OPTIMIZATION OF A FEEDBACK WORKING VACATION QUEUE WITH REVERSE BALKING AND REVERSE RENEGING

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Abstract

This paper analyzes a steady-state finite buffer $M/M/1$ feedback queue with reverse balking, reverse reneging and multiple working vacations. The concept of reverse balking and reverse reneging evolves from investment businesses wherein more the number of customers associated with a firm less the probability of balking of a customer and similar is the case of reverse reneging. Furthermore, if a customer is dissatisfied with the service provided, he or she may chose to rejoin the queue as a feedback customer. The server exits for working vacations whenever the system becomes empty instead of staying idle in the system. Vacation times and service times during working vacations are all independent random variables following exponential distribution. The model's steady-state system length distributions are calculated using the matrix approach. Some performance characteristics and cost optimization using ant colony optimization (ACO) are presented. Sensitivity analysis is performed using numerical results which are shown in the form of tables and graphs.

Keywords: reverse balking, reverse reneging, feedback, multiple working vacations, ACO

1. Introduction

Queueing models with server vacations have been actively researched and successfully used in manufacturing and production systems, service systems, communication systems, and other fields over the last three decades. Working vacations (WV) are a form of vacation policy established by Servi and Finn [13] wherein the server can provide service at a reduced rate rather than shutting down altogether during the vacation period. Wu and Takagi [16] and Baba [2] extended the $M/M/1/WV$ queue to $M/G/1$ and $GI/M/1$ queues with working vacations, respectively. Krishnamoorthy and Sreenivasan [8] analyzed an $M/M/2$ queue with one of the two servers in working vacations. A survey on WV queues has been presented by Chandrasekaran et al. [4].

There is a growing trend to examine queueing systems from an economic perspective in order to address customers' unhappiness with waiting and desire for service. Customer impatience has a damaging influence on businesses since it causes them to lose potential consumers, which has a negative impact on the entire company. Balking and reneging are two queueing concepts that are commonly used to depict customer impatience. In balking, if a customer sees a large queue ahead of him, he may resist at joining the queueing system. In the case of reneging, the customer joins the queue, waits for his service, and then departs the system without receiving service if the wait time exceeds his expectations. The situation of impatient customers in a server vacation period was investigated by Altman and Yechiali [1]. Yue, Yue and Xu [17] analyzed the single server queueing systems with customer impatience and WV. A Markovian queueing system with balking, reneging and WV has been studied by Vijaya Laxmi et al. [15].

In the above mentioned queueing models, the size of the system or the length of the queue influences balking and reneging. The larger the system, the more balking occurs, and the same is true with reneging. However, in the case of investment enterprizes, the number of customers with a certain firm becomes an intriguing and appealing feature for potential investors. As a result, the likelihood of joining such a company is high. In this scenario, the larger the system size, the greater the number of consumers who join it. As a result, when the system size is high, the chance of balking is low which is referred to as "reverse balking". Furthermore, having a large number of investors with an investment firm instils trust in investors and helps them to complete the term of their policies/bonds. That instance, when a firm has a big number of investing consumers, waiting customers will have more patience. When seen as a queueing system, it is obvious that as the queue becomes longer, fewer consumers would renege, a phenomenon known as "reverse reneging". Jain et al. [7] first incorporated the concept of reverse balking in queueing theory. Kumar and Som [10] developed the concept of reverse reneging and incorporated into an $M/M/1$ queueing system with reverse balking. A heterogeneous two server queue with reverse balking and reneging has been studied by Bouchentouf and Messabihi [3].

In queueing theory, feedback refers to a dissatisfied client rejoining the queue owing to poor service quality. Rework is another example of a queue with feedback in industrial processes. Tackacs [14] studied a single server queue with feedback to determine the stationary process for the queue size. Shanthakumaran and Thangaraj [12] considered a single server feedback queue with impatient customers. An $M/M/1$ feedback queueing model with retention of reneged customers and balking has been studied by Kumar and Sharma [9]. Kumar et al. [11] developed an $M/M/1/N$ feedback queueing system with reverse balking.

