RT&A, No 1 (67)
Volume 17, March 2022

The Reliability Performance of the Exponential Inverted Marshall-Olkin-G Family of Distributions: Non-Bayesian Properties and Applications

Joseph Thomas Eghwerido and Eferhonore Efe-Eyefia

Department of Statistics, Federal University of Petroleum Resources Effurun, Delta State, Nigeria
School of Mathematics, Cardiff University, UK
eghwerido.joseph@fupre.edu.ng
efe-eyefiae@cardiff.ac.uk

Abstract

This article introduces a class of generator for enhancing the performance, productivity and flexibility of statistical distributions called the exponential Inverted Marshall-Olkin-G (EMA-G) distribution. The characteristics of the new class of generator were obtained and examined. Some special models of the proposed model were investigated. The Bernstein function of the EMA-G model was also obtained in a closed form. The maximum likelihood method was adopted to obtain the parameters estimate of the formulated EMA-G distribution model. The flexibility, productivity, tractability, applicability and viability of the new contemporary class of distribution were examined by Monte Carlo simulation. A two real life data sets was used to illustrate the empirical performance and flexibility, productivity, tractability of the generator. The up-to-the-minute outcomes of the new generator indicated that the EMA-G density gives a better fit compare to some existing statistical generators in literature using their goodness-of-fit.

Keywords: Bernstein function, Exponential distribution, Generating function, Generator, Marshall-Olkin characterization, Vehicle fatalities.

1. Introduction

Statistical distributions have unraveled the behaviour, characteristics and nature of life time processes. However, these scenarios depend on the flexibility, productivity, performance and tractability of the underlying probability used in analysing these processes. Hence, the performance, productiveness and flexibility can be enhanced either by adding a new parameter or compounding the probability density function (pdf) involved. One of such methods for high productivity is using the T-X family approach called exponential Inverted Marshall-Olkin generator(EMA-G) distribution. This method negates the exponentiated method in existing literature by using the T-X approach in developing the underlying exponential generator.

Despite the emerging statistical generators in literature, newer generators are still being proposed to improve productivity and performance of lifetimes scenarios. However, many models have been proposed in literature. These include the works of [3] who proposed the Gompertz-G model. [17] proposed the logistic-X generator. [8] proposed the Weibull-G generator. [9] proposed the Kumaraswamy-G generator. [10] proposed the alpha power Marshall-Olkin-G generator. [11] proposed the transmuted alpha power-G generator. [14] proposed the alpha power transformation method of adding a new parameter. The beta transmuted-H generator was proposed in [1]. Kumaraswamy Marshall-Olkin generator was proposed in [2]. [4] proposed the transmuted Weibull-G generator. The transmuted odd log-logistic-G generator was proposed in [5]. [6] proposed the log-gamma family of distributions. The exponentiated generalized-G Poisson generator was proposed in [7]. [12] proposed the bivariate Gumbel-G generator. [16] proposed the Topp Leone odd Lindley-G generator. Marshall-Olkin generalized-G generator was proposed in [20]. The transmuted Topp-Leone-G generator was proposed in [18]. Burr-X generator was proposed in [19]. Of most important is the works of [13] who proposed the exponentiated generalized Marshall-Olkin distribution. However, the exponentiated family of distribution is contrary and different from the EMA-G family of distributions.

Thus, [15] proposed a one parameter model for adding a contemporary parameter with a pdf $g(t) = \frac{dG}{dt}$ such that $G(t)$ is associated cdf for a random variable t. Then, its pdf can be expressed as

$$g(t) = \frac{\beta m(t)}{\left[1 - \bar{\beta}\bar{M}(t)\right]^2} \quad for \quad \beta > 1, \tag{1}$$

with a tilt parameter $\bar{\beta} = (1 - \beta)$ and $\bar{M}(t) = 1 - M(t)$. The cumulative distribution function (cdf) that corresponds to Equation (1) is expressed as

$$G(t) = \frac{M(t)}{\left[1 - \bar{\beta}\bar{M}(t)\right]} \quad for \quad \beta > 1. \tag{2}$$

Redefining (1) and (2), we have the inverted Marshall-Olkin cdf and pdf as

$$M(t) = \frac{\beta G(t)}{\left(1 - \bar{\beta}G(t)\right)}. \tag{3}$$

and

$$m(t) = \beta^{-1} g(t)\left(1 - \bar{\beta}M(t)\right)^2 \quad for \quad \beta > 1, \tag{4}$$

where $g(t)$ and $G(t)$ are the parents pdf and cdf.

However, using the T-X characterization proposed by [3], the pdf of the exponential-G can be expressed as

$$g(t) = \alpha f(t)\bar{F}(t)^{\alpha-1} \quad for \quad \alpha > 0. \tag{5}$$

The cdf that corresponds to Equation (3) is expressed as

$$G(t) = 1 - \bar{F}(t)^{\alpha} \quad for \quad \alpha > 0. \tag{6}$$

The study introduces a generator for enhancing the performance and flexibility of distribution with a better goodness-of-fit to real life data. The EMA-G model was applied to the Weibull, Burrxii and Frechet distributions in a bid to investigate their performance and flexibility with glass fibers data obtained from the UK National Physical Laboratory, breaking stress of carbon fiber data and data from the Highway Traffic Safety Administration of accidents fatality rate in the United States real life data (Vehicle fatalities in South Caroline for 2012, www.fars.nhtsa.dot.gov/states). It is also motivated as a result of inefficiency in researched existing literature in distribution theory and some results obtained from Weibull Frechet, Gompertz Weibull, Gompertz Burrxii, transmuted Gompertz, Gompertz Frechet, Kumaraswamy Frechet models and to mention but for a few.

The aim of this study is to introduce a new class of generator called the exponential Marshall-Olkin-G (EMA-G) distribution using both the T-X and Marshall-Olkin characterizations that is different from the exponentiated generalized Marshall-Olkin of [13].

2. The EMA-G method

Let $g(t)$ and $G(t)$ be the pdf and cdf of the Marshall-Olkin distribution respectively for a random variable $T \in \Re$. Then, the pdf of the new class of generator is defined as

$$f(t) = \alpha\beta^{\alpha} \frac{m(t)[1 - M(t)]^{\alpha-1}}{[\beta + (1-\beta)M(t)]^{\alpha+1}}, \quad \alpha > 0 \quad \beta > 1, \tag{7}$$

with a baseline cdf and pdf given as $M(t)$ and $m(t)$ respectively. The cdf of Equation (5) is defined as

$$F(t) = 1 - \left[\frac{\beta(1 - M(t))}{\beta + (1-\beta)M(t)}\right]^{\alpha}, \quad \alpha > 0 \quad \beta > 1. \tag{8}$$

However, when $\beta = 1$, $\alpha > 0$, we obtained the exponential-G family of distribution. Also, when $\beta > 1$, $\alpha = 0$, we obtained the usual Marshall-Olkin transformation of adding a one parameter.

The survival rate, hazard rate, cumulative hazard rate, odd and reversed hazard rate functions of the EMA-G distribution can be expressed respectively as

$$S(t) = \left[\frac{\beta(1 - M(t))}{\beta + (1-\beta)M(t)}\right]^{\alpha},$$

$$h(t) = \frac{\alpha\ m(t)}{\left[1 - M(t)\right]\left[\beta + (1-\beta)M(t)\right]},$$

$$H(t) = -\alpha\left[\log(\beta(1 - M(t))) - \log(\beta + (1-\beta)M(t))\right],$$

$$O(t) = \frac{1 - \left[\frac{\beta(1-M(t))}{\beta+(1-\beta)M(t)}\right]^{\alpha}}{\left[\frac{\beta(1-M(t))}{\beta+(1-\beta)M(t)}\right]^{\alpha}},$$

and

$$r(t) = \frac{\alpha\beta^{\alpha}\frac{m(t)[1-M(t)]^{\alpha-1}}{[\beta+(1-\beta)M(t)]^{\alpha+1}}}{1 - \left[\frac{\beta(1-M(t))}{\beta+(1-\beta)M(t)}\right]^{\alpha}}.$$

The quantile function t_u for a given EMA-G density when $u \in (0, 1)$ is defined as

$$t_u = M^{-1}\beta(1 - (1-u)^{\frac{1}{\alpha}})(\beta + (1-u)^{\frac{1}{\alpha}}(1-\beta))^{-1}. \tag{9}$$

The skewness and kurtosis of the EAP-G density can be obtained respectively as

$$SK(t_u) = \frac{t_{0.25} + t_{0.75} - 2t_{0.5}}{t_{0.75} - t_{0.25}},$$

$$KU(t_u) = \frac{t_{0.875} + t_{0.125} - t_{0.625} - t_{0.375}}{t_{0.75} - t_{0.25}}.$$

The performance of the skewness and kurtosis of the EMA-G models are given in Table 1 with the skewness as (SK), kurtosis denoted as (KU), 25^{th} percent as (Q_1), the median as (M), and the 75^{th} percent as (Q_3) for some EMA-G models. The data set were generated with the quantile function given in Equation (7) with different parameter values cases. The simulation sub-model are Weibull, Burrxii and Frechet. The results of the simulation show that increase in parameter estimates increases the skewness and kurtosis and decreases the median and the quarters for Weibull, Burrxii and Frechect models. However, increase in parameter estimate increases the quarters in Weibull, Frechet and Burrxii models. The EMA-G Burrxii model is left skewed. Otherwise right skewed and the parameter values increases. The kurtosis increases as parameter increases.

