STATISTICAL PROPERTIES AND APPLICATION OF A TRANSFORMED LIFETIME DISTRIBUTION: INVERSE MUTH DISTRIBUTION

Agni Saroj[1], Prashant K. Sonker[2] and Mukesh Kumar[*3]

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[1,2]Department of Statistics, Banaras Hindu University, Varanasi, 221005, India.
[3]Department of Statistics, MMV, Banaras Hindu University, Varanasi 221005, India.
E-mail: [1]agni.saroj4@bhu.ac.in, [2]prashant.s4@bhu.ac.in,[*3]mukesh.mmv@bhu.ac.in
[*]Corresponding Author

Abstract

In this paper, we have proposed a transformed distribution called inverse Muth (IM) distribution. The expressions for probability density function (pdf), cumulative distribution function (cdf), reliability and hazard function of this distribution are well defined. The statistical properties such as, quantile function, moments, skewness and kurtosis are derived. The methods of estimation such as maximum likelihood estimation (MLE) and maximum product spacing estimation (MPSE) are used to estimate the parameters. The IM distribution is positively skewed and its behavior of hazard rate is upside-down bathtub (UBT) shape. The important finding of the study is that the moments of IM distribution do not exist. A real dataset (the active repair time for airborne communication transceiver) used for application purpose, after taking a natural extension of IM distribution. It is expected that the proposed model would be used as a life time model in field of reliability and its applicability.

Keywords: Inverse Muth distribution, quantile function, maximum likelihood estimation, maximum product spacing estimation, real data analysis.

1. Introduction

In the statistical literature, there are lots of distribution exist, which are very useful in various fields of science with its applicability. The application of statistical distributions gives the well explanation about the probabilistic behavior of random phenomenon and plays an important role to analyze the different types of data from various fields.

In the field of reliability, the various lifetime distributions derived which are preferred in reliability analyses or lifetime investigation see Martz & Waller [1], and the behavior of failure rate observed to be as increasing , decreasing and bathtub shape. Some distributions (Maxwell, normal, Gompertz, etc.) are having only increasing failure rate whereas Gamma, Weibull and other distributions gives increasing, decreasing as well as constant failure rate. In many situations failure rate increases consistently, after reaching the peak, it starts to decrease which is discussed in Bennett [2], Langlands et. al. [3]. Such type of failure rate is named as UBT failure rate given in Sharma et. al. [4]. Muth distribution is defined on a continuous random variable and introduced by Muth [5] in 1977 for reliability analysis. Let us consider that a random variable Y follow Muth distribution with the shape parameter α and its pdf is defined as

$$f(y;\alpha) = \begin{cases} (e^{\alpha y} - \alpha).exp\left\{\alpha y - \frac{1}{\alpha}(e^{\alpha y} - 1)\right\} & y > 0, \quad \alpha \in (0,1] \\ 0 & otherwise \end{cases} \tag{1}$$

The cdf is given by,

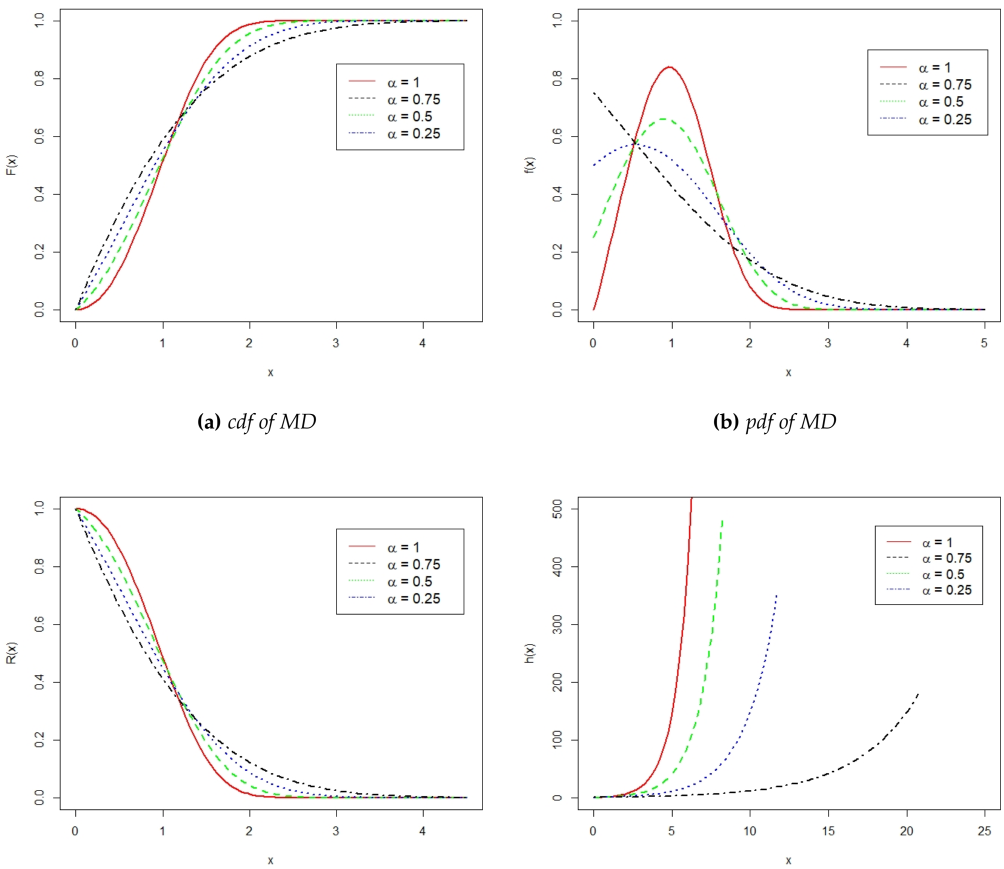
$$F(y;\alpha) = 1 - exp\left\{\alpha y - \frac{1}{\alpha}.\left(e^{\alpha y} - 1\right)\right\} \qquad y > 0, \quad \alpha \in (0,1] \tag{2}$$

