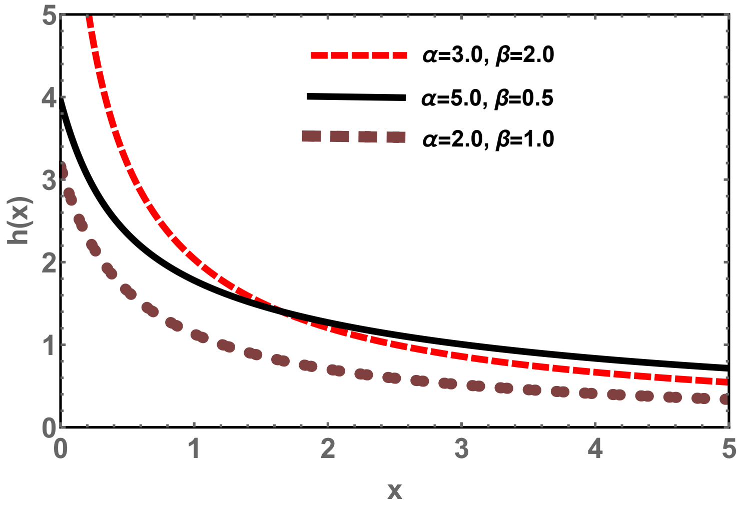

Figure 2: *Hazard rate plot*

3.1. Theoretical explanation of the shape of the hazard rate function

We follow Glaser [8] for theoretical explanation of the shape of the hazard rate. Suppose $f(t)$ is the pdf of some distribution and $f'(t)$ is the first derivative of $f(t)$. Then

$$\nu(t) = \frac{-f'(t)}{f(t)}$$

For our proposed distribution

$$\nu(x) = \beta\left[\alpha + (\alpha+1)(1+\beta x)^{-1}\right]$$

and

$$\nu'(x) = -\beta^2(\alpha+1)(1+\beta x)^{-2} \tag{8}$$

based on Glaser [8] we get a result from Equation (8):
$\nu'(x) < 0$ for all $x > 0$ when $\alpha \geq 0$. Then the distribution has decreasing failure rate (DFR).

4. Some analytical characteristics

Here we discuss some of the analytical characteristics of our proposed distribution.

4.1. Moments

The moments of a random variable, if they exist, are useful for estimating measures of central tendency, dispersion, and shapes. The r^{th} raw moments of the proposed distribution is

$$E(X^r) = \frac{\alpha\beta}{e-1}\int_0^\infty x^r(1+\beta x)^{-(\alpha+1)}e^{(1+\beta x)^{-\alpha}}dx.$$

After transformation, we get,

$$E(X^r) = \frac{1}{\beta^r(e-1)}\int_0^1 (1-u^{\frac{1}{\alpha}})^r u^{-\frac{r}{\alpha}} e^u du,$$

applying binomial expansion, then

$$E(X^r) = \frac{1}{\beta^r(e-1)}\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}\int_0^1 u^{\frac{1}{\alpha}(i-r)}e^u du.$$

Expanding exponential term, and we get the r^{th} raw moment as

$$E(X^r) = \frac{1}{\beta^r(e-1)} \sum_{i=0}^{\infty} \frac{(-1)^i}{j!} \binom{r}{i} \frac{\alpha}{\alpha j + \alpha - r + i}.$$

4.2. Moment generating function

The moment generating function of the proposed distribution is

$$M_X(t) = \frac{\alpha}{(e-1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!\ n!} \frac{t^m}{\beta^m} \frac{(n\alpha + \alpha - 2)!}{(n+\alpha)!} \tag{9}$$

4.3. Characteristic function

The characteristic function of the proposed distribution is obtained as

$$\phi_X(t) = \frac{\alpha}{(e-1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!\ n!} \frac{(it)^m}{\beta^m} \frac{(n\alpha + \alpha - 2)!}{(n+\alpha)!} \tag{10}$$

where $i = \sqrt{-1}$.

4.4. Quantile function

The quantile function is useful when generating random observations from a distribution. It can also be utilized in estimating measures of shapes (skewness and kurtosis) when the moments of the random variable do not exist. The p^{th} quantile function of the proposed distribution is obtained as

$$Q(p) = \frac{1}{\beta} \left[\left(1 + \log(1 - \frac{p(e-1)}{e}) \right)^{\frac{1}{\alpha}} - 1 \right] \tag{11}$$

We can easily find the first, second and third quartile functions after substituting $p = \frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$ in Equation (11).