Table 1: *Results for goodness-of-fit with skewness, kurtosis, first quantile, median and third quantile for different parament values cases for the EMA-G models*

Distribution	Parameter				SK	KU	Q_1	M	Q_3
	α	β	λ	μ					
Weibull	0.5	1.0	0.5		0.5675	1.6591	0.6620	3.8436	15.3745
		1.5	1.0		0.2136	0.4911	0.8109	1.8325	3.4094
		2.0	2.0	0.5	0.0258	0.0631	1.0384	1.5536	2.0962
		3.5	3.0		-0.0584	-0.0987	1.3414	1.7297	2.0751
		5.0	5.0		-0.1064	-0.1864	1.5269	1.7783	1.9814
		10.0	8.0		-0.1474	-0.2556	1.7290	1.8931	2.0150
	1.5	1.0	0.5		0.5675	1.6591	0.1324	0.7687	3.0748
		1.5	1.0		0.2136	0.4911	0.1621	0.3665	0.6818
		2.0	2.0	2.5	0.0258	0.0631	0.2076	0.3107	0.4192
		3.5	3.0		-0.0584	-0.0987	0.2682	0.3459	0.4150
		5.0	5.0		-0.1064	-0.1864	0.3053	0.3556	0.3962
		10.0	8.0		-0.1474	-0.2556	0.3458	0.3786	0.4030
	2.5	1.0	0.5		0.5675	1.6591	0.0662	0.3843	1.5374
		1.5	1.0		0.2136	0.4911	0.0810	0.1832	0.3409
		2.0	2.0	5.0	0.0258	0.0631	0.1038	0.1553	0.2096
		3.5	3.0		-0.0584	-0.0987	0.1341	0.1729	0.2075
		5.0	5.0		-0.1064	-0.1864	0.1526	0.1778	0.1981
		10.0	8.0		-0.1474	-0.2556	0.1729	0.1893	0.2015
Burrxii	0.5	1.0	0.5		0.9932	257.8495	4.6677	225.00	65025
		1.5	1.0		0.8000	5.1096	0.2500	2.2500	20.25
		2.0	2.0	0.5	0.5992	1.9455	0.0208	0.1240	0.5358
		3.5	3.0		0.5141	1.4381	0.0111	0.0579	0.2035
		5.0	5.0		0.4690	1.2171	0.0028	0.0138	0.0442
		10.0	8.0		0.4122	1.0082	0.0015	0.0070	0.0201
	1.5	1.0	0.5		0.5922	2.6458	1.3608	2.9541	9.1752
		1.5	1.0		0.2162	0.5446	0.7578	1.1760	1.8250
		2.0	2.0	2.5	0.0621	0.1398	0.4609	0.6587	0.8827
		3.5	3.0		0.0099	0.0311	0.4072	0.5656	0.7273
		5.0	5.0		-0.0182	-0.0243	0.3093	0.4247	0.5360
		10.0	8.0		-0.0487	-0.0799	0.2753	0.3711	0.4580
	2.5	1.0	0.5		0.4070	1.2491	1.1665	1.7187	3.0290
		1.5	1.0		0.1094	0.2515	0.8705	1.0844	1.3509
		2.0	2.0	5.0	-0.0182	-0.0287	0.6789	0.8116	0.9395
		3.5	3.0		-0.0617	-0.1093	0.6381	0.7521	0.8528
		5.0	5.0		-0.0859	-0.1521	0.5562	0.6517	0.7321
		10.0	8.0		-0.1112	-0.1945	0.5246	0.6091	0.6767
Frechet	0.5	1.0	0.5		0.9109	16.6310	0.7316	6.0414	120.0417
		1.5	1.0		0.7497	4.5117	0.8285	3.8322	24.8331
		2.0	2.0	0.5	0.6019	2.3041	0.9609	3.1910	12.1653
		3.5	3.0		0.5644	1.9297	1.6578	5.4552	19.0959
		5.0	5.0		0.5134	1.5901	2.2939	6.8833	21.1601
		10.0	8.0		0.5063	1.5110	4.6143	14.4425	44.4367

Table 1 – *Continued from previous page*

Distribution	Parameter				SK	KU	Q_1	M	Q_3
	α	β	λ	μ					
	1.5	1.0	0.5		0.4075	1.2536	0.5395	0.8230	1.4963
		1.5	1.0		0.2640	0.6746	0.9630	1.3082	1.9011
		2.0	2.0	2.5	0.1796	0.4158	1.7272	2.1958	2.8697
		3.5	3.0		0.1465	0.3277	2.6644	3.3811	4.3439
		5.0	5.0		0.1214	0.2635	4.2785	5.3301	6.6723
		10.0	8.0		0.1052	0.2254	7.1662	9.0032	11.2725
	2.5	1.0	0.5		0.2934	0.7643	0.5194	0.6414	0.8649
		1.5	1.0		0.1826	0.4294	0.9813	1.1437	1.3788
		2.0	2.0	5.0	0.1174	0.2589	1.8586	2.0956	2.3957
		3.5	3.0		0.0862	0.1864	2.8272	3.1848	3.6099
		5.0	5.0		0.0663	0.1405	4.6252	5.1624	5.7759
		10.0	8.0		0.0490	0.1042	7.5716	8.4868	9.4963

In Table 1, increase in parameter decreases the skewness, kurtosis and the quartiles with EMA-GWb model.

Theorem 1. The EMA-G density behaviour can be examined by investigating the characteristics of $f(t), W'(t)$ and $W''(t)$; where $W(t) = In\ f(t)$.

Proof. Given that $W(t) = In\ f(t)$, then

$$W(t) = \alpha \log \beta + \log \alpha + \log m(t) + (\alpha - 1)\log(1 - M(t)) - (\alpha + 1)\log(\beta + (1 - \beta)M(t)).$$

Thus,

$$W'(t) = \frac{m'(t)}{m(t)} - \frac{(\alpha - 1)m(t)}{1 - M(t)} - (\alpha + 1)\frac{(1 - \beta)m(t)}{\beta + (1 - \beta)M(t)}.$$

However, $F(t)$ is monotonically decreasing for all t if $W' < 0$ for all t. The mode is obtained when W'' for α, β. More so, if $f(t)''$ changes sign from negative to positive and to negative and again positive as t increases viz-a-viz, then, the pdf of the EMA-G distribution will be bimodal. ■

3. Special models

Some examples of the EMA-G family of distributions will be investigated for various parameter cases. This is to enable us examine the model performance, flexibility and the goodness-of-fit. The models examined include the Weibull (Wb), Frechet (F) and Burrxii (Br) distributions.

3.1. The EMA-GWb distribution

Let T be a random variable with the pdf and cdf (for $t \geq 0$), say $m(t) = \lambda\mu^{\lambda}t^{\lambda-1}exp(-(\mu t)^{\lambda})$ and $M(t) = 1 - exp(-(\mu t)^{\lambda})$ respectively, (for $\mu > 0, \lambda > 0$) of the Weibull density function. Then, the pdf,cdf and hazard rate function of the EMA-GWb distribution for are expressed respectively as

$$f(t) = \alpha\beta^{\alpha}\frac{\lambda\mu^{\lambda}t^{\lambda-1}exp(-(\mu t)^{\lambda})[exp(-(\mu t)^{\lambda})]^{\alpha-1}}{[\beta + (1 - \beta)(1 - exp(-(\mu t)^{\lambda}))]^{\alpha+1}}, \quad \alpha > 0 \quad \beta > 1, \tag{10}$$

The corresponding cdf is defined as

$$F(t) = 1 - \left[\frac{\beta(exp(-(\mu t)^{\lambda}))}{\beta + (1 - \beta)(1 - exp(-(\mu t)^{\lambda}))}\right]^{\alpha}, \quad \alpha > 0 \quad \beta > 1, \tag{11}$$

and

$$h(t) = \frac{\alpha\ \lambda\mu^{\lambda}t^{\lambda-1}exp(-(\mu t)^{\lambda})}{\Big[exp(-(\mu t)^{\lambda})\Big]\Big[\beta + (1-\beta)(1-exp(-(\mu t)^{\lambda}))\Big]}. \tag{12}$$

Figure 1 shows the density functions for the EMA-GWb density for selected values of parameters α, β, λ and μ. The plot in Figure 1 shows that the EMA-GWb density could be increasing, decreasing or skewed depending on the values of the parameters.