It has mainly focused on strictly positive memory in Muth [5]. The basic statistical properties of Muth distribution are discussed by Jodra et. al. [6]. The reliability function and hazard function are given by respectively

$$R(t) = P[Y \geq t] = exp\left\{\alpha t - \frac{1}{\alpha}.\left(e^{\alpha t} - 1\right)\right\} \qquad t > 0, \quad \alpha \in (0,1] \tag{3}$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{\left(e^{\alpha t} - \alpha\right).exp\left\{\alpha t - \frac{1}{\alpha}\left(e^{\alpha t} - 1\right)\right\}}{exp\left\{\alpha t - \frac{1}{\alpha}.\left(e^{\alpha t} - 1\right)\right\}} \qquad t > 0, \quad \alpha \in (0,1] \tag{4}$$

At different values of parameter α pdf, cdf, reliability and hazard functions are plotted in Figure 1.



(a) *cdf of MD*

(b) *pdf of MD*

(c) *reliability function of MD*

(d) *hazard function of MD*

Figure 1: *pdf, cdf, reliability and hazard functions of Muth Distribution.*

A natural extension is also considered in Jodra et. al. [6] by adding a scale parameter named as Scaled Muth distribution. A transformed distribution for Muth distribution called power Muth (PM) distribution proposed by Jodra et. al. [7]. The exponentiated PM distribution and Inverse PM distribution will be proposed by Irshad et. al. [8] and Chesneau & Agiwal [9]. Some other literature on Muth distribution are discussed in Almarashi & Elgarhy [10], Al-Babtain et.al.

[11], Bicer et. at. [12]. In Figure 1, the hazard rate shows the failure rate is increasing. It has explained that the failure rate occurs in UBT shape when we take the inverse transformation of usual distributions given Sharma et. al. [4]. In the case of Invese PM distribution it is found that the behavior of hazard rate is in UBT shape. In this article, we have proposed a transformed distribution which is termed as the IM distribution. All the work of this article is arranged in different sections as: In section 2, statistical properties of proposed distribution are discussed. In section 3, we obtained the estimates of the parameter α using MLE and MPSE. In section 4, we have computed the expression for asymptotic confidence interval in case of MLE and MPSE. In section 5, the scale transformation of IM distribution has taken to estimate the parameters. In section 6, the simulation study has done to compute the estimates of parameters for both IM and scaled inverse Muth (SIM) distributions respectively. In section 7, the real data analysis is done to show the applicability of SIM distribution. Finally, the conclusion of this article is written in section 8.

2. Inverse Muth Distribution

Let Y be a random variable follows the Muth distribution with pdf in equation (1) and cdf in equation (2), on taking inverse transformation as $X=\frac{1}{Y}$, the pdf of IM distribution is obtained as

$$f(x;\alpha) = \begin{cases} \frac{1}{x^2}(e^{\alpha/x} - \alpha).exp\left\{\frac{\alpha}{x} - \frac{1}{\alpha}(e^{\alpha/x} - 1)\right\} & x > 0, \quad \alpha \in (0,1] \\ 0 & otherwise \end{cases} \tag{5}$$

The cdf is given by,

$$F(x;\alpha) = exp\left\{\frac{\alpha}{x} - \frac{1}{\alpha}(e^{\alpha/x} - 1)\right\} \qquad x > 0, \quad \alpha \in (0,1] \tag{6}$$

Some statistical properties of IM distribution are discussed as below:

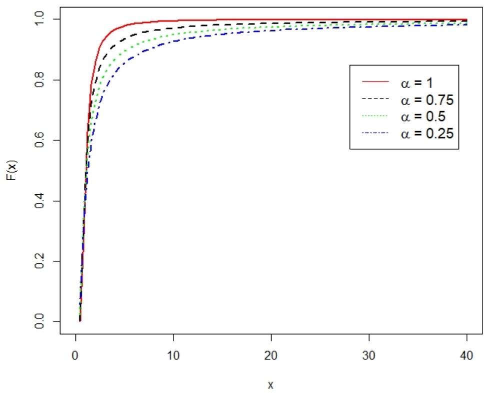
2.1. Reliability and Hazard Function of IM Distribution

Importance of any lifetime distribution is based on its reliability and hazard rate. By using equation (5) and (6) the reliability and hazard function of the IM distribution are obtained as

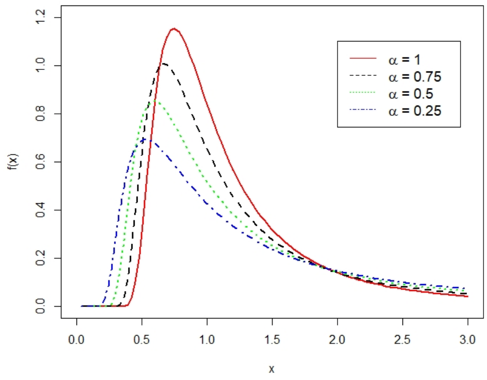
$$R(t) = 1 - exp\left\{\frac{\alpha}{t} - \frac{1}{\alpha}(e^{\alpha/t} - 1)\right\} \qquad t > 0, \quad \alpha \in (0,1] \tag{7}$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{(e^{\alpha/t} - \alpha).exp\left\{\frac{\alpha}{t} - \frac{1}{\alpha}(e^{\alpha/t} - 1)\right\}}{t^2.\left(1 - exp\left\{\frac{\alpha}{t} - \frac{1}{\alpha}(e^{\alpha/t} - 1)\right\}\right)} \qquad t > 0, \quad \alpha \in (0,1] \tag{8}$$

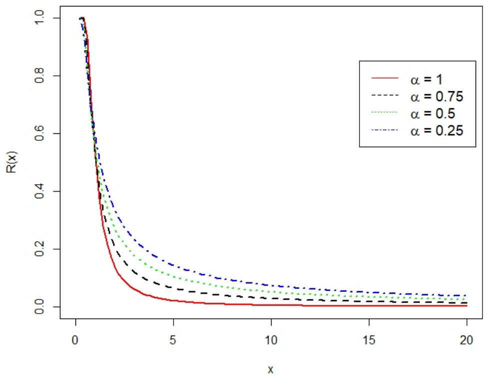
The above equation (7) and (8) show the reliability and hazard function respectively and the graphical representation of these are given in Figure 2. We observed the behavior of hazard rate as UBT shape in Figure 2. As increases the value of parameter α, the peak of hazard rate also increases.