To the best of our knowledge, the impatient attitude of customers in the reverse view has not been explored in working vacations queues. Therefore, we intend to embed reverse balking and reverse reneging in a feedback WV queue. In this article, we explore a finite buffer feedback WV queue in which customers may balk or renege owing to impatience in the reverse notion. The inter-arrival times, service times during regular service period, during WV period and vacation times are presumed to be exponentially distributed. The matrix form solution of the steady-state probabilities is found by putting the steady-state equations in block matrix form. The model's performance metrics, cost analysis using ACO are obtained. Tables and graphs have been used to demonstrate certain numerical findings.

The rest of the paper is laid out as follows. The queueing model is described in Section 2, followed by the steady-state equations and their solution in Section 3. In Section 4, we offer different model performance metrics as well as a cost model. Section 5 contains the sensitivity analysis followed by conclusions in Section 6.

2. Model description

Consider an $M/M/1/N$ feedback queueing system with reverse balking, reverse reneging and WV. According to a Poisson process with an arrival rate λ, customers arrive one at a time. When the system is unoccupied, a new customer has a probability q of joining the system and a $p = (1-q)$ probability of not joining. When there are i customers ahead of him in the system, let b_i indicate the probability that the customer will join the queue or balk with probability $1 - b_i$. Furthermore, we assume that $b_0 = q$ and $b_N = 0$. The assumption of reverse balking has been incorporated with $b_{i+1} > b_i$, $1 \leq i \leq N-1$.

After joining the queue each customer will wait a certain length of time which is exponentially distributed with mean $1/\alpha$. When there are i customers in the system, the average rate of reverse reneging of a customer is given by $(N-(i-1))\alpha,\ ,1 \le i \le N$.

If a customer receives service and finds it unsatisfactory, it can return to the system as a feedback customer with a probability q_1 or depart with a probability $p_1 = 1 - q_1$.

A single server serves the customers on a first-come first-served basis with a service rate that follows an exponential distribution with mean $1/\mu$. When the system gets empty, the server takes WV. If there are waiting customers in the line after a vacation expires, the server resumes regular service; otherwise, he departs for another WV. During the vacation time, the server stays active and provides service at a different service rate to the arriving customers. This type of working vacation is called multiple working vacations (MWV).

The vacation times and service times during WV are assumed to follow Poisson distribution with parameter ϕ and η, respectively. The inter-arrival times, vacation times, service times during regular service and during working vacation are mutually independent.

3. Analysis of the model

In this section, the Markov process is used to build the steady-state probability equations and the matrix technique is adopted to determine steady-state probabilities. Let $\pi_{0,i}, 0 \le i \le N$, be the probability that the server is on WV when there are i customers in the system, and $\pi_{1,i}, 1 \le i \le N$, be the probability that there are i customers in the system while the server is in regular service period. The steady-state equations are derived using the Markov process as:

$$\begin{aligned}
\lambda b_0 \pi_{0,0} &= u_1 \pi_{0,1} + v_1 \pi_{1,1}, && (1)\\
z_i \pi_{0,i} &= \lambda b_{i-1} \pi_{0,i-1} + u_{i+1} \pi_{0,i+1},\ 1 \le i \le N-1, && (2)\\
z_N \pi_{0,N} &= \lambda b_{N-1} \pi_{0,N-1}, && (3)\\
t_1 \pi_{1,1} &= v_2 \pi_{1,2} + \phi \pi_{0,1}, && (4)\\
t_i \pi_{1,i} &= \lambda b_{i-1} \pi_{1,i-1} + v_{i+1} \pi_{1,i+1} + \phi \pi_{0,i},\ 2 \le i \le N-1, && (5)\\
v_N \pi_{1,N} &= \lambda b_{N-1} \pi_{1,N-1} + \phi \pi_{0,N}, && (6)
\end{aligned}$$

where for $1 \le i \le N, u_i = \eta p_1 + (N-i+1)\alpha;\ v_i = \mu p_1 + (N-i+1)\alpha;\ z_i = \lambda b_i + \phi + u_i;\ t_i = \lambda b_i + v_i$.

3.1. Matrix solution

In this subsection, the steady-state probabilities $\pi_{j,i}, j = 0,1; j \le i \le N$, are obtained by solving the system of equations (1) to (6) using matrices.