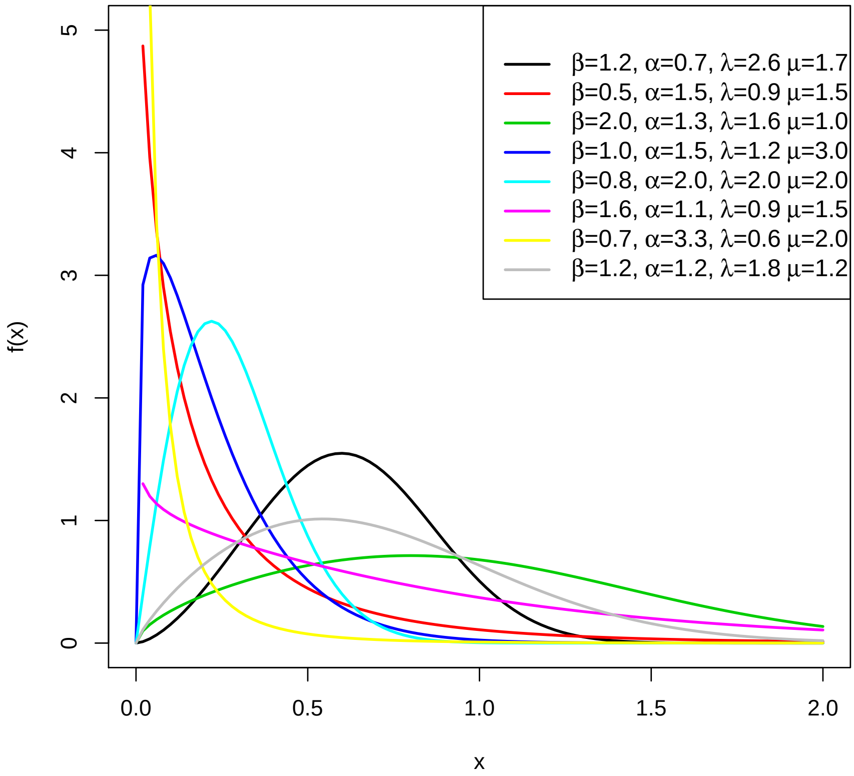

Figure 1: *The plots of the EMA-GWb model for some parameter values cases*

3.2. The EMA-GBr distribution

Let the pdf and cdf say (for $\lambda > 0, \mu > 0$) of the Burrxii density be $m(t) = \lambda\mu t^{\mu-1}(1+t^{\mu})^{-(\lambda+1)}$ and $M(t) = 1-(1+t^{\mu})^{-\lambda}$, respectively. Then, the pdf, cdf and hazard rate function of the EMA-GBr density are expressed respectively as

$$f(t) = \alpha\beta^{\alpha}\frac{\lambda\mu t^{\mu-1}(1+t^{\mu})^{-(\lambda+1)}[(1+t^{\mu})^{-\lambda}]^{\alpha-1}}{[\beta+(1-\beta)(1-(1+t^{\mu})^{-\lambda})]^{\alpha+1}}, \quad \alpha > 0 \quad \beta > 1, \tag{13}$$

The corresponding cdf is defined as

$$F(t) = 1 - \left[\frac{\beta((1+t^{\mu})^{-\lambda})}{\beta+(1-\beta)(1-(1+t^{\mu})^{-\lambda})}\right]^{\alpha}, \quad \alpha > 0 \quad \beta > 1, \tag{14}$$

and

$$h(t) = \frac{\alpha\ \lambda \mu t^{\mu-1}(1+t^{\mu})^{-(\lambda+1)}}{\Big[(1+t^{\mu})^{-\lambda}\Big]\Big[\beta+(1-\beta)M(t)\Big]}. \tag{15}$$

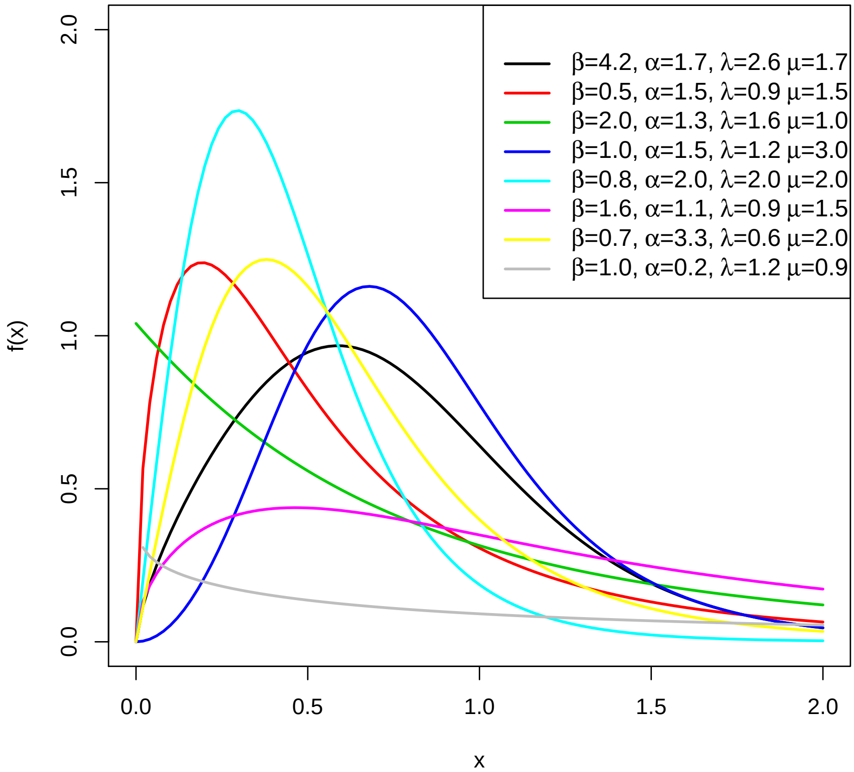

Figure 2: *The plots of the EMA-GBr model for some parameter values cases*

Figure 2 shows the EMA-GBr density plot for selected values of parameters α, β, λ and μ. The plot shows that the EMA-GBr density is increasing and decreasing.

3.3. The EMA-GF distribution

Let the pdf of the Frechet is expressed as $m(t) = \mu\lambda^{\mu}t^{-\mu-1}e^{-(\frac{\lambda}{t})^{\mu}}$ and the cdf as $M(t) = e^{-(\frac{\lambda}{t})^{\mu}}$ for positive parameters λ and μ. Then the pdf, cdf and hazard rate function of the EMA-GFr model $\alpha \in \Re - \{1\}$ are expressed respectively as

$$f(t) = \alpha\beta^{\alpha}\frac{\mu\lambda^{\mu}t^{-\mu-1}e^{-(\frac{\lambda}{t})^{\mu}}[1-e^{-(\frac{\lambda}{t})^{\mu}}]^{\alpha-1}}{[\beta+(1-\beta)e^{-(\frac{\lambda}{t})^{\mu}}]^{\alpha+1}}, \quad \alpha > 0 \quad \beta > 1, \tag{16}$$

The corresponding cdf is defined as

$$F(t) = 1 - \left[\frac{\beta(1-e^{-(\frac{\lambda}{t})^{\mu}})}{\beta+(1-\beta)e^{-(\frac{\lambda}{t})^{\mu}}}\right]^{\alpha}, \quad \alpha > 0 \quad \beta > 1, \tag{17}$$

and

$$h(t) = \frac{\alpha\ \mu\lambda^{\mu}t^{-\mu-1}e^{-(\frac{\lambda}{t})^{\mu}}}{\left[1 - e^{-(\frac{\lambda}{t})^{\mu}}\right]\left[\beta + (1-\beta)e^{-(\frac{\lambda}{t})^{\mu}}\right]}. \tag{18}$$