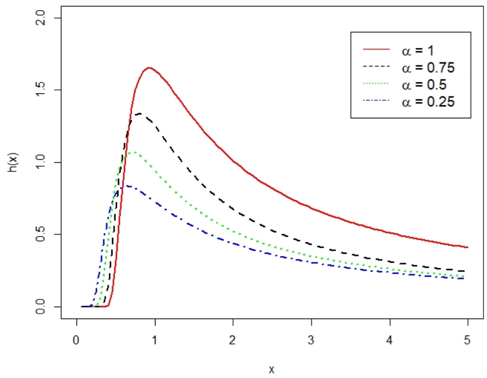
(a) *cdf of IM distribution*



(b) *pdf of IM distribution*



(c) *reliability function of IM distribution*



(d) *hazard function of IM distribution*

Figure 2: *pdf, cdf, reliability and hazard functions of IM Distribution.*

2.2. Quantile Function

Quantile function for the cdf $F_X(x)$ is defined as,

$$Q_X(u) = inf\{x \in R : F_X(x) \geq u\} \qquad 0 < u < 1 \tag{9}$$

It shows u^{th} quantile of an integer valued random variable, is also an integer. It indicates that if $F_X(x)$ be a continuous and strictly increasing, then quantile function of X is defined as

$$Q_X(u) = F_X^{-1}(u) \qquad 0 < u < 1 \tag{10}$$

To find the quantile function for the IM distribution, it has to solve $F(x, \alpha) = u$; $x > 0$ with respect to x for any $\alpha \in (0, 1]$ and $u \in (0, 1)$ i.e.

$$u = exp\left(\frac{\alpha}{x} + \frac{1}{\alpha} - \frac{1}{\alpha}e^{\frac{\alpha}{x}}\right)$$

$$log(u) - \frac{\alpha}{x} - \frac{1}{\alpha} = -\frac{1}{\alpha}e^{\frac{\alpha}{x}} \tag{11}$$

Multiplying by $e^{(log(u) - \frac{\alpha}{x} - \frac{1}{\alpha})}$ on both side in equation (11), we get

$$\left(log(u) - \frac{\alpha}{x} - \frac{1}{\alpha}\right).e^{\left(log(u) - \frac{\alpha}{x} - \frac{1}{\alpha}\right)} = -\frac{e^{-\frac{1}{\alpha}}.u}{\alpha} \tag{12}$$

To solve equation (12), here we use a generalized integro-exponential function, Lambert-W function. It has applicability in computer algebra system and in mathematics given by Corless et. al. [13]. The Lambert W function is defined as the solution of,

$$W(z).exp(W(z)) = z \tag{13}$$

Where, z is complex function. If z is a real number such that If z is a real number such that $z \geq -\frac{1}{e}$ then W(z) becomes a real function having two possible real branches. If the real branch taking value in $(-\infty, -1]$ is called negative branch and denoted by $W_{-1}(z)$ where $-\frac{1}{e} \leq z \leq 0$. The real root branch taking values in $[-1, \infty)$ is called the principle branch and denoted by $W_0(z)$ where $z \geq -\frac{1}{e}$, we shall use the negative branch which is satisfies the following properties, $W_{-1}(\frac{-1}{e}) = -1$, $W_{-1}(z)$ is decreasing as z increases to 0 and $W_{-1}(z)$ tends to $-\infty$ as z tends to 0 see Jodra [14].
By using equations (12) and (13), we obtained that $(log(u) - \frac{\alpha}{x} - \frac{1}{\alpha})$ is the Lambert-W function of the real argument $(-\frac{e^{-\frac{1}{\alpha}}.u}{\alpha})$, then, the explicit expression for Q_x in terms of Lambert-W function.

$$x = \frac{\alpha^2}{\alpha.log(u) - \alpha.W\left(\frac{-e^{-\frac{1}{\alpha}}.u}{\alpha}\right) - 1} \tag{14}$$

It gives the Quantile function of IM distribution.
Now for any $\alpha \in (0, 1]$, $x > 0$ and $u \in (0, 1)$ it ensure that,

$$\left(log(u) - \tfrac{\alpha}{x} - \tfrac{1}{\alpha}\right) < -1$$

And it also be checked that,

$$\left(\tfrac{-e^{-\frac{1}{\alpha}}.u}{\alpha}\right) \in \left(-\tfrac{1}{e}, 1\right)$$

By using the negative branch of Lambert W function the Quantile function of IM distribution in terms of negative branch of Lambert W function as,

$$x_u = \frac{\alpha^2}{\alpha.log(u) - \alpha.W_{-1}\left(\frac{-e^{-\frac{1}{\alpha}}.u}{\alpha}\right) - 1} \tag{15}$$

Where, x_u gives the u^{th} quantile of IM distribution.