Let $\mathbf{\Pi} = (\mathbf{\Pi_0}, \mathbf{\Pi_1})$ be the steady-state probability vector, where $\mathbf{\Pi_0} = (\pi_{0,0}, \pi_{0,1}, \pi_{0,2}, ..., \pi_{0,N})$ and $\mathbf{\Pi_1} = (\pi_{1,1}, \pi_{1,2}, ..., \pi_{1,N})$. The equations (1) to (6) can be written in matrix form as

$$\mathbf{\Pi Q} = \mathbf{0}, \quad (7)$$

$$\mathbf{\Pi e} = 1, \quad (8)$$

where $\mathbf{e}$ is a column vector with each component equal to unity and the Markov process's transition rate matrix $\mathbf{Q}$ has the block form:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A_{vv}} & \mathbf{A_{vb}} \\ \mathbf{A_{bv}} & \mathbf{A_{bb}} \end{pmatrix}.$$

The elements of the matrices $\mathbf{A_{vv}}$, $\mathbf{A_{vb}}$, $\mathbf{A_{bv}}$ and $\mathbf{A_{bb}}$ are given by

$$\mathbf{A_{vv}} = \begin{cases} -\lambda b_0 &, \text{ if } i = j = 1, \\ \lambda b_{i-1} &, \text{ if } i = j-1, j \ge 2, \\ u_{i-1} &, \text{ if } i = j+1, \\ -z_{i-1} &, \text{ if } i = j, j \ge 2, \\ 0 &, \text{ otherwise} \end{cases}$$

$$\mathbf{A_{vb}} = \begin{cases} \phi &, \text{ if } i = j+1, \\ 0 &, \text{ otherwise} \end{cases}$$

$$\mathbf{A_{bv}} = \begin{cases} v_1 &, \text{ if } i = j = 1, \\ 0 &, \text{ otherwise} \end{cases}$$

$$\mathbf{A_{bb}} = \begin{cases} -t_i &, \text{ if } i = j, \\ \lambda b_i &, \text{ if } i = j-1, j \geq 2, \\ v_i &, \text{ if } i = j+1, \\ 0 &, \text{ otherwise.} \end{cases}$$

$\mathbf{A_{vb}}$ is a $(N+1) \times N$ matrix, $\mathbf{A_{bv}}$ is a $N \times (N+1)$ matrix, $\mathbf{A_{vv}}$ and $\mathbf{A_{bb}}$ are square matrices of orders $N+1$ and N, respectively.
Based on the partition $\mathbf{\Pi} = (\mathbf{\Pi_0}, \mathbf{\Pi_1})$, equations (7) and (8) can be written as:

$$\mathbf{\Pi_0 A_{vv}} + \mathbf{\Pi_1 A_{bv}} = \mathbf{0}, \tag{9}$$

$$\mathbf{\Pi_0 A_{vb}} + \mathbf{\Pi_1 A_{bb}} = \mathbf{0}, \tag{10}$$

$$\mathbf{\Pi_0 e_0} + \mathbf{\Pi_1 e_1} = 1, \tag{11}$$

where $\mathbf{e_0}, \mathbf{e_1}$ are column vectors of order $N+1$ and N, respectively, with each component as 1.
From (9), we have

$$\mathbf{\Pi_0} = -\mathbf{\Pi_1 A_{bv} A_{vv}}^{-1}. \tag{12}$$

Using (12) in (10) and (11), we get

$$\mathbf{\Pi_1} \left(\mathbf{I} - \mathbf{A_{bv} A_{vv}}^{-1} \mathbf{A_{vb} A_{bb}}^{-1} \right) = \mathbf{0}, \tag{13}$$

$$\mathbf{\Pi_1} \left(\mathbf{e_1} - \mathbf{A_{bv} A_{vv}}^{-1} \mathbf{e_0} \right) = 1. \tag{14}$$