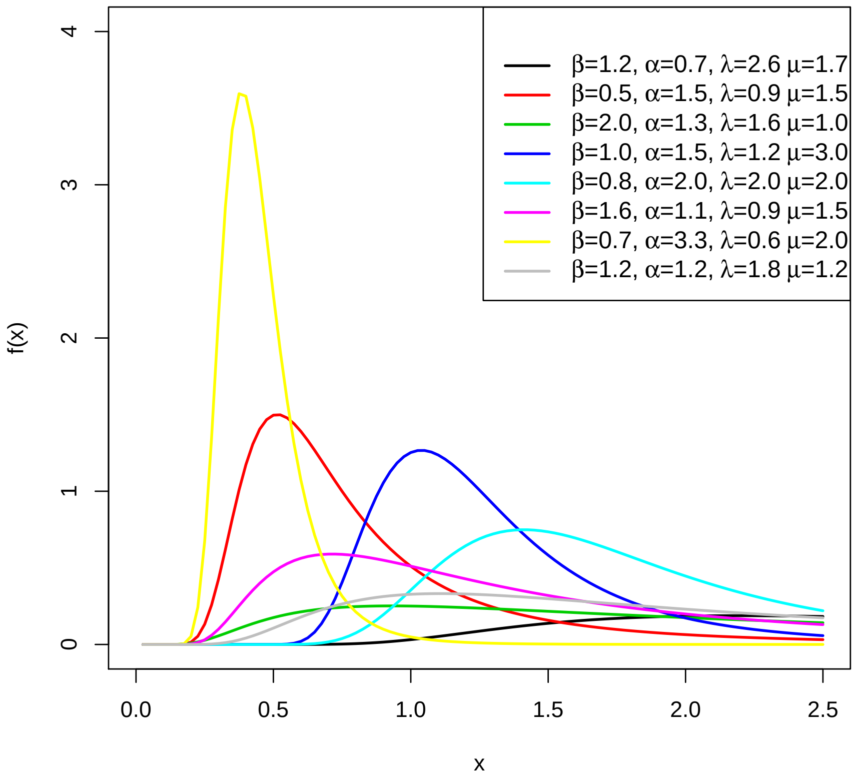

Figure 3: *The plots of the EMA-GF model for some parameter values cases*

Figure 3 shows the density functions for selected values of parameters β, α, λ and μ. The density plot shows that the EMA-GFr distribution can be increasing, decreasing or unimodal and skewed to the left.

4. Statistical useful representation

A useful representation of the EMA-G family of distributions will be derived in this section. The representation is used to study the statistical characteristics of the EMA-G distribution. This representation will help to simplify the properties of the proposed EMA-G model. However, for $\tau > 0$, $(a-b)^{\tau} = \sum_{\eta=0}^{\tau}(-1)^{\eta}\binom{\tau}{\eta}a^{\tau-\eta}b^{\eta}$. Thus, the pdf and cdf of the EMA-G density can be defined respectively as

$$f(t) = \sum_{r=0}^{\infty}\sum_{\eta=0}^{\alpha-1}\alpha(-1)^{\eta+r}\binom{\alpha-1}{\eta}\binom{\alpha+r}{r}\beta^{-r-1}(1-\beta)^{r}m(t)M^{\eta+r}(t) \tag{19}$$

and

$$F(t) = 1 - \sum_{r=0}^{\infty}\sum_{\eta=0}^{\alpha}(-1)^{\eta+r}\binom{\alpha}{\eta}\binom{\alpha+r-1}{r}\beta^{-r}(1-\beta)^{r}M^{\eta+r}(t). \tag{20}$$

RT&A, No 1 (67)
Volume 17, March 2022

5. Statistical properties of the EMA-G distribution

This section investigated the statistical properties of the EMA-G family of distributions. These properties include the moments, generating function, entropies, probability weighted moment, moments of the residual and reversed residual lifes and order statistics.

5.1. The moments of the EMA-G distribution

The k^{th} moment of the EMA-G density with random variable T for is expressed as

$$\begin{aligned}\mu'_k &= \sum_{r=0}^{\infty}\sum_{\eta=0}^{\alpha-1} Avio_{r,\eta}\int_0^{\infty} t^k m(t) M^{\eta+r}(t)dt \\ &= \sum_{r=0}^{\infty}\sum_{\eta=0}^{\alpha-1} Avio_{r,\eta} D_t,\end{aligned} \tag{21}$$

where

$$Avio_{r,\eta} = \alpha(-1)^{\eta+r}\binom{\alpha-1}{\eta}\binom{\alpha+r}{r}\beta^{-r-1}(1-\beta)^r, \quad D_t = \int_0^{\infty} t^k m(t) M^{\eta+r}(t)dt.$$

The mean of Equation (21) is obtained when $k = 1$. The central moment of the random variable T, say μ_ψ and the cumulants (Ko_ψ) of the random variable T can be obtained respectively as

$$\mu_\psi = \sum_{v=0}^{\psi}(-1)^v\binom{\psi}{v}\mu_1'^{v}\mu'_{\psi-r}, \tag{22}$$

and

$$Ko_\psi = \mu'_\psi - \sum_{v=0}^{\psi-1}\binom{\psi-1}{v-1}K_r\mu'_{\psi-r}, \tag{23}$$

with $Ko_1 = \mu'_1$.

The d^{th} incomplete moment for, say $\rho_d(s)$ of the EAP-G density can be obtained as

$$\begin{aligned}\rho_d(s) &= \sum_{r=0}^{\infty}\sum_{\eta=0}^{\alpha-1} Avio_{r,\eta}\int_0^{s} t^s m(t) M^{\eta+r}(t)dt \\ &= \sum_{r=0}^{\infty}\sum_{\eta=0}^{\alpha-1} Avio_{r,\eta} D_s,\end{aligned} \tag{24}$$

where

$$D_s = \int_0^{s} t^s m(t) M^{\eta+r}(t)dt.$$

However, the Bonferroni and Lorenz curve can be obtained respectively as

$$B(p) = \frac{\rho_1(t_p)}{p\mu'_1},$$

and

$$L(p) = \frac{\rho_1(t_p)}{\mu'_1},$$

where t_p is evaluated numerically from the quantile function in Equation (8) for probability p

More so, the mean deviation about the median, say M of T can be obtained as

$$\delta_M = \int_0^{\infty} |T-M| f(t)dt = \mu'_1 - \rho_1(M).$$

Also, the mean deviation about the mean of random variable T can be expressed

$$\delta_\mu = \int_0^{\infty} |T-\mu'_1| f(t)dt = 2\mu'_1 F(\mu'_1) - \rho_1\mu'_1,$$

with $\mu'_1 = E[T]$ and $F(\mu'_1)$ evaluated numerically from Equation (6).

5.2. Probability weighted moments (PWM)

The estimate of the estimators of the parameters and the quantiles of the generalized distributions can be derived using the PWMs of the EMA-G density. The $(s,r)th$ PWM of T, say for $s \geq 0,\ r \geq 1$ is given as

$$\begin{aligned} PWM_{r,s} &= \int_0^\infty t^r f(t) F(t)^s dt \\ &= \sum_{r=0}^{\infty}\sum_{\eta}^{\alpha} B_{r,\eta} Avio_{r,\eta} \int_0^\infty t^s m(t) M^{(\eta+r)(\phi+1)}(t)dt \\ &= \sum_{r=0}^{\infty}\sum_{\eta}^{\alpha} B_{r,\eta} Avio_{r,\eta} Qp_s \end{aligned} \tag{25}$$

where

$$B_{r,\eta} = (-1)^{\eta+r}\binom{\alpha}{\eta}\binom{\alpha+r-1}{r}\beta^{-r}(1-\beta)^r, \quad Qp = \int_0^\infty t^s m(t) M^{(\eta+r)(\phi+1)}(t)dt.$$

5.3. Generating function

The probability generating function of EMA-G density function of a random variable T is expressed as

$$\begin{aligned} M(x) &= \sum_{\delta=0}^{\infty} \frac{(\log x)^\delta}{\delta!} \int_1^\infty t^\delta f(t)dt \quad for \quad |x| > 1, \quad t > 0 \\ &= \sum_{\delta=0}^{\infty}\sum_{r=0}^{\infty}\sum_{\eta=0}^{\alpha-1} \frac{(\log x)^\delta}{\delta!} Avio_{r,\eta} L_\delta, \end{aligned} \tag{26}$$

where

$$L_\delta = \int_1^\infty t^\delta m(t) M^{\eta+r}(t)dt.$$

More so, the moment generating function of the random variable T is given as

$$M_T(x) = \sum_{r=0}^{\infty}\sum_{\eta=0}^{\alpha-1} Avio_{r,\eta} R_\psi \tag{27}$$

where

$$R_\psi = \int_0^\infty e^{tx} m(t) M^{(\eta+r)(\phi+1)}(t)dt.$$

5.4. Moments of the residual life and reversed residual life

The η^{th} moment of the residual life, say $b_\eta(x) = E[(T-x)^\eta | T > x]$ for $\eta = 1, \ldots$ uniquely determines $M(t)$ (see [**?**]). However, the η^{th} moment of the residual life is given as

$$m_\eta(x) = \frac{1}{1-F(x)} \sum_{r=0}^{\infty}\sum_{\eta=0}^{\alpha-1}\sum_{p=0}^{\eta} (-1)^{\eta-p} x^{\eta-p} \binom{\eta}{p} Avio_{r,\eta} \xi_p, \tag{28}$$