2.3. Moments of the IM distribution

Let X be a random variable follows IM distribution with pdf in equation (5) then the k^{th} raw moment is defined as:

$$\mu'_k = \int_0^\infty x^k.f\,(x;\alpha)\,dx$$

$$\mu'_k = \int_0^\infty x^k.\frac{1}{x^2}\left(e^{\alpha/x} - \alpha\right) e^{\left\{\frac{\alpha}{x} - \frac{1}{\alpha}.(e^{\alpha/x} - 1)\right\}}dx$$

$$I = \mu'_k = \int_0^\infty x^{k-2}.\left(e^{\alpha/x} - \alpha\right) e^{\left\{\frac{\alpha}{x} - \frac{1}{\alpha}.(e^{\alpha/x} - 1)\right\}}dx$$

$$I = \int_0^a x^{k-2}.\left(e^{\alpha/x} - \alpha\right) e^{\left\{\frac{\alpha}{x} - \frac{1}{\alpha}.(e^{\alpha/x} - 1)\right\}}dx + \int_a^\infty x^{k-2}.\left(e^{\alpha/x} - \alpha\right) e^{\left\{\frac{\alpha}{x} - \frac{1}{\alpha}.(e^{\alpha/x} - 1)\right\}}dx$$

$$I = I_1 + I_2$$

Where,

$$I_1 = \int_0^a x^{(k-2)} \cdot (e^{\frac{\alpha}{x}} - \alpha) \cdot e^{\{\frac{\alpha}{x} - \frac{1}{\alpha} \cdot (e^{\frac{\alpha}{x}} - 1)\}} dx$$

$$I_2 = \int_a^\infty x^{(k-2)} \cdot (e^{\frac{\alpha}{x}} - \alpha) \cdot e^{\{\frac{\alpha}{x} - \frac{1}{\alpha} \cdot (e^{\frac{\alpha}{x}} - 1)\}} dx$$

Now proceeding with integration I_2

$$I_2 = \int_a^\infty x^{(k-2)} \cdot (e^{\frac{\alpha}{x}} - \alpha) \cdot e^{\{\frac{\alpha}{x} - \frac{1}{\alpha} \cdot (e^{\frac{\alpha}{x}} - 1)\}} dx$$

To check the convergence or divergence of integral I_2, we use the limit comparison test which state that if

1. f(x) and g(x) $>$ 0 on [a,∞)
2. f(x) and g(x) both are continuous on [0,∞) and
3. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L > 0$ where, L is some finite positive number.

then$\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ either both converge or both diverge.

For I_2, Let,

$$f_1(x) = \int_a^\infty x^{(k-2)} \cdot (e^{\frac{\alpha}{x}} - \alpha) \cdot e^{\{\frac{\alpha}{x} - \frac{1}{\alpha} \cdot (e^{\frac{\alpha}{x}} - 1)\}} \text{ and,}$$

$$g_1(x) = x^{(k-2)}$$

$f_1(x)$ and $g_1(x) > 0$ as well as continuous for [a,∞) for k =1, 2, 3,...
now,

$$\lim_{x\to\infty} \frac{f_1(x)}{g_1(x)} = \lim_{x\to\infty} (e^{\frac{\alpha}{x}} - \alpha) \cdot e^{\{\frac{\alpha}{x} - \frac{1}{\alpha} \cdot (e^{\frac{\alpha}{x}} - 1)\}}$$

$$= (1-\alpha) \cdot e^0$$
$$= (1-\alpha) > 0 \qquad \alpha \in (0,1]$$

$\int_a^\infty g_1(x)dx = \int_a^\infty x^{(k-2)}dx = \int_a^\infty \frac{1}{x^{-(k-2)}} dx$

$\because \int_a^\infty \frac{1}{x^n} dx$ is convergent if n $>$ 1 and divergent for $n \leq 1$.

So, $\int_a^\infty \frac{1}{x^{-(k-2)}} dx$ is convergent if (2-k) $>$ 1 or k $<$1. But we have k $>$ 0 (k = 1, 2, 3,...).
Then it shows that $\int_a^\infty g_1(x)dx$ is divergent for all $k \geq 1$ and by using limit comparison test for convergence of an improper integral, $\int_a^\infty f_1(x)dx$ is also divergent i.e. integral
$I_2 = \int_a^\infty x^{(k-2)} \cdot (e^{\frac{\alpha}{x}} - \alpha) \cdot e^{\{\frac{\alpha}{x} - \frac{1}{\alpha} \cdot (e^{\frac{\alpha}{x}} - 1)\}} dx$ is divergent for all the value of $k \geq 1$.
By using the property of convergence of integral, if we have an integral $I = I_1 + I_2$ then I is convergent iff I_1 and I_2 both are convergent. If any one of the I_1 and I_2 is divergent then the integral I is also divergent. Thus we found that integral I also become a divergent. Hence the moment for the IM distribution does not exist.

2.4. Measures of Skewness and Kurtosis

In the above section, we found that the moment of the IM distribution does not exist, so we cannot obtain Pearson's measure of skewness and kurtosis based on moments. Therefore by using the quantile function, it may be possible to obtain Galton's measures of skewness and Moor's measures of kurtosis mentioned in Gilchrist [15]. These measures are defined as:

$$G(\alpha) = \frac{x_{3/4}(\alpha) + x_{1/4}(\alpha) - 2x_{1/2}(\alpha)}{x_{3/4}(\alpha) - x_{1/4}(\alpha)} \tag{16}$$

$$K(\alpha) = \frac{x_{7/8}(\alpha) - x_{5/8}(\alpha) + x_{3/8}(\alpha) - x_{1/8}(\alpha)}{x_{3/4}(\alpha) - x_{1/4}(\alpha)} \tag{17}$$

Where, $x_{i/4}$; $i = 1, 2, 3$ denote the i^{th} quartile and $x_{i/8}$; $i = 1, 2, ..., 7$ denote the i^{th} octile for this distribution. Galton's measure of skewness G (.) lies between (-1,1). If G (.) > 0 it is called positive or right skewed and if G (.) < 0 it is called negative skewed. For a perfect symmetrical distribution, G(.) = 0. Galton's measures of skewness $G(\alpha)$ and Moor's measures of kurtosis $K(\alpha)$ for IM distribution are calculated at different value of α in Table 1. From the Table 1, we observed that all values of skewness are greater than zero for different values of parameter, thus IM distribution is a positive or right skewed distribution.