4.5. Order statistic

Order statistics are important for estimating summary statistics such as the minimum, maximum, and range of a data set. They are also used in quality control testing and reliability to forecast failure of future items based on the times of few early failures. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the proposed distribution and $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ denote the corresponding order statistics. The pdf of the r^{th} order statistic $f_r(x)$ is given by

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!} F^{r-1}(x)[1 - F(x)]^{n-r} f(x),$$

and the pdf of the r^{th} order statistic of our proposed model is obtained as

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!} \frac{\alpha\beta}{(e-1)^n} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+j}}{j!k!} \binom{(r-1)}{j}$$
$$[i(1-(1+\beta x)^{-\alpha})]^j (1+\beta x)^{-(\alpha k + \alpha + 1)} \left(e^{(1+\beta x)^{-\alpha}} - 1 \right)^{n-r} \tag{12}$$

Substitute $r = 1$ and $r = n$ in Equation (12), we get the pdf of the smallest and the largest order statistics respectively.

The cdf of the r^{th} order statistic is

$$F_r(x) = \sum_{j=r}^{n} \binom{n}{j} F^j(x)[1 - F(x)]^{n-j},$$

and the cdf $F_r(x)$ of r^{th} order statistic of the new distribution is obtained by using the Equation (3) as,

$$F_r(x) = \sum_{j=r}^{n} \binom{n}{j} \frac{e^j}{(e-1)^n} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+k}}{k!} \binom{j}{i} [j(1-(1+\beta x)^{-\alpha})]^k \left(e^{(1+\beta x)^{-\alpha}} - 1\right)^{n-j}. \quad (13)$$

The cdf of $X_{(1)}$ and $X_{(n)}$ are obtained by putting $r = 1$ and $r = n$ respectively in Equation (13).

5. Ordering

Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behavior. There are different types of stochastic orderings that are useful in ordering random variables in terms of different properties. Here we consider four different stochastic orders, namely, the usual, the hazard rate, the mean residual life, and likelihood ratio order for KM-Lomax random variables. If X and Y are two random variables with cumulative distribution functions F_X and F_Y, respectively, then X is said to be smaller than Y in the

- stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$ for all x
- hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x
- mean residual life order ($X \leq_{mrl} Y$) if $m_X(x) \geq m_Y(x)$ for all x
- likelihood ratio order ($X \leq_{lr} Y$) if $\frac{f_X(x)}{f_Y(x)}$ decreases in x

The implication between the ordering is $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \Rightarrow X \leq_{st} Y$. The KM-Lomax distribution is ordered with respect to the strongest "likelihood ratio" ordering as shown in the following theorem. It shows the flexibility of the proposed distribution.

Theorem 1. Let $X \sim$ KML(α_1, β_1) and $Y \sim$ KML(α_2, β_2) if $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 \geq \beta_2$ and if $\beta_1 = \beta_2 = \beta$ and $\alpha_1 \geq \alpha_2$, then $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Proof. The likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1\beta_1(1+\beta_1 x)^{-(\alpha_1+1)} e^{(1+\beta_1 x)^{-\alpha_1}}}{\alpha_2\beta_2(1+\beta_2 x)^{-(\alpha_2+1)} e^{(1+\beta_2 x)^{-\alpha_2}}} \quad (14)$$

and

$$\log\frac{f_X(x)}{f_Y(x)} = \log\alpha_1 + \log\beta_1 - (\alpha_1+1)\log(1+\beta_1 x) + (1+\beta_1 x)^{-\alpha_1}$$
$$- \log\alpha_2 - \log\beta_2 + (\alpha_2+1)\log(1+\beta_2 x) - (1+\beta_2 x)^{-\alpha_2}$$

thus,

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = -(\alpha_1+1)\frac{\beta_1}{1+\beta_1 x} - \alpha_1\beta_1(1+\beta_1 x)^{-(\alpha_1+1)}$$
$$+ (\alpha_2+1)\frac{\beta_2}{1+\beta_2 x} + \alpha_2\beta_2(1+\beta_2 x)^{-(\alpha_2+1)} \quad (15)$$