where

$$\xi_p = \int_x^\infty t^p m(t) M^{\eta+r}(t)dt.$$

Similarly, the η^{th} moment of the reversed residual life, say $M_\eta(t) = E[(x-T)^\eta | T \leq x]$ for $\eta > 0$, and $\eta = \ldots$ uniquely determines $F(t)$ is given as

$$M_\eta(x) = \frac{1}{F(x)} \sum_{r=0}^{\infty}\sum_{\eta=0}^{\alpha-1}\sum_{u=0}^{\eta} (-1)^{\eta-u} x^{\eta-u} \binom{\eta}{u} Avio_{r,\eta} \vartheta_u, \tag{29}$$

where

$$\vartheta_u = \int_0^x t^u m(t) M^{\eta+r}(t) dt.$$

5.5. Order statistics

Let $T_1, T_2, \ldots, T_\eta$ is a random sample of size η from the $f(t)$ distribution and $T_{(1)}, T_{(2)}, \ldots, T_{(\eta)}$ be the corresponding order statistics. Then, the probability density function of the k^{th} order statistic $T_{(k)}$, say $f_k(t)$ is given as

$$\begin{aligned} f_k(t) =& \frac{\eta!}{(k-1)!(\eta-k)!}\left[1-\left[\frac{\beta(1-M(t))}{\beta+(1-\beta)M(t)}\right]^{\alpha}\right]^{k-1}\left[\left[\frac{\beta(1-M(t))}{\beta+(1-\beta)M(t)}\right]^{\alpha}\right]^{\eta-k} \\ &\times \alpha\beta^{\alpha}\frac{m(t)[1-M(t)]^{\alpha-1}}{[\beta+(1-\beta)M(t)]^{\alpha+1}}. \end{aligned} \tag{30}$$

The minimum order statistics is obtained when $k = 1$, while that of the maximum order statistics is obtained when $k = n$.

5.6. Entropies

The Renyi entropy of the EMA-G random variable T measures the variation of the uncertainty is given as

$$Ren_\delta = \frac{1}{1-\delta}\log\sum_{r=0}^{\infty}\sum_{\eta=0}^{\delta(\alpha-1)}\alpha^{\delta}(-1)^{\eta+r}\binom{\delta(\alpha-1)}{\eta}\binom{\delta(\alpha+1)+r-1}{r}\beta^{-r-1}(1-\beta)^r Q_{Ren}, \tag{31}$$

where $Q_{Ren} = \int_{-\infty}^{\infty} m^{\delta}(t) M^{\eta+r}(t) dt$. $\delta > 0$, $\delta \neq 1$.

The δ entropy $D_\delta(T)$ for $\delta > 0$, $\delta \neq 1$ is expressed as

$$D_\delta = \frac{1}{1-\delta}\log\left[1-\sum_{r=0}^{\infty}\sum_{\eta=0}^{\delta(\alpha-1)}\alpha^{\delta}(-1)^{\eta+r}\binom{\delta(\alpha-1)}{\eta}\binom{\delta(\alpha+1)+r-1}{r}\beta^{-r-1}(1-\beta)^r\right]. \tag{32}$$

6. Estimation

In this section, we shall examine the Bernstein function and maximum likelihood estimation of the EMA-G density. The maximum likelihood estimators of the model performance will be investigated in terms of their means, biases, variance and mean squared errors using the Monte Carlo simulation method. However, real life applications were also provided to examine the flexibility, performance and potential of the EMA-G density.

6.1. The Bernstein estimation

The Bernstein polynomials were developed as a probabilistic proof of the Weierstrass Approximation Theorem (WAT) for continuous function say, $f(t)$ on the closed interval $[a, b]$ is defined in **[?]** as

$$B_\varphi(t, f) = \sum_{\phi=1}^{\varphi} f\left(a + \frac{\phi-1}{\varphi-1}(b-a)\right)\alpha\beta^{\alpha}\frac{m(t)[1-M(t)]^{\alpha-1}}{[\beta+(1-\beta)M(t)]^{\alpha+1}}, \tag{33}$$

which converges to the true function, i.e. $\|B_\varphi(t, f) - f(t)\|_\infty \equiv \sup_{y\leq t\leq z} \mid B_\varphi(t, f) - f(t) \mid \longrightarrow 0$, as $\varphi \longrightarrow \infty$.

However, for $a = 0, b = 1$, and re-scaling Equation (33), we have the pdf of EAP-G model as

$$f_\varphi(t) = \frac{\sum_{\phi=1}^{\varphi} f\left(\frac{\phi-1}{\varphi-1}\right)\alpha\beta^{\alpha}\frac{m(t)[1-M(t)]^{\alpha-1}}{[\beta+(1-\beta)M(t)]^{\alpha+1}}}{\sum_{\phi=1}^{\varphi} f\left(\frac{\phi-1}{\varphi-1}\right)}. \tag{34}$$

6.2. Maximum likelihood estimation

The maximum likelihood method is used to obtain the parameters estimates of the EMA-G density. Let $\mathbf{t} = (t_1, t_2, t_2, \ldots, t_{\eta-1}, t_\eta)$ be a random sample from the EMA-G density with unknown parameter vector ε. Then, the log-likelihood function ℓ of the EMA-G can be expressed as

$$\begin{aligned}\ell =&\eta \log \alpha + \eta\alpha \log \beta + \sum_{i=1}^{\eta} \log m(t) + (\alpha - 1) \sum_{i=1}^{\eta} \log(1 - M(t)) \\ &- (\alpha + 1) \sum_{i=1}^{\eta} \log(\beta + (1 - \beta)M(t)).\end{aligned} \tag{35}$$

The partial derivative of Equation (35) with respect to parameters $\alpha, \beta, \varepsilon$ and equating to zero gives

$$\frac{\partial \ell}{\partial \alpha} = \frac{1}{\eta} + \eta \log \beta + \sum_{i=1}^{\eta} \log(1 - M(t)) - \sum_{i=1}^{\eta} \log(\beta + (1 - \beta)M(t)) = 0, \tag{36}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{\eta\alpha}{\beta} - (\alpha + 1) \sum_{i=1}^{\eta} \frac{1 - M(t)}{\beta + (1 - \beta)M(t)} = 0, \tag{37}$$

and

$$\frac{\partial \ell}{\partial \varepsilon} = \sum_{i=1}^{\eta} \frac{m'_\varepsilon(t)}{m(t)} - (\alpha - 1) \sum_{i=1}^{\eta} \frac{m(t)}{1 - M(t)} + \frac{(\alpha + 1)(1 - \beta)m(t)}{\beta + (1 - \beta)M(t)} = 0. \tag{38}$$

The unknown parameters estimate can obtained by solving the nonlinear Equations in (36), (37) and (38) numerically using the Newton-Raphson algorithm in R, Matlap, Maple and Mathematica.

6.3. Simulations study

In order to examine performance of the EMA-G density, a Monte Carlo simulation is performed and examined. The distributions considered include the Burrxii (Br), Frechet (F) and Weibull (Wb) distributions.

The simulation study is carried out using n sample size by computing their mean estimates (MEs), biases, variance and means squared errors (MSEs) of the maximum likelihood estimate MLEs ($\beta, \alpha, \lambda, \mu$) using Equation (7). The random samples used are $5, 10, 30, 50, 100, 150, 200, 300, 400,$ and 500. The simulation was performed using $(\hat{\beta} = 1.0, \hat{\alpha} = 1.0, \hat{\lambda} = 1.5, \hat{\mu} = 1.5)$. The bias is estimated by (for $Q = \alpha, \beta, \lambda, \mu$)

$$\hat{B}ias_Q = \frac{1}{5000} \sum_{\rho=1}^{5000} \Big(\hat{Q}_\rho - Q \Big).$$

Also, the MSE is obtained as

$$\hat{M}SE_U = \frac{1}{5000} \sum_{\rho=1}^{5000} \Big(\hat{Q}_\rho - Q \Big)^2.$$

The results of the simulation is shown in Table 2. The results indicate that increase in sample sizes decreases the mean, bias, variance and MSE and tends to zero.