Table 1: *Skewness and kurtosis of IM distribution*

α	Skewness	Kurtosis
0.1	0.4759	2.1413
0.2	0.4741	2.1385
0.3	0.4695	2.1301
0.4	0.4607	2.1108
0.5	0.4465	2.0733
0.6	0.4264	2.0109
0.7	0.4008	1.9207
0.8	0.3710	1.8080
0.9	0.3388	1.6861
1.0	0.3060	1.5698

3. Parameter Estimation

3.1. Maximum likelihood estimation

Let $x_1, x_2, ..., x_n$ be a random sample of size of n from IM distribution with unknown parameter α having pdf equation (5). Likelihood function for the sample $x_1, x_2, ..., x_n$ as follows,

$$L(x;\alpha) = \prod_{i=1}^{n} \frac{1}{x_i^2}\left(e^{\alpha/x_i} - \alpha\right).exp\left(\frac{\alpha}{x_i} - \frac{1}{\alpha}(e^{\alpha/x_i} - 1)\right) \tag{18}$$

$$log(L(x;\alpha)) = -2\sum_{i=1}^{n} log(x_i) + \sum_{i=1}^{n} log\left(e^{\alpha/x_i} - \alpha\right) + \sum_{i=1}^{n}\left(\frac{\alpha}{x_i} - \frac{1}{\alpha}(e^{\alpha/x_i} - 1)\right) \tag{19}$$

MLE is the value of unknown parameter α which maximize the equation (18). To get estimated value of α, we take partial derivative of equation (19) w.r.t. α and equating to zero i.e.

.

$$\frac{\partial}{\partial\alpha}\, log(x,\alpha) = 0$$

$$\sum_{i=1}^{n} \frac{\left(e^{\alpha/x_i} - 1\right)}{x_i\left(e^{\alpha/x_i} - \alpha\right)} + \sum_{i=1}^{n} \frac{1}{x_i} + \frac{1}{\alpha^2}\sum_{i=1}^{n} e^{\alpha/x_i} - \frac{1}{\alpha}\sum_{i=1}^{n} \frac{e^{\alpha/x_i}}{x_i} - \frac{n}{\alpha^2} = 0 \tag{20}$$

Now we have to solve equation (20) to get $\hat{\alpha}_{ml}$ and check that this solution to maximizes equation (18) following condition has to be satisfies:

$$\left[\frac{\partial^2}{\partial\alpha^2} log(L(x;\alpha))\right]_{\alpha=\hat{\alpha}_{ml}} < 0 \tag{21}$$

Where, $\hat{\alpha}_{ml}$ is the estimated value of α which obtained from equation (20). We observed that it is not in closed form, so we cannot solve it analytically. Newton-Raphson iteration method used which gives the numerical solution of equation (20) for α.

3.2. Maximum product spacing

Maximum product spacing estimation (MPSE) method is an alternative to MLE which is proposed by Cheng & Amin [16] and Ranneby [17]. MLE does not give better performance or fails in the case of three or more parameters exist, remarked in Cheng & Traylor [18], and MLE does not perform satisfactorily for heavy tailed distribution which is discussed in Pitman [19]. Let us consider $x_1, x_2, ..., x_n$ be a random sample of size 'n' drawn from the IM distribution having cdf in equation (6).
Let $x_{i:n}$ be i^{th} order statistic and the spacing function $Di's$ is defined as,

$$D_i = \Big[F(x_{i:n};\alpha) - F(x_{(i-1):n};\alpha)\Big] \tag{22}$$

For x_0 and x_{n+1} , $F(x_0;\alpha) = 0$ and $F(x_{n+1};\alpha) = 1$ respectively.
at i=1,

$$D_1 = exp\left(\frac{\alpha}{x_1} - \frac{1}{\alpha}\left(e^{\alpha/x_1} - 1\right)\right) \tag{23}$$

at i=n+1,

$$D_{n+1} = 1 - F(x_{n:n};\alpha)$$

$$D_{n+1} = 1 - exp\left(\frac{\alpha}{x_n} - \frac{1}{\alpha}\left(e^{\alpha/x_n} - 1\right)\right) \tag{24}$$

For $i = 2, 3, ..., n$ the expression is

$$D_i = \Big[F(x_{i:n};\alpha) - F(x_{(i-1):n};\alpha)\Big]$$

$$D_i = exp\left(\frac{\alpha}{x_i} - \frac{1}{\alpha}\left(e^{\alpha/x_i} - 1\right)\right) - exp\left(\frac{\alpha}{x_{i-1}} - \frac{1}{\alpha}\left(e^{\alpha/x_{i-1}} - 1\right)\right) \tag{25}$$

Then the product of spacing function is defined as

$$S = \prod_{i=1}^{n+1} D_i \tag{26}$$

MPSE is the value of α which maximize the product spacing function given in equation (26).
Taking the log of both side of equation (26)

$$log(S) = \sum_{i=1}^{n+1} log(D_i)$$

$$log(S) = log(D_1) + log(D_{n+1}) + \textstyle\sum_{i=2}^{n} log(D_i)$$

$$\begin{aligned} log(S) = {} & log\left[exp\left(\frac{\alpha}{x_1} - \frac{1}{\alpha}\left(e^{\alpha/x_1} - 1\right)\right)\right] + log\left[1 - exp\left(\frac{\alpha}{x_n} - \frac{1}{\alpha}\left(e^{\alpha/x_n} - 1\right)\right)\right] \\ & + \sum_{i=2}^{n} log\left[exp\left(\frac{\alpha}{x_i} - \frac{1}{\alpha}\left(e^{\alpha/x_i} - 1\right)\right) - exp\left(\frac{\alpha}{x_{i-1}} - \frac{1}{\alpha}\left(e^{\alpha/x_{i-1}} - 1\right)\right)\right] \end{aligned} \tag{27}$$

To find the estimated value of α which maximize the equation (26) we use the method of optimization. For this we have to differentiate the equation (27) w.r.t. α and equate to zero,

$$\frac{\partial}{\partial \alpha}\big(log(S)\big) = 0 \tag{28}$$

On solving the above equation it found an estimated value of $\alpha = \hat{\alpha}_{mp}$, and to satisfy the condition of maximization by the value $\alpha = \hat{\alpha}_{mp}$, i.e.

$$\left[\frac{\partial^2}{\partial\alpha^2} log(S)\right]_{\alpha=\hat{\alpha}_{mp}} < 0 \tag{29}$$

The expression given in equation (27) and (28) together is not easy to solve and it is not in closed form. For the solution of this and to find the estimated value of α which maximize the product of spacing function given in equation (26) by satisfying the condition in equation (29) and we have used some numerical method to find the numerical solution of equation (28).