The matrices $\mathbf{A_{bv}}$ and $\mathbf{A_{vb}}$ can be written as

$$\mathbf{A_{bv}} = \begin{pmatrix} v_1 & \mathbf{O_1} \\ \mathbf{O_2} & \mathbf{O_3} \end{pmatrix}_{N \times (N+1)}, \quad \mathbf{A_{vb}} = \phi \begin{pmatrix} \mathbf{O_1} \\ \mathbf{I}_{N \times N} \end{pmatrix}_{(N+1) \times N},$$

where $\mathbf{O_1}, \mathbf{O_2}$ and $\mathbf{O_3}$ are zero matrices of order $1 \times N$, $(N-1) \times 1$ and $(N-1) \times N$, respectively.
Let $\mathbf{A_{vv}}^{-1} = [a_{i,j}]_{(N+1)\times(N+1)}$ and $\mathbf{w}$ denote the first row of $\mathbf{A_{vv}}^{-1}$, i.e., $\mathbf{w} = (a_{11}, a_{12}, ..., a_{1,N+1})$, then

$$\mathbf{A_{bv} A_{vv}}^{-1} = \begin{pmatrix} v_1 \mathbf{w} \\ \mathbf{O_4} \end{pmatrix}_{N \times (N+1)}, \tag{15}$$

where $\mathbf{O_4}$ is a zero matrix of order $(N-1) \times (N+1)$.
Now,

$$\mathbf{A_{vb} A_{bb}}^{-1} = \phi \begin{pmatrix} \mathbf{O_1} \\ \mathbf{A_{bb}}^{-1} \end{pmatrix}. \tag{16}$$

From (15) and (16), we have

$$\mathbf{A_{bv} A_{vv}}^{-1} \mathbf{A_{vb} A_{bb}}^{-1} = v_1 \phi \begin{pmatrix} \mathbf{w_0 A_{bb}}^{-1} \\ \mathbf{O_3} \end{pmatrix}, \tag{17}$$

where $\mathbf{w_0} = (a_{12}, a_{13}, \ldots, a_{1,N+1})$.
Let us partition $\mathbf{\Pi_1}$ as $\left[\pi_{1,1}, \widetilde{\mathbf{\Pi_1}}\right]$ where $\widetilde{\mathbf{\Pi_1}} = [\pi_{1,i}, 2 \leq i \leq N]_{1 \times (N-1)}$. From (13) and (17), we have

$$\left[\pi_{1,1}, \widetilde{\mathbf{\Pi_1}}\right] = \left[\pi_{1,1}, \widetilde{\mathbf{\Pi_1}}\right] \begin{pmatrix} v_1 \phi \mathbf{w_0 A_{bb}}^{-1} \\ \mathbf{O_3} \end{pmatrix}.$$

Hence, the system length probabilities of regular service period are given by

$$\pi_{1,i} = \pi_{1,1} v_1 \phi \mathbf{w_0} \mathbf{A_{bb}}^{-1} \epsilon_i, 1 \le i \le N,$$

where ϵ_i is a column vector whose i^{th} component is unity and the remaining components are zero. From (12) and (15), the system length probabilities of server being in WV are given by

$$[\pi_{0,0}, \pi_{0,1}, \ldots, \pi_{0,N}] = -[\pi_{1,1}, \widetilde{\mathbf{\Pi_1}}] \begin{pmatrix} v_1 \mathbf{w} \\ \mathbf{O_4} \end{pmatrix}.$$

Hence,

$$\pi_{0,i} = -\pi_{1,1} v_1 \mathbf{w} \epsilon_{i+1}, 0 \le i \le N.$$

Using the normalization condition $\sum_{j=0}^{1} \sum_{i=j}^{N} \pi_{j,i} = 1$, the only unknown $\pi_{1,1}$ is obtained as

$$\pi_{1,1} = \left(v_1 \phi \sum_{i=1}^{N} \mathbf{w_0} \mathbf{A_{bb}}^{-1} \epsilon_i - v_1 \sum_{i=0}^{N} \mathbf{w} \epsilon_{i+1} \right)^{-1}.$$

This completes the evaluation of steady-state probabilities.