7. Real life data application

This section investigated the empirical flexibility and performance of the EMA-G model with a three real life data set. The test statistics of the EMA-GBr was also compared with the Kumaraswamy Burr-XII (KBur), beta Burrxii (BBur), transmuted Burr-XII, lognormal Burr-XII

Table 2: *Monte Carlo simulation results for parameter estimates*

Distribution	n	ME	Bias	Variance	MSE
Weibull	05	1.18, 1.28, 2.07, 1.99	0.78, 0.28, 0.57, 0.30	1.53, 2.02, 2.43, 1.41	1.56, 2.11, 0.76, 1.50
	10	1.26, 1.09, 1.82, 1.54	0.46, 0.09, 0.32, 0.14	1.09, 1.29, 1.20, 1.04	1.16, 1.30, 0.31, 1.07
	30	1.12, 0.96, 1.65, 1.49	0.32, -0.03, 0.15, 0.05	0.57, 0.48, 1.07, 0.46	0.67, 0.49, 0.09, 0.46
	50	1.02, 0.13, 0.61, 1.48	0.32, -0.06, 0.11, 0.02	0.39, 0.31, 0.24, 0.29	0.49, 0.31, 0.06, 0.29
	100	0.68, 0.09, 0.58, 1.47	0.32, -0.09, 0.08, 0.01	0.24, 0.17, 0.09, 0.18	0.35, 0.18, 0.03, 0.18
	150	0.45, 0.08, 0.57, 0.50	0.32, -0.11, 0.07, -0.01	0.18, 0.12, 0.07, 0.12	0.28, 0.13, 0.02, 0.12
	200	0.12, 0.07, 0.57, 0.51	0.31, -0.12, 0.07, -0.01	0.14, 0.09, 0.07, 0.09	0.24, 0.11, 0.01, 0.09
	300	0.02, 0.07, 0.56, 0.51	0.32, -0.12, 0.06, -0.01	0.11, 0.06, 0.06, 0.07	0.21, 0.08, 0.01, 0.07
	400	0.02, 0.07, 0.55, 0.51	0.32, -0.12, 0.05, -0.01	0.09, 0.05, 0.01, 0.05	0.20, 0.07, 0.01, 0.05
	500	0.02, 0.07, 0.55, 0.51	0.32, -0.12, 0.05, -0.01	0.08, 0.04, 0.01, 0.04	0.18, 0.06, 0.01, 0.04
Burrxii	05	1.65, 1.30, 2.18, 0.95	0.65, 0.30, 0.68, -0.54	2.06, 3.38, 0.95, 2.15	2.50, 3.48, 1.42, 2.45
	10	1.62, 1.22, 1.85, 1.15	0.62, 0.22, 0.35, 0.34	1.71, 2.56, 0.32, 1.60	2.10, 2.61, 0.45, 1.72
	30	1.53, 1.08, 1.64, 1.34	0.53, 0.08, 0.14, 0.15	0.80, 0.87, 0.09, 0.75	1.09, 0.88, 0.11, 0.78
	50	0.48, 1.02, 0.60, 0.57	0.48, 0.02, 0.10, 0.12	0.54, 0.57, 0.06, 0.51	0.78, 0.57, 0.07, 0.53
	100	0.44, 0.96, 0.57, 0.41	0.44, -0.03, 0.07, 0.08	0.31, 0.28, 0.03, 0.27	0.34, 0.28, 0.03, 0.28
	150	0.42, 0.94, 0.05, 0.34	0.42, -0.05, 0.05, 0.05	0.21, 0.18, 0.02, 0.17	0.29, 0.19, 0.02, 0.17
	200	0.41, 0.93, 0.05, 0.14	0.41, -0.06, 0.05, 0.05	0.17, 0.14, 0.01, 0.13	0.14, 0.15, 0.01, 0.14
	300	0.41, 0.92, 0.04, 0.08	0.41, -0.07, 0.04, 0.04	0.12, 0.10, 0.01, 0.09	0.09, 0.10, 0.01, 0.09
	400	0.01, 0.92, 0.04, 0.06	0.41, -0.07, 0.04, 0.03	0.10, 0.08, 0.01 0.07	0.07, 0.09, 0.01, 0.07
	500	0.01, 0.92, 0.03, 0.01	0.41, -0.07, 0.03, 0.02	0.08, 0.06, 0.01 0.05	0.05, 0.07, 0.01 0.05
Frechet	05	1.97, 1.33, 1.75, 1.04	0.97, 0.33, 1.25, -0.45	3.21, 5.96, 0.13, 3.19	4.16, 6.07, 0.19, 3.40
	10	1.94, 1.40, 1.63, 1.09	0.94, 0.40, 0.13, 0.40	2.75, 4.74, 0.05, 2.86	3.63, 4.90, 0.07, 3.03
	30	1.72, 1.23, 1.54, 1.25	0.72, 0.23, 0.04, 0.24	1.07, 1.69, 0.01, 1.09	1.60, 1.75, 0.02, 1.15
	50	0.64, 1.15, 0.53, 0.81	0.64, 0.15, 0.03, 0.18	0.61 0.92 0.01 0.59	1.02, 0.95, 0.01, 0.62
	100	0.12, 1.02, 0.51, 0.69	0.52, 0.02, 0.01, 0.10	0.22, 0.30, 0.01, 0.25	0.49, 0.30, 0.01 0.26
	150	0.08, 0.98, 0.51, 0.51	0.48, -0.01, 0.01, 0.08	0.13, 0.19, 0.01, 0.18	0.37, 0.19, 0.01 0.18
	200	0.07, 0.96, 0.51, 0.42	0.47, -0.03, 0.01, 0.07	0.08, 0.12, 0.01 0.13	0.31, 0.12, 0.00 0.13
	300	0.05, 0.94, 0.51, 0.22	0.45, -0.05, 0.01, 0.07	0.05, 0.07, 0.00, 0.09	0.26 0.07, 0.00, 0.09
	400	0.04, 0.93, 0.51, 0.13	0.44, -0.06, 0.01, 0.06	0.04, 0.05, 0.00, 0.06	0.24, 0.05, 0.00, 0.06
	500	0.04, 0.93, 0.51, 0.03	0.44, -0.06, 0.01, 0.06	0.03, 0.04, 0.00, 0.05	0.23, 0.04, 0.00, 0.05

(LogBur) and Gompertz Burrxii distributions. More so, the goodness-of-fit of the EMA-GWb was compared with Gompertz Weibull (GW), alpha power Weibull (APW), transmuted Weibull (TW), Kumaraswamy Weibull and alpha power inverted Weibull (APIW) distributions. Also, the goodness-f-fit of the EMA-GF distribution is compared with the transmuted Marshall-Olkin Frechet (TMFr), Weibull Frechet (WFr), Kumaraswamy Frechet (KFr), exponentiated Frechet (EFr), and Marshall-Olkin Frechet (MFr) distributions. Finally, the goodness-of-fit of the EMA-G models are compared with Exponentiated shifted exponential (ESE), Kumaraswamy Frechet (KFr), gamma extended Frechet (GaFr), Generalized Lindley (GL) , beta Frechet (BFr), Alpha power inverted exponential (APIE) distribution, and the Generalized inverted generalized exponential (GIGE) distributions using the Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Hannan and Quinn Information Criteria (HQIC), Anderson Darling (A), and Cramer-von Mises (W) test statistics.

The first data as used in [**?**], [**?**], [**?**], [**?**], [**?**], [**?**] [10], and [**?**] consist of 63 workers at the UK National Physical Laboratory observations of strength of 1.5cm glass fibers in [**?**]. The results of the test statistics are shown in Table 3.

Table 3: *The statistics rating of the EMA-G distribution with glass fibres dataset with standard errors in parentheses*

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
EMA-GBr	$\hat{\alpha} = 4.15(3.04)$ $\hat{\beta} = 197.58(131.80)$ $\hat{\lambda} = 1.93(0.76)$	**35.89**	**36.58**	**44.46**	**39.26**	**0.19**	**1.06**