4. Asymptotic Confidence Interval

We have obtained both MLE and MPSE of the parameter which are not in explicit form. So the exact distribution of the estimator is quite difficult to obtain. The authors Cheng & Amin [16], Ghosh & Jammalamadaka [20], Anatolyev [21] and Singh et. al. [22] have used MPSE method in their papers and explained the MPSE method is asymptotically equivalent to MLE method. By using the concept of large sample theory we may write the asymptotic distribution for the estimators as,

$$\big(\hat{\theta} - \theta\big) \equiv N\big(0, I^{-1}(\hat{\theta})\big); \tag{30}$$

where,
$\hat{\theta}$ is the estimate of parameter
θ is the true value of parameter
$I^{-1}(\hat{\theta})$ is the inverse of Fisher information matrix

For m parameters $\theta_1, \theta_2, \theta_3, ..., \theta_m$ involved in a distribution the m x m Fisher information matrix is defined as

$$I(\hat{\theta}) = \begin{bmatrix} I_{1,1} & I_{1,2} & \cdots & I_{1,m} \\ I_{2,1} & I_{2,2} & \cdots & I_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ I_{m,1} & I_{m,2} & \cdots & I_{m,m} \end{bmatrix}$$

Where, $I_{i,j} = -E\left(\frac{\partial^2(L)}{\partial\theta_i\partial\theta_j}\right); \quad i, j = 1, 2, 3..., m$
And the estimated variance for $\hat{\theta}$ is given by:

$$Var(\hat{\theta}) = I_{i,j}^{-1} = -E\left(\frac{\partial^2(L)}{\partial\theta^2}\right)^{-1}_{\theta=\hat{\theta}}; \quad here \;\; i = j \tag{31}$$

This is the diagonal element of the inverse of Fisher information matrix. Therefore, the two sided $100(1-\alpha^*)$ % confidence interval for the θ is

$$\hat{\theta} \;\pm\; Z_{\alpha^*/2}\sqrt{Var(\hat{\theta})}; \tag{32}$$

Where, α^* is the level of significance and $Z_{\alpha^*/2}$ is upper $\alpha^*/2$ % point of standard normal distribution.
For the IM distribution, asymptotic confidence interval defined for the MLE is defined as:

$$\hat{\alpha}_{ml} \;\pm\; Z_{\alpha^*/2}\sqrt{Var(\hat{\alpha}_{ml})} \tag{33}$$

In the case of MPSE defined as :

$$\hat{\alpha}_{mp} \;\pm\; Z_{\alpha^*/2}\sqrt{Var(\hat{\alpha}_{mp})} \tag{34}$$

5. Scale transformation of IM distribution

We take a natural transformation (extension) of random variable by including a scale parameter say $\beta > 0$. The scale transformation is taken as $Z = \beta X$. Then the cdf of Z is given as,

$$F_Z(z) = exp\left\{\frac{\alpha \cdot \beta}{z} - \frac{1}{\alpha}\left(e^{\left(\frac{\alpha \cdot \beta}{z}\right)} - 1\right)\right\}; \quad \alpha \in (0,1] \;\; and \;\; \beta > 0 \tag{35}$$

the pdf is given by

$$f_Z(z;\alpha,\beta) = \frac{1}{\beta \cdot z^2}\left(e^{\left(\frac{\alpha \cdot \beta}{z}\right)} - \alpha\right) exp\left\{\frac{\alpha \cdot \beta}{z} - \frac{1}{\alpha}\left(e^{\left(\frac{\alpha \cdot \beta}{z}\right)} - 1\right)\right\}; \quad \alpha \in (0,1] \;\; and \;\; \beta > 0 \tag{36}$$

Since, the distribution of Z is obtained by the scaling transformation of X which follows the IM distribution with parameter α. So the new distribution of Z is called scaled inverse Muth (SIM) distribution. Here, it is noticeable that Z comes from X follows IM distribution, on taking scale transformation by adding a scale parameter β, thus SIM distribution has some properties as similar to IM distribution, like as moments of this distribution also does not exist etc. The quantile function for SIM is defined as:

$$Q_Z(u;\alpha,\beta) = \beta \cdot Q(u;\alpha); \quad 0 < u < 1$$

where $Q(u;\alpha)$ is the quantile function for IM distribution. So it becomes as

$$z_u = \frac{\beta \cdot \alpha^2}{\alpha.log(u) - \alpha.W_{-1}\left(\frac{-e^{-\frac{1}{\alpha}}.u}{\alpha}\right) - 1} \tag{37}$$

6. Simulation study

We have given numerical illustration of the results based on simulation study. We calculated the estimates of parameters, bias and confidence limit for parameter, based on generated random sample from IM distribution. The method of estimation MLE and MPS are used to compare the MSE of parameters. Less MSE gives more efficient method of estimation. We generated 10000 random samples for different sizes to find the estimates for each sample and calculated their MSE and bias using formula :

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2 \;\; and \;\; bias = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha), \quad where \;\; N = 10000$$

R-codes are used to all the numerical computation. To compute the numerical values first we generated a uniform random sample $U = u_1, u_2, u_3, ..., u_n$ of size n then generated random sample from both distribution by using their quantile function where 'u' is the uniform random sample. For each value of u_i we get x_i. In equation (15) and (37) $W_{-1}()$ is the lambert-W function which is calculated by "lambertWm1( )" command from package "lamW" in 'R', Adler [23].