4. Performance measures

Once the steady-state probabilities are determined, several model performance measures may be calculated. The average number of customers in the system (l_s), the probability that the server is busy with regular service (p_b) and the probability that the server is in WV (p_{wv}) are given by

$$l_s = \sum_{i=1}^{N} i \left(\pi_{0,i} + \pi_{1,i} \right) \; ; \; p_b = \sum_{i=1}^{N} \pi_{1,i} \; ; \; p_{wv} = \sum_{i=0}^{N} \pi_{0,i}.$$

The average reverse balking rate (br), the average reverse reneging rate (rr) and the average rate of loosing a customer due to impatience (lr) are obtained as

$$br = \sum_{i=0}^{N} \lambda(1-b_i)\pi_{0,i} + \sum_{i=1}^{N} \lambda(1-b_i)\pi_{1,i} \; ; \; rr = \sum_{i=1}^{N} \alpha(N-i+1) \left(\pi_{0,i} + \pi_{1,i} \right) \; ; \; lr = br + rr.$$

4.1. Cost model

The total expected cost function per unit time is formulated in this subsection with service rates as the decision variables. Our goal is to figure out the best service rates that minimize the total expected cost function. The cost parameters are assumed to be:

- $C_{ls}-$ holding cost per unit time,
- $C_{lr}-$ cost incurred when a customer is lost due to impatience,
- $C_{\mu}-$ cost per service during regular service period,
- $C_{\eta}-$ cost per service during WV period,
- $C_{f\mu}-$ cost per service for a feedback customer during regular service period,
- $C_{f\eta}-$ cost per service for a feedback customer during WV period.

The total expected cost (tec) is defined as:

$$tec = C_{ls} l_s + C_{lr} lr + \mu(C_{\mu} + q_1 C_{f\mu}) + \eta(C_{\eta} + q_1 C_{f\eta}).$$

Analytical optimization of the aforementioned cost model is a tedious job due to the complexity of the cost function. As a result, we have used the ACO developed by Colorni et al. [5] and Dorigo et al. [6] to find the best values for μ and η. A brief algorithm of ACO is given below:

Algorithm for ACO

Step 1: Consider a suitable number of ants in the colony (B). Assume a set of permissible discrete values for each of the n design variables x_{ij} as $x_{i1}, x_{i2}, ..., x_{ip}$ $(i = 1, 2, ..., n)$. Assume equal amounts of pheromone $\tau_{ij}^{(1)}$ initially along all the arcs. The superscript to τ_{ij} denotes the iteration number. For simplicity, $\tau_{ij}^{(1)}$ is assumed to be 1. Set the iteration number l = 1.

Step 2: (a) Compute the probabilities (p_{ij}) of selecting the arc x_{ij} as

$$p_{ij}^{(l)} = \frac{\tau_{ij}^{(l)} D_{ij}^{(\beta)}}{\sum_{m=1}^{p} [\tau_{im}^{(l)} D_{im}^{(\beta)}]}; \; i = 1, 2, ..., n; \; j = 1, 2, ..., p,$$

where τ_{ij} is a pheromone amount between arc i and arc j, D_{ij} is a reciprocal of the distance between arc i and arc j, β is the parameter that allow a user control on the relative importance of trail versus visibility.

(b) The specific path (or discrete values) chosen by the k^{th} ant can be determined using random numbers generated in the range (0, 1). For this, we find the cumulative probability ranges associated with different paths based on the probabilities given by above equation. The specific path chosen by ant k will be determined using the roulette-wheel selection process in step 3(a).

Step 3: (a) Generate B random numbers $r_1, r_2, ..., r_B$ in the range (0, 1), one for each ant. Determine the discrete value or path assumed by ant k for variable i as the one for which the cumulative probability range (found in step 2 (b)) includes the value r_i.

(b) Repeat step 3 (a) for all design variables $i = 1, 2, ..., n$.

(c) Evaluate the objective function values corresponding to the complete paths (design vectors $X^{(k)}$ or values of x_{ij} chosen for all design variables $i = 1, 2, ..., n$ by ant k, $k = 1, 2, ..., B$):

$$f_k = f(X^{(k)}); \; k = 1, 2, ..., B.$$

Determine the best and worst paths among the B paths chosen by different ants as follows:

$$f_{best} = \min_{k=1,2,...,B} f_k, \; f_{worst} = \max_{k=1,2,...,B} f_k.$$

Step 4: Test for the convergence of the process. The process is assumed to have converged if all the B ants take the same best path. If convergence is not achieved, assume that all the ants return home and start again in search of food. Set the new iteration number as l = $l + 1$, and update the pheromone on different arcs as