Table 3 – *Continued from previous page*

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
	$\hat{\mu} = 3.99(1.14)$						
GBur	$\hat{\alpha} = 15.09(69.05)$ $\hat{\beta} = 36.95(98.24)$ $\hat{a} = 2.06(0.64)$ $\hat{b} = 0.65(0.69)$	37.00	36.70	44.58	39.38	0.17	1.00
KBur	$\hat{a} = 15.52(7.31)$ $\hat{b} = 132.22(145.98)$ $\hat{\alpha} = 1.36(0.57)$ $\hat{\beta} = 1.03(0.30)$	47.20	47.89	55.78	50.57	0.42	2.29
BBur	$\hat{\alpha} = 15.09(69.05)$ $\hat{\beta} = 36.95(98.24)$ $\hat{a} = 2.06(0.64)$ $\hat{b} = 0.65(0.69)$	67.34	68.03	76.00	70.80	0.71	3.86
TBur	$\hat{\alpha} = -0.92(0.11)$ $\hat{\beta} = 0.58(0.14)$ $\hat{\lambda} = 5.80(1.22)$	85.37	85.77	91.80	87.90	0.98	5.33
LoGBur	$\hat{\alpha} = 87.39(260.09)$ $\hat{\beta} = 10.04(13.70)$ $\hat{a} = 10.04(13.71)$ $\hat{b} = 0.37(0.59)$	305.08	305.49	315.49	309.30	32.11	197.6
EMA-GF	$\hat{\alpha} = 23.43(19.43)$ $\hat{\beta} = 0.01(0.01)$ $\hat{\lambda} = 0.73(0.08)$ $\hat{\mu} = 23.52(11.09)$	**31.85**	**32.53**	**36.84**	**33.64**	**0.21**	**1.32**
WFr	$\hat{\alpha} = 0.40(0.81)$ $\hat{\beta} = 0.30(0.30)$ $\hat{a} = 1.49(4.77)$ $\hat{b} = 16.85(20.48)$	38.80	39.48	47.38	42.17	0.25	1.36
KFr	$\hat{\alpha} = 2.12(4.56)$ $\hat{\beta} = 0.74(0.07)$ $\hat{a} = 5.51(7.98)$ $\hat{b} = 857.35(153.94)$	47.63	48.31	56.18	52.84	0.31	0.57
EFr	$\hat{\alpha} = 7.82(2.95)$ $\hat{\beta} = 1.01(0.14)$ $\hat{\mu} = 132.83(116.64)$	50.50	50.70	56.70	52.80	0.31	0.58
TMFr	$\hat{\alpha} = 0.66(0.06)$ $\hat{\beta} = 0.16(0.34)$ $\hat{a} = 6.88(0.61)$ $\hat{b} = 376.27(246.84)$	56.51	57.11	65.10	59.81	0.16	1.29
MFr	$\hat{\beta} = 0.17(0.045)$ $\hat{\gamma} = 6.48(0.56)$ $\hat{\mu} = 161.612(91.50)$	57.11	57.51	63.51	59.61	0.22	2.80
EMA-GW	$\hat{\alpha} = 1.18(0.72)$ $\hat{\beta} = 21.83(6.98)$ $\hat{\lambda} = 0.91(0.25)$ $\hat{\mu} = 2.98(1.22)$	**31.98**	**32.67**	**40.55**	**35.35**	**0.09**	**0.56**
KW	$\hat{\alpha} = 0.55(0.01)$ $\hat{\beta} = 0.23(0.01)$ $\hat{a} = 0.74(0.01)$	35.413	36.11	43.99	38.79	0.16	0.87

Table 3 – *Continued from previous page*

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
	$\hat{b} = 7.10(0.01)$						
TW	$\hat{\alpha} = -0.51(0.28)$ $\hat{\beta} = 0.66(0.04)$ $\hat{\lambda} = 5.17(0.68)$	36.69	37.38	45.26	40.06	0.22	1.13
APW	$\hat{\alpha} = 6.57(8.04)$ $\hat{\beta} = 0.16(0.10)$ $\hat{\lambda} = 4.74(0.82)$	38.19	38.59	44.62	40.72	0.18	0.97
GW	$\hat{\alpha} = 0.23(0.82)$ $\hat{\beta} = 0.01(0.05)$ $\hat{a} = 0.80(0.52)$ $\hat{b} = 5.62(0.51)$	38.38	39.07	46.95	41.75	0.24	1.29
APIW	$\hat{\alpha} = 61.03(48.15)$ $\hat{\beta} = 0.79(0.17)$ $\hat{\lambda} = 3.82(0.30)$	82.59	83.00	89.02	85.13	0.99	5.30

Figures 4 and 5 shows the empirical densities and cdfs with the glass fiber data set for some models. The quantile-quantile plots of some of the models for glass data are shown in Figure 6. However, the plots of the EMA-G models performed favourably when compared to some existing models.

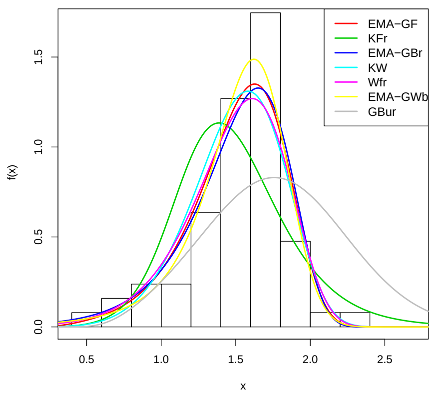

Figure 4: *EMA-G density empirical pdf plots for glass fiber data*

The second data consist of data set obtained from the National Highway Traffic Safety Administration on fatal accidents that occur on roads in the United States. The data represent the number of vehicle fatalities for 39 counties in South Carolina for 2012 (www-fars.nhtsa.dot.gov/States) as used in [**?**]. The test statistics are shown in Table 4. Figures 7 and 8 show the empirical density and cdf of the EMA-G model.

Table 4: *The statistics rating of the EMA-G distribution with vehicle fatalities dataset with standard errors in parentheses*

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
EMA-GWb	$\hat{\alpha} = 0.13(0.04)$ $\hat{\beta} = 3.89(2.85)$ $\hat{\lambda} = 0.41(0.04)$ $\hat{\mu} = 1.02(0.04)$	**314.12**	**315.29**	**320.77**	**316.50**	**0.03**	**0.26**

Table 4 – *Continued from previous page*

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
BW	$\hat{\alpha}=2.45(2.13)$ $\hat{\beta}=1.33(0.01)$ $\hat{a}=0.80(0.16)$ $\hat{b}=12.90(0.01)$	315.56	316.74	322.22	317.95	0.14	0.47
APW	$\hat{\alpha}=0.01(0.02)$ $\hat{\beta}=0.01(0.01)$ $\hat{\lambda}=1.33(0.22)$	316.15	316.84	321.14	317.94	0.15	0.50
TW	$\hat{\alpha}=0.43(0.51)$ $\hat{\beta}=0.04(0.02)$ $\hat{\lambda}=1.34(0.17)$	316.41	317.09	321.40	318.20	0.09	0.87
GW	$\hat{\alpha}=0.01(0.01)$ $\hat{\beta}=4.73(2.46)$ $\hat{a}=0.28(0.58)$ $\hat{b}=0.19(0.04)$	318.15	319.32	324.80	320.53	0.13	0.80
APIW	$\hat{\alpha}=69.69(107.78)$ $\hat{\beta}=4.25(1.88)$ $\hat{\lambda}=1.25(0.14)$	319.74	320.43	324.73	321.53	0.12	0.73
EMA-GBr	$\hat{\alpha}=3.22(3.75)$ $\hat{\beta}=145.95(112.32)$ $\hat{\lambda}=0.40(0.86)$ $\hat{\mu}=3.33(6.82)$	**314.43**	**315.61**	**321.08**	**316.82**	**0.03**	**0.25**
KUBur	$\hat{a}=26.04(53.60)$ $\hat{b}=63.01(0.55)$ $\hat{\alpha}=1.77(2.57)$ $\hat{\beta}=0.23(0.18)$	324.87	326.05	331.52	327.26	0.04	0.85
BBur	$\hat{\alpha}=90.30(214.17)$ $\hat{\beta}=78.59(223.29)$ $\hat{a}=0.81(1.90)$ $\hat{b}=0.18(0.24)$	326.24	327.42	332.89	328.63	0.05	0.85
LoGBur	$\hat{\alpha}=57.63(133.08)$ $\hat{\beta}=33.79(52.95)$ $\hat{a}=1.65(2.58)$ $\hat{b}=0.23(0.31)$	326.13	327.31	332.79	328.52	0.77	0.92
EMA-GF	$\hat{\alpha}=0.01(0.02)$ $\hat{\beta}=2.51(2.23)$ $\hat{\lambda}=0.98(0.98)$ $\hat{\mu}=0.54(0.27)$	**290.95**	**291.37**	**301.37**	**295.16**	**0.01**	**0.02**
WFr	$\hat{\alpha}=4.24(6.48)$ $\hat{\beta}=60.30(65.46)$ $\hat{\lambda}=1.28(0.37)$ $\hat{\mu}=2.20(01.72)$	311.27	312.44	317.92	313.65	0.03	0.26
KFr	$\hat{\alpha}=5.52(0.00)$ $\hat{\beta}=78.42(71.71)$ $\hat{a}=0.26(0.05)$ $\hat{b}=8.09(0.00)$	314.44	315.62	321.10	316.83	0.03	0.24