6.1. Simulation study for IM distribution

To generate the random sample from IM distribution, we have used the quantile function equation (15). We used different sample size n = (15, 25, 50, 75, 100, 125) for each true value of parameter α= (0.3, 0.5, 0.7). In Table 2, we have given average value of MLE and MPSE of parameter α along with their respective MSEs, average value of bias, average length of confidence interval (CI) and average of the upper limit (UL) and lower limit (LL) of confidence interval for α= 0.3, 0.5, and 0.7. The output of simulation study is based on Table 2, explained as: for both method of estimation, MSE decreases as the sample size increases. For the small value of shape parameter α, MPSE has less MSE than MLE only for small sample, and for the large sample, MLE has less MSE than MPSE. From Table 2, it is observed that for large value of α within its range $\alpha \in (0,1]$, MLE has less MSE than MPSE to all sample size. In the case of MLE, bias is positive for each value of parameter and mostly negative in MPSE method. As usual, the average length of the CI decreases as the sample size increases for both the method MLE and MPSE. In Table 2, somewhere we found that LL of CI and UL of CI is going to outside of range of $\alpha \in (0,1]$, but IM distribution is defined for only $\alpha \in (0,1]$. For this we take 0.0000^* for $LL < 0$ and 1.0000^* for $UL > 1$.

6.2. Simulation study for SIM distribution

To generate the random sample from SIM distribution we have used the quantile function equation (37). We have used different sample size n = (15, 25, 50, 75, 100, 125) for different value of shape parameter α and scale parameter β. All the numerical value of average value of MLE and MPSE of parameter α and β along with their respective MSEs, average value of bias, average length of CI and average of the upper limit (UL) and lower limit (LL) of CI estimates presented in Table [3, 4, 5, 6, 7]. From these Tables, we can observe that MSE of the estimates of shape parameter α and scale parameter β, decreases as the sample size increases in case of MLE as well as in MPSE. At the fixed value of β and small value α, MPSE gives less MSE than MLE. It indicates that MPSE gives better estimates than MLE. For large value of $\alpha \in (0,1]$ at the same β, MLE gives less MSE than MPSE for all different sample sizes. Length of the CI decreases as the sample size increases in both the cases MLE and MPSE. MLE has mostly positive bias whereas MPSE has mostly negative bias. 0.0000^* and 1.0000^* defined same as above in section 6.1.

Table 2: *MLE and MPS estimate for α = 0.3, 0.5 and 0.7*

	n	MLE						MPS					
		Est.	bias	MSE	CI			Est.	bias	MSE	CI		
					LL	UL	length				LL	UL	length
α= 0.3	15	0.3954	0.0954	0.0497	0.0000*	0.8375	0.8375	0.2827	-0.0173	0.0323	0.0000*	0.7870	0.7870
	25	0.3544	0.0544	0.0307	0.0069	0.7019	0.6950	0.2661	-0.0339	0.0251	0.0000*	0.6493	0.6493
	50	0.3227	0.0227	0.0155	0.0776	0.5679	0.4903	0.2557	-0.0443	0.0152	0.0000*	0.5192	0.5192
	75	0.3149	0.0149	0.0105	0.1161	0.5136	0.3976	0.2624	-0.0376	0.0105	0.0531	0.4716	0.4185
	100	0.3108	0.0108	0.0078	0.1405	0.4811	0.3406	0.2662	-0.0338	0.0087	0.0885	0.4438	0.3554
	125	0.3097	0.0097	0.0060	0.1584	0.4610	0.3026	0.2755	-0.0245	0.0065	0.1194	0.4317	0.3123
α=0.5	15	0.5441	0.0441	0.0371	0.9732	0.1149	0.8583	0.3996	-0.1004	0.0454	0.0000*	0.9135	0.9135
	25	0.5329	0.0329	0.0261	0.8612	0.2046	0.6566	0.4250	-0.0750	0.0321	0.0541	0.7960	0.7419
	50	0.5157	0.0157	0.0133	0.7429	0.2884	0.4545	0.4464	-0.0536	0.0168	0.2025	0.6904	0.4879
	75	0.5098	0.0098	0.0090	0.6943	0.3252	0.3691	0.4545	-0.0455	0.0110	0.2615	0.6476	0.3861
	100	0.5099	0.0099	0.0068	0.6683	0.3515	0.3168	0.4654	-0.0346	0.0082	0.3015	0.6293	0.3278
	125	0.5094	0.0094	0.0054	0.6509	0.3679	0.2830	0.4678	-0.0322	0.0061	0.3219	0.6137	0.2918
α = 0.7	15	0.7007	0.0007	0.0253	0.3139	1.0000*	0.6861	0.5485	-0.1515	0.0527	0.0980	0.9990	0.9010
	25	0.7083	0.0083	0.0189	0.4124	1.0000*	0.5876	0.6048	-0.0952	0.0307	0.2790	0.9307	0.6518
	50	0.7097	0.0097	0.0110	0.5028	0.9165	0.4137	0.6435	-0.0565	0.0143	0.4256	0.8614	0.4357
	75	0.7079	0.0079	0.0076	0.5389	0.8769	0.3380	0.6558	-0.0442	0.0098	0.4802	0.8313	0.3511
	100	0.7079	0.0079	0.0056	0.5614	0.8543	0.2928	0.6678	-0.0322	0.0065	0.5173	0.8183	0.3011
	125	0.7041	0.0041	0.0042	0.5731	0.8350	0.2618	0.6732	-0.0268	0.0051	0.5392	0.8073	0.2681

Est.: Estimate; MSE: Mean Square Error; CI: Confidence interval; UL: Upper limit; LL: Lower limit.