$$\tau_{ij} \;\; = \;\; \tau_{ij}^{(old)} + \sum_{k} \Delta\tau_{ij}^{(k)},$$

where $\tau_{ij}^{(old)}$ denotes the pheromone amount of the previous iteration left after evaporation, $\Delta\tau_{ij}^{(k)}$ is the amount of pheromone deposited on arc i and arc j by the best ant k and are taken as

$$\tau_{ij}^{(old)} \;\; = \;\; (1 - \rho)\tau_{ij},$$

$$\Delta\tau_{ij}^{(k)} \;\; = \;\; \begin{cases} \dfrac{\xi f_{best}}{f_{worst}}; & \text{if } (i, j) \in \text{global best tour}, \\ 0\,; & \text{otherwise}, \end{cases}$$

where $\rho \in$ (0, 1] is the evaporation rate (also known as the pheromone decay factor) and ξ is the parameter used to control the scale of the global updating of the pheromone. With the new values of τ_{ij}, go to step 2. Steps 2, 3, and 4 are repeated until the process converges. In some cases, the iterative process may be stopped after completing a prespecified maximum number of iterations (l_{max}).

The complexity of the algorithm is $O(l\varrho^2 B)$ where l is the number of iterations, ϱ is the number of nodes and B is the number of ants.

Table 1: *Various performance measures of the model for different λ and q_1*

	λ=1.0		λ=1.7		λ=2.4	
	$q_1 = 0.6$	$q_1 = 0.2$	$q_1 = 0.6$	$q_1 = 0.2$	$q_1 = 0.6$	$q_1 = 0.2$
l_s	0.038536	0.027681	0.092328	0.052341	0.407215	0.094482
p_b	0.002170	0.001060	0.006109	0.002123	0.036461	0.004367
p_{wv}	0.997830	0.998930	0.993891	0.997876	0.963538	0.995632
br	0.947901	0.948532	1.606110	1.610020	2.237570	2.266750
rr	0.034617	0.025827	0.061203	0.044884	0.092589	0.065255
lr	0.982518	0.974359	1.667310	1.654901	2.330160	2.332010

Table 2: *Effect of α on the performance measures*

	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 1.5$
l_s	0.015341	0.008061	0.005467
p_b	0.000244	0.000071	0.000033
p_{wv}	0.999756	0.999929	0.999966
br	1.613650	1.614230	1.614530
rr	0.073773	0.079008	0.080914
lr	1.687420	1.693310	1.695440

5. Sensitivity analysis

In this section, tables and graphs have been used to display certain numerical results. We fix the capacity of the system as $N = 10$ and the balking function is taken as $b_i = i/N, 1 \leq i \leq N-1,\ b_N = 0$. The various parameters of the model are chosen to be $\lambda = 1.7, \mu = 2.0, \eta = 1.2, \phi = 0.1, q = 0.05, \alpha = 0.1, q_1 = 0.3$, unless they are considered as variables or their values are mentioned in the respective tables and figures. For employing the *ACO*, we have arbitrarily chosen the following: $n = 2, B = 3, \varrho = 40, l = 100, \beta = 0.5, \xi = 2, \rho = 0.5$ and the distances between the arcs are obtained using the *RandomReal* function of *Mathematica* software.

Table 1 shows the model's performance metrics for various values of λ and q_1. All the performance measures, with the exception of p_{wv} and br, drop as q_1 lowers, whereas p_{wv} and br rise as q_1 decreases for fixed λ. Further, increase in λ results in a drop in p_{wv}, whereas increase in λ results in the increase of the remaining performance metrics.

Table 2 shows the influence of α on the model's performance measures. With the rise of α, a rising trend can be noticed in p_{wv}, br, rr and lr while a declining trend can be found in l_s and p_b.

Figure 1 shows the influence of μ on the server's state probabilities for various values of the vacation parameter (ϕ). The picture illustrates that when μ grows, the probability of the server being busy with regular service (p_b) decreases while the probability of the server being on vacation (p_{wv}) increases. Furthermore, as the vacation parameter (ϕ) is increased, p_b grows while p_{wv} decreases for any μ.

The impact of λ on the average number of customers in the system (l_s) in models with and without reverse balking and reverse reneging is shown in Figure 2. From the graph, one may observer that in either of the models l_s increases with the increase of the arrival rate λ. Further, the queue lengths are lower in models with reverse balking and reverse reneging when compared to models without reverse balking and reverse reneging.