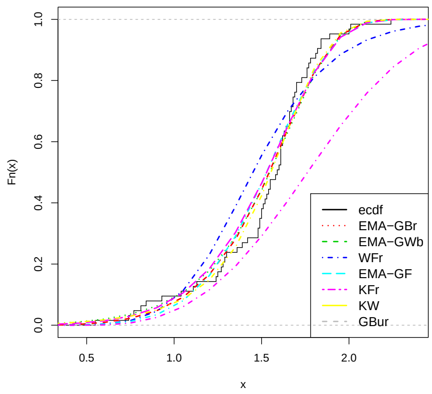

Figure 5: *EMA-G density empirical cdf plots for glass fiber data*

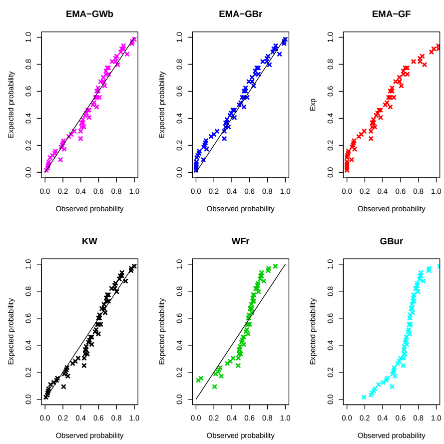

Figure 6: *EMA-G density Q-Q plots for glass data*

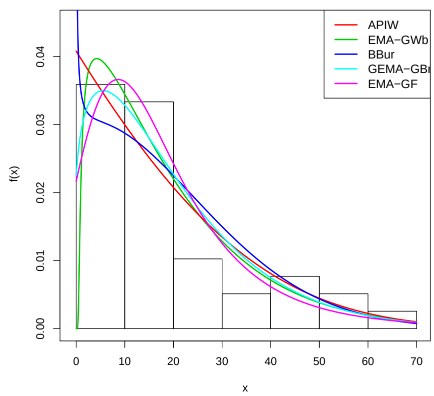

Figure 7: *EMA-G density empirical cdf plots for vehicle fatalities data*

7.1. Discussion

Two real life data sets were used to examine the performance of the EMA-G models. However, a model is said to perform better than another if its value of the lowest Akaike Information Criteria

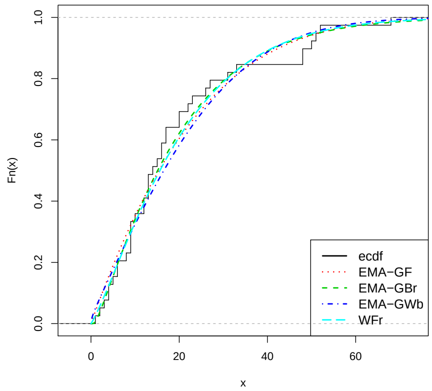

Figure 8: *EMA-G density empirical cdf plots for vehicle fatalities data*

(AIC) is the smallest. However, in the real data cases investigated, the EMA-G distributions have the lowest AIC value in glass fibres data and vehicle fatalities data respectively. Hence, it is said to be better for these data sets under consideration and competes favourably with other existing model for the data used.

8. Conclusion

This study introduces a new class of generator called EMA-G distribution in probability theory. This generator extends the performance of some existing generators like the Gompertz, Weibull, Frechet generators. Basic characteristics of the EMA-G distribution were examined. The EMA-G generator was expressed as a linear form of the baseline distribution. The entropy and PWMs of the proposed distribution were derived. The unknown parameters of the EMA-G density were obtained by maximum likelihood. A simulation study of the EMA-G model was illustrated using the Monte Carlo method. The simulation shows that the shape of the proposed distribution could be skewed, unimodal, increasing or decreasing (depending on the value of the parameters cases). The new distribution was applied to a real life data. It shows that the EMA-G distribution performed better than some existing models in literature like APIE, APIW, GW, TW, APW, KW, BBur, KBur, LoGBur, TMFr, TGGz, TGz, KGz, WFr, TMFr, EFr, and MFr.

References

[1] Afify, A. Z., Yousof, H. M., and Nadarajah, S. (2017). The beta transmuted-H family of distributions: properties and applications. *Stasistics and its Inference*, 10:505–520.

[2] Alizadeh, M., Tahir, M. H., Cordeiro, G. M., Mansoor, M., Zubair, M., and Hamedani, G. G. (2015). The Kumaraswamy Marshal-Olkin family of distributions. *Journal of the Egyptian Mathematical Society*, 23(3):546–557.

[3] Alizadeh, M., Cordeiro, G. M., Pinho, L. G. B., and Ghosh, I. (2017). The Gompertz-G family of distributions. *Journal of Statistical Theory and Practice*, 11(1):179 – 207.

[4] Alizadeh, M., Rasekhi, M., Yousof, H. M., and Hamedani, G. G. (2018). The transmuted Weibull-G family of distributions. *Hacettepe Journal of Mathematics and Statistics*, 47(6): 1–20.

[5] Alizadeh, M., Yousof, H. M., Jahanshahiz, S. M. A., Najibi, S. M., and Hamedani, G. G.(2020). The transmuted odd log-logistic-G family of distributions. *Journal of Statistics and Management systems*, doi:10.1080/09720510.2019.1685228.

[6] Amini, M., Mir-Mostafaee, S. M. T. K., and Ahmadi., J. (2014). Log-gamma- generated families of distributions. *Statistics*, 48(4): 1–20.

[7] Aryal, G. R. and Yousof, H. M. (2017). The exponentiated generalized-G poisson family of distributions. *Economic Quality Control*, 32(1):1–17.

[8] Bourguignon, B. M., Silva, R., and Cordeiro, G. M. (2014). The Weibull-G family of probability distributions. *Journal of Data Science*, 12: 53–68.

[9] Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81(7): 883–898.

[10] Eghwerido, J. T., Oguntunde, P. E., and Agu, F. I. (2021). The alpha power Marshall-olkin-G distribution: properties, and applications. *Sankhya A - The Indian Journal ofStatistics*, https://doi.org/10.1007/s13171-020-00235-y.

[11] Eghwerido, J. T., Zelibe, S. C., and Efe-Eyefia, E. (2020c). The transmuted alpha power-G family of distributions. *Journal of Statistics and Management Systems*, https://doi.org/10.1080/09720510.2020.1794528.

[12] Eliwa, M. S. and El-Morshedy, M. (2018). Bivariate Gumbel-G family of distributions: statistical properties, bayesian and non-bayesian estimation with application. *Annals of Data Science*, doi/10.1007/s40745-018-00190-4.

[13] Handique, L. and Chakraborty, S. and de Andrade, T. A. (2019). The Exponentiated Generalized Marshall-Olkin Family of Distribution: Its Properties and Applications. *Annals of Data Science*, 6(3): 391–411.

[14] Mahdavi, A. and Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. *Communications in Statistics - Theory and Methods*, 46(13): 6543–6557.

[15] Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84: 641–652.

[16] Reyad, H., Alizadeh, M., Jamal, F., and Othman, S. (2018). The Topp Leone odd Lindley-G family of distributions: Properties and applications. *Journal of Statistics and Management Systems*, 21(7):1273–1297.

[17] Tahir, M. H., Cordeiro, G. M., Alzaatreh, A., Mansoor, M., and Zubair, M. (2016). The logistic-X family of distributions and its applications. *Communications in Statistics-Theory and Methods*, 45(24): 7326–7348.

[18] Yousof, H. M., Alizadeh, M., Jahanshahiand, S. M. A., Ramires, T. G., Ghosh, I., and Hamedani, G. G. (2017). The transmuted Topp-Leone-G family of distributions: theory, characterizations and applications. *Journal of Data Science*, 15: 6723–740.

[19] Yousof, H. M., Afify, A. Z., Hamedani, G. G., and Aryal, G. (2017a). The Burr-X generator of distributions for lifetime data. *Journal of Statistical Theory and Applications*, 16:288–305.

[20] Yousof, H. M., Afify, A. Z., Alizadeh, M., Nadarajah, S., Aryal, G. R., and Hamedani,G. G. (2018). The Marshall-Olkin generalized-G family of distributions with applications. *Statistica*, 78(3): 273–295.

APPENDIX

Abbreviations

A = Anderson Darling
AIC = Akaike Information Criteria
APIE = Alpha power inverted exponential
APIW = alpha power inverted Weibull
APW = alpha power Weibull
BBur = beta Burrxii
BFr = beta Frechet
BIC = Bayesian Information Criteria
CAIC = Consistent Akaike Information Criteria
EFr = exponentiated Frechet

RT&A, No 1 (67)
Volume 17, March 2022

EMA-G = Exponential Marshall-Olkin-G
EMA-GBr = Exponential Marshall-Olkin-G Burrxii
EMA-GF = Exponential Marshall-Olkin-G Frechet
EMA-GWb = Exponential Marshall-Olkin-G Weibull
ESE = Exponentiated shifted exponential
GaFr = gamma extended Frechet
GBur = Gompertz Burrxii
GIGE = Generalized inverted generalized exponential
GL = Generalized Lindley
GW = Gompertz Weibull
HQIC = Hannan and Quinn Information Criteria
KBur = Kumaraswamy Burrxii
KFr= Kumaraswamy Frechet
KWb = Kumaraswamy Weibull
LogBur = lognormal Burrxii
MFr = Marshall-Olkin Frechet
TBur = transmuted Burrxii
TMFr = Marshall-Olkin Frechet
TW = transmuted Weibull
W = Cramer-von Mises
WFr = Weibull Frechet
UK = United Kingdom