1. Case I: $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 \geq \beta_2$
$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} \leq 0 \Rightarrow X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.
2. Case II: $\beta_1 = \beta_2 = \beta$, $\alpha_1 \geq \alpha_2$
$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} \leq 0 \Rightarrow X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

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6. Estimation of the parameters of the model

In this section we estimate the parameters involved in the distribution using maximum likelihood estimation method. This is one of the most popular methods used for estimation. The likelihood function is defined as,

$$L(x;\lambda) = \prod_{i=1}^{n} f(x_i, \lambda)$$

In our distribution,

$$L(x;\alpha,\beta) = \left(\frac{\alpha\beta}{e-1}\right)^n \prod_{i=0}^{n}(1+\beta x_i)^{-(\alpha+1)} e^{\sum_{i=0}^{n}(1+\beta x_i)^{-\alpha}}.$$

The log-likelihood function of the distribution is given by,

$$\log\ L(x;\alpha,\beta) = -n\ \log(e-1) + n\ \log\alpha + n\log\beta - (\alpha+1)\sum_{i=1}^{n}\log(1+\beta x_i) + \sum_{i=1}^{n}(1+\beta x_i)^{-\alpha}.$$

We proceed as follows. First we find partial derivatives of the log-likelihood function with respect to the parameters α and β. The partial derivatives are

$$\frac{\partial \log\ L}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log(1+\beta x_i) + \sum_{i=1}^{n}\log(1+\beta x_i)^{-\alpha}\log(1+\beta x_i),$$

and

$$\frac{\partial \log\ L}{\partial\beta} = \frac{n}{\beta} - (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{1+\beta x_i} - \alpha\sum_{i=1}^{n} x_i(1+\beta x_i)^{-(\alpha+1)}.$$

Two non-linear equations can be obtained by equating these partial derivatives to zero, the solutions for which provide the maximum likelihood estimates of the parameters. The Newton-Raphson method can be used to solve this equation with the help of the available statistical packages. We use R [17] language for finding the numerical solution of the non-linear system of equations.

7. Application

In this section we are showing the flexibility of the proposed distribution using two real-life data sets. The first data set is the uncensored data set corresponding to intervals in days between 109 successive coal-mining disasters in Great Britain, for the period 1875-1951, published by Maguire et al. [15] and the data set is given in Table 1. The second data set is of Wheaton River obtained from Choulakian and Stephens [4] and presented in Table 2.

We use AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), HQC (Hannan-Quinn Information Criterion) and K-S (Kolmogorov-Smirnov) test value for the comparison. The distribution which shows minimum AIC, BIC, HQC and K-S test value is the sign of a better fit for the data set. The AIC, BIC and HQC are defined as

$$\text{AIC} = -2\ \log(\widehat{L}) + 2m,$$

$$\text{BIC} = -2\ \log(\widehat{L}) + m\ \log(n),$$

and

$$\text{HQC} = -2\ \log(\widehat{L}) + 2m\ \log(\log(n)),$$

where n is the sample size, m is the number of parameters, and $\widehat{L}$ is the maximum value of the likelihood function for the considered distribution. Here R [17] language is used for all the computation. We compare the proposed distribution with the following distributions,

Table 1: *Flood Level Data.*

1	4	4	7	11	13	15	15	17	18	19	19
20	20	22	23	28	29	31	32	36	37	47	48
49	50	54	54	55	59	59	61	61	66	72	72
75	78	78	81	93	96	99	108	113	114	120	120
120	123	124	129	131	137	145	151	156	171	176	182
188	189	195	203	208	215	217	217	217	224	228	233
255	271	275	275	275	286	291	312	312	312	315	326
326	329	330	336	338	345	348	354	361	364	369	378
390	457	467	498	517	566	644	745	871	1312	1357	1613
1630											

Table 2: *Wheaton River Data.*

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0
12.0	9.3	1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1
2.5	14.4	1.7	37.6	0.6	2.2	39.0	0.3	15.0	11.0
7.3	22.9	1.7	0.1	1.1	0.6	9.0	1.7	7.0	20.1
0.4	2.8	14.1	9.9	10.4	10.7	30.0	3.6	5.6	30.8
13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7	64.0
1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3	9.7	27.5
2.5	27.0								