Figure 3 displays the effect of service rates μ and η on the average rate of loosing a customer (lr). With the increase of both μ and η, the average rate of loosing a customer decreases. We can carefully setup the service rates μ and η in the system in order to ensure the minimum average rate of loosing a customer due to impatience.

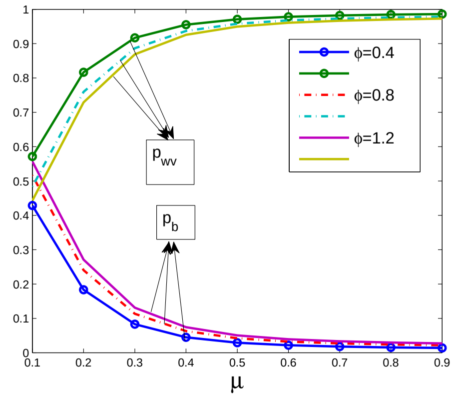

Figure 1: *μ versus p_{wv} and p_b*

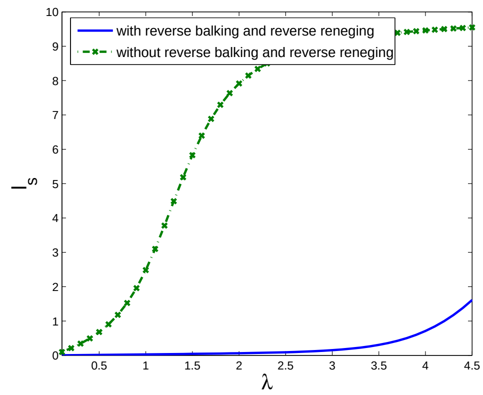

Figure 2: *Impact of λ on l_s*

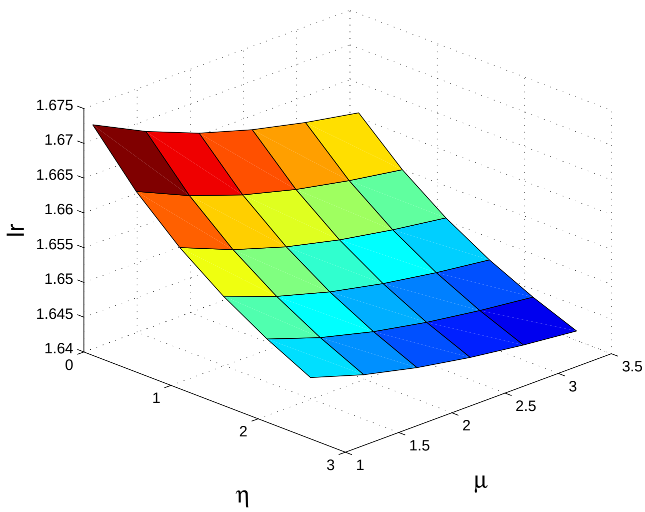

Figure 3: *Effect of μ and η on lr*

Table 3: *Optimum service rates and the corresponding minimum tec.*

	ϕ	0.06	0.08	0.1
Case 1	(μ^*, η^*)	(0.606627, 0.393620)	(0.648424, 0.295251)	(0.679539, 0.213562)
	tec^*	65.9207	65.3714	64.7454
Case 2	(μ^*, η^*)	(0.613050, 0.539007)	(0.656637, 0.446063)	(0.689743, 0.366756)
	tec^*	70.8644	70.5699	70.1348
Case 3	(μ^*, η^*)	(0.608498, 0.390333)	(0.650509, 0.291586)	(0.681784, 0.209613)
	$, tec^*$	57.5689	57.0077	56.3712
Case 4	(μ^*, η^*)	(0.557353, 0.457533)	(0.597739, 0.360028)	(0.628019, 0.278463)
	tec^*	68.8265	68.4826	68.0101
Case 5	(μ^*, η^*)	(0.695438, 0.199822)	(0.737154, 0.101221)	(0.768531, 0.018769)
	tec^*	67.9667	66.7260	65.5257
Case 6	(μ^*, η^*)	(0.590782, 0.413392)	(0.632152, 0.315211)	(0.662999, 0.233546)
	tec^*	66.8187	66.3317	65.7522
Case 7	(μ^*, η^*)	(0.622554, 0.356552)	(0.664430, 0.257807)	(0.695557, 0.175969)
	tec^*	66.3707	65.703	64.9789