1. DUS-Lomax distribution Deepthi and Chacko [6] with cdf,

$$F(x) = \frac{1}{(e-1)}\left[e^{(1-(1+\theta x)^{-\alpha})} - 1\right], \qquad x > 0,\ \alpha,\ \theta > 0$$

2. Lomax distribution Lomax [13] with cdf,

$$F(x) = 1 - (1+\theta x)^{-\alpha}, \quad x > 0,\ \alpha,\ \theta > 0$$

3. KM-Exponential (KME) distribution Kavya and Manoharan [10] with cdf,

$$F(x) = \frac{e}{e-1}[1 - e^{-(1-e^{-\lambda x})}], \quad x > 0,\ \lambda > 0$$

4. KM-Weibul (KMW) distribution Kavya and Manoharan [10] with cdf,

$$F(x) = \frac{e}{e-1}[1 - e^{-(1-e^{-(x\beta)^{\alpha}})}], \qquad x > 0, \quad \alpha, \beta > 0$$

5. Weibull distribution Weibull [22] with cdf,

$$F(x) = 1 - e^{-(\beta x)^{\alpha}}, \qquad x > 0, \quad \alpha, \beta > 0$$

The values of AIC, BIC, HQC and K-S test for distributions based on the first data set are given in Table 3.

From Table 2, we can see that the new model shows the lowest AIC, BIC and HQC values among all the distributions considered here. The K-S test value of Lomax distribution is smaller than KM-Lomax distribution. In general we can say that our proposed model shows better fit to the data compared to other distributions given in this study. The plot of empirical cdf along with other cdf of the distributions for the first data set is given in Fig. 3. for a better understanding of

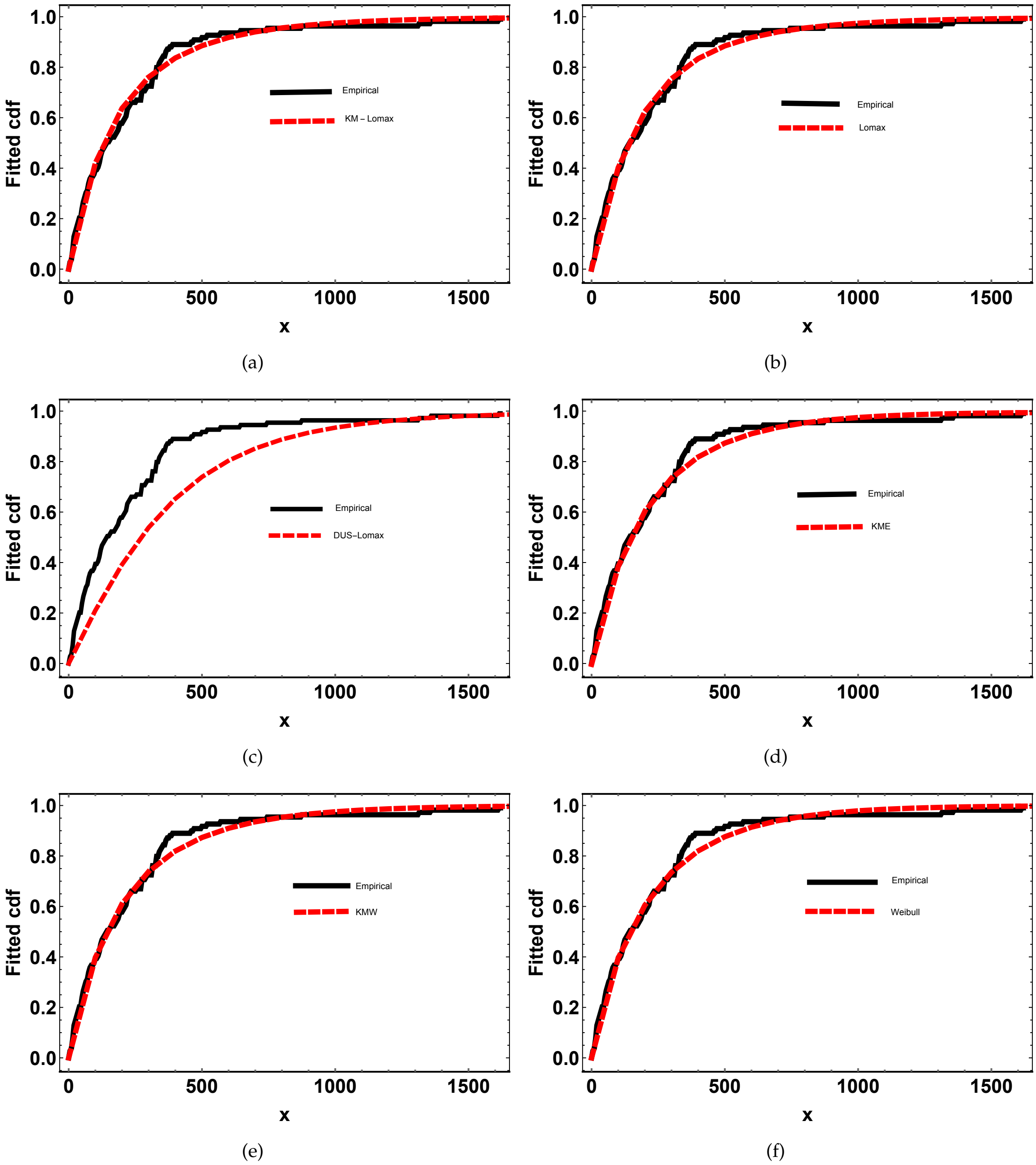

Figure 3: *Comparison plot for the first data set.*

the result. Compared to the distributions mentioned in Mahdavi [14] for this particular data set, the proposed model gives better result.

The comparison table of the considered models for the second data set is given in Table 4.

Based on the AIC, BIC and HQC values, we can conclude that the proposed model gives the best fit to the data set compared to other Lomax, KM family and Weibull distributions considered here. The K-S test value of the KMW distribution is slightly lower than the KM-Lomax distribution. The empirical and fitted cdf plot of distributions considered for the comparison for the second data set is given in Fig. 4.