Table 4: *Optimum service rates and the corresponding model characteristics for various model parameters*

		(μ^*, η^*)	tec^*	l_s	p_b	p_{wv}	lr
	1.5	(0.560865, 0.053717)	53.0653	0.291992	0.027470	0.972530	1.486520
$\lambda =$	2.0	(0.856029, 0.449138)	82.0273	0.355744	0.032820	0.967180	1.955930
	2.5	(1.145080, 0.832582)	110.272	0.415343	0.037775	0.962225	2.421721
	0.1	(0.533546, 0.289963)	55.1871	0.231776	0.021178	0.978822	1.672162
$q_1 =$	0.2	(0.597385, 0.258192)	59.5429	0.269601	0.024895	0.975105	1.673490
	0.3	(0.679539, 0.213562)	64.7454	0.318140	0.029674	0.970325	1.674791
	0.04	(1.021179, 0.862099)	91.8238	0.332031	0.034453	0.965547	1.625062
$\alpha =$	0.08	(0.782366, 0.436608)	73.5377	0.320756	0.030862	0.969137	1.659655
	0.12	(0.572881, 0.022225)	81.3452	0.304472	0.027703	0.972297	1.687794

Table 3 presents the optimum values of the service rates (μ^*, η^*) that minimize the total expected cost (tec) for different values of ϕ and for the following cost values:
Case 1: $C_{ls} = 40,\ C_{lr} = 15,\ C_\mu = 25,\ C_\eta = 20,\ C_{f\mu} = 22,,\ C_{f\eta} = 18,$
Case 2: $C_{ls} = 60,\ C_{lr} = 15,\ C_\mu = 25,\ C_\eta = 20,\ C_{f\mu} = 22,,\ C_{f\eta} = 18,$
Case 3: $C_{ls} = 40,\ C_{lr} = 10,\ C_\mu = 25,\ C_\eta = 20,\ C_{f\mu} = 22,,\ C_{f\eta} = 18,$
Case 4: $C_{ls} = 40,\ C_{lr} = 15,\ C_\mu = 30,\ C_\eta = 20,\ C_{f\mu} = 22,,\ C_{f\eta} = 18,$
Case 5: $C_{ls} = 40,\ C_{lr} = 15,\ C_\mu = 25,\ C_\eta = 27,\ C_{f\mu} = 22,,\ C_{f\eta} = 18,$
Case 6: $C_{ls} = 40,\ C_{lr} = 15,\ C_\mu = 25,\ C_\eta = 20,\ C_{f\mu} = 27,,\ C_{f\eta} = 18,$
Case 7: $C_{ls} = 40,\ C_{lr} = 15,\ C_\mu = 25,\ C_\eta = 20,\ C_{f\mu} = 22,,\ C_{f\eta} = 22.$

One may observe from the table that for any set of cost values with the increase of ϕ, μ^* increases while tec^* and η^* decrease.

The values of the service rates that minimize the total expected cost are presented in Table 4 along with the corresponding performance metrics for $\lambda,\ q_1,\ \alpha$ and the cost values in Case 1. It is clear from the table that an increase in λ or q_1 results in the increase of $\mu^*,\ tec^*,\ l_s,\ p_b$ and lr while p_{wv} decreases with λ or q_1. One may note that η^* increases with λ and decreases with q_1. On the otherhand increase in α leads to the decrease of all the values except p_{wv} and lr.

6. Conclusions

We investigated a Markovain feedback queue with reverse balking, reverse reneging, and working vacations in this study. Using the matrix technique, we have obtained the steady-state probabilities. Different performance measures, cost analysis using ACO and numerical findings in the form of tables and graphs are sketched out to show the influence of the system parameters. The provided approach has the potential to be utilized in a variety of investment business areas, including insurance, mutual funds, banking and so on. The current model may be expanded to a renewal input feedback queue with working vacations under reverse balking and reverse reneging in future.

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