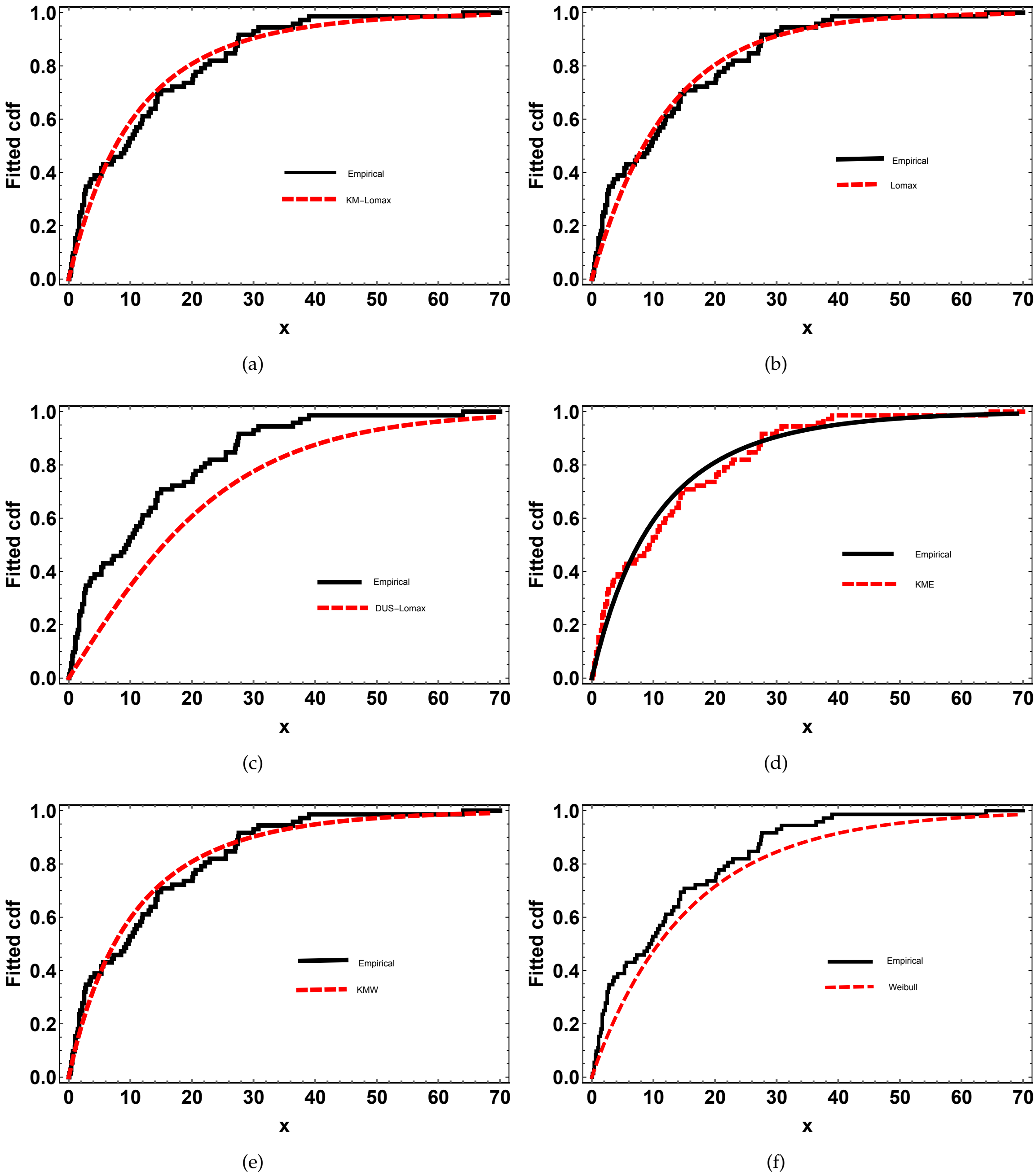

Figure 4: *Comparison plot for the second data set.*

8. Conclusions

In the present work, we introduce a new modified Lomax distribution using KM transformation. The main advantage of the new model is that which is parsimonious in parameter. So we can use the model more conveniently. The behavior of the hazard rate function is studied and the shape of the hazard function is shown both graphically and theoretically. Some of properties of the new model like moments, mgf, characteristic function, quantile function and order statistics are derived. stochastic ordering of the KM Lomax distribution is discussed and established the condition for the stronger mode of ordering viz likelihood ratio. We consider two real data sets to

Table 3: *Maximum likelihood (ML) estimates, K-S test value, AIC, BIC, and HQC of the fitted models.*

Model	ML estimates	K-S value	AIC	BIC	HQC
KM-Lomax	$\widehat{\alpha} = 9.4097, \widehat{\beta} = 0.0004$	0.0720	1040.647	1046.03	1042.83
DUS-Lomax	$\widehat{\alpha} = 9.4008, \widehat{\theta} = 0.0004$	0.2703	1187.936	1193.319	1190.119
Lomax	$\widehat{\alpha} = 4.9251, \widehat{\theta} = 0.0011$	0.0642	1405.426	1410.809	1407.609
KME	$\widehat{\lambda} = 0.0033$	0.0761	1404.864	1407.555	1405.955
KMW	$\widehat{\alpha} = 0.9685, \widehat{\beta} = 0.0033$	0.0772	1406.654	1412.037	1408.837
Weibull	$\widehat{\alpha} = 0.8848, \widehat{\beta} = 0.0046$	0.0784	1407.545	1412.927	1409.728

Table 4: *Maximum likelihood (ML) estimates, K-S test value, AIC, BIC, and HQC of the fitted models.*

Model	ML estimates	K-S value	AIC	BIC	HQC
KM-Lomax	$\widehat{\alpha} = 243.7254, \widehat{\beta} = 0.000258$	0.11	286.6547	291.2081	288.4674
DUS-Lomax	$\widehat{\alpha} = 251.4388, \widehat{\theta} = 0.00025$	0.24	364.0633	368.6166	365.876
Lomax	$\widehat{\alpha} = 80.7719, \widehat{\theta} = 0.00102$	0.14	508.2621	512.8155	510.0748
KME	$\widehat{\lambda} = 0.0632$	0.11	506.025	508.3017	506.9313
KMW	$\widehat{\alpha} = 0.9722, \widehat{\beta} = 0.0633$	0.10	507.9317	512.4851	509.7444
Weibull	$\widehat{\alpha} = 0.9012, \widehat{\beta} = 0.08597$	0.11	506.9973	511.5506	508.81

show the suitability of the model. The proposed model shows better fit to the data sets compared to other models in the literature. We can conclude that the proposed model can be used as a useful lifetime model for decreasing failure rate.

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