Table 3: *MLE and MPS estimate for $\alpha = 0.3$ and $\beta = 2$*

	n	MLE						MPS					
		Est.	bias	MSE	CI			Est.	bias	MSE	CI		
					UL	LL	length				UL	LL	length
$\alpha = 0.3$	15	0.4302	0.1302	0.0606	0.8548	0.0056	0.8492	0.3325	0.0325	0.0386	0.8613	0.0000*	0.8613
	25	0.3856	0.0856	0.0369	0.7171	0.0540	0.6630	0.3128	0.0128	0.0268	0.7014	0.0000*	0.7014
	50	0.3422	0.0422	0.0181	0.5782	0.1063	0.4719	0.2886	-0.0114	0.0156	0.5496	0.0276	0.5220
	75	0.3254	0.0254	0.0113	0.5184	0.1324	0.3861	0.2887	-0.0113	0.0105	0.4960	0.0815	0.4146
	100	0.3219	0.0219	0.0085	0.4888	0.1549	0.3339	0.2774	-0.0226	0.0075	0.4552	0.0997	0.3555
	125	0.3224	0.0224	0.0077	0.4716	0.1733	0.2983	0.2937	-0.0063	0.0070	0.4500	0.1373	0.3127
$\beta = 2$	15	2.0570	0.0570	0.1831	2.7382	1.3758	1.3624	1.9418	-0.0582	0.1756	2.6697	1.2139	1.4558
	25	2.0256	0.0256	0.1004	2.5621	1.4890	1.0730	1.9670	-0.0330	0.0881	2.5401	1.3940	1.1460
	50	2.0114	0.0114	0.0499	2.3998	1.6229	0.7769	1.9697	-0.0303	0.0437	2.3773	1.5621	0.8152
	75	2.0082	0.0082	0.0328	2.3284	1.6879	0.6405	1.9694	-0.0306	0.0304	2.2993	1.6395	0.6598
	100	2.0112	0.0112	0.0243	2.2892	1.7332	0.5560	1.9750	-0.0250	0.0247	2.2636	1.6864	0.5771
	125	1.9968	-0.0032	0.0185	2.2433	1.7503	0.4929	1.9732	-0.0268	0.0193	2.2262	1.7203	0.5058

Table 4: *MLE and MPS estimate for $\alpha = 0.5$ and $\beta = 2$*

	n	MLE						MPS					
		Est.	bias	MSE	CI			Est.	bias	MSE	CI		
					UL	LL	length				UL	LL	length
$\alpha = 0.5$	15	0.5675	0.0675	0.0432	0.9683	0.1668	0.8015	0.4415	-0.0585	0.0417	0.9266	0.0000*	0.9266
	25	0.5524	0.0524	0.0317	0.8630	0.2417	0.6213	0.4630	-0.0370	0.0288	0.8150	0.1110	0.7040
	50	0.5315	0.0315	0.0172	0.7520	0.3110	0.4411	0.4776	-0.0224	0.0162	0.7138	0.2413	0.4724
	75	0.5163	0.0163	0.0108	0.6971	0.3354	0.3617	0.4740	-0.0260	0.0109	0.6641	0.2839	0.3802
	100	0.5171	0.0171	0.0087	0.6736	0.3606	0.3130	0.4833	-0.0167	0.0084	0.6459	0.3207	0.3252
	125	0.5123	0.0123	0.0070	0.6525	0.3721	0.2804	0.4833	-0.0167	0.0065	0.6280	0.3387	0.2893
$\beta = 2$	15	2.0673	0.0673	0.1349	2.6636	1.4710	1.1926	1.9923	-0.0077	0.1134	2.6630	1.3216	1.3414
	25	2.0262	0.0262	0.0766	2.4799	1.5725	0.9074	1.9826	-0.0174	0.0797	2.4791	1.4860	0.9931
	50	2.0042	0.0042	0.0347	2.3231	1.6854	0.6377	1.9838	-0.0162	0.0332	2.3209	1.6466	0.6743
	75	2.0093	0.0093	0.0236	2.2724	1.7463	0.5261	1.9884	-0.0116	0.0231	2.2624	1.7144	0.5480
	100	2.0038	0.0038	0.0172	2.2303	1.7774	0.4529	1.9930	-0.0070	0.0180	2.2275	1.7584	0.4691
	125	2.0014	0.0014	0.0143	2.2043	1.7985	0.4058	1.9917	-0.0083	0.0170	2.2007	1.7826	0.4180

Table 5: *MLE and MPS estimate for $\alpha = 0.7$ and $\beta = 2$*

	n	MLE						MPS					
		Est.	bias	MSE	CI			Est.	bias	MSE	CI		
					UL	LL	length				UL	LL	length
α=0.7	15	0.7055	0.0055	0.0289	1.0000*	0.3277	0.6723	0.5808	-0.1192	0.0437	1.0000*	0.1437	0.8563
	25	0.7167	0.0167	0.0218	1.0000	0.4270	0.5730	0.6253	-0.0747	0.0269	0.9437	0.3068	0.6369
	50	0.7187	0.0187	0.0142	0.9237	0.5138	0.4099	0.6600	-0.0400	0.0149	0.8762	0.4438	0.4324
	75	0.7118	0.0118	0.0097	0.8795	0.5440	0.3355	0.6717	-0.0283	0.0100	0.8457	0.4977	0.3480
	100	0.7111	0.0111	0.0077	0.8566	0.5656	0.2910	0.6786	-0.0214	0.0080	0.8283	0.5289	0.2994
	125	0.7089	0.0089	0.0059	0.8393	0.5784	0.2609	0.6795	-0.0205	0.0061	0.8131	0.5458	0.2673
β=2	15	2.0806	0.0806	0.1020	2.6046	1.5566	1.0480	2.0263	0.0263	0.0918	2.6177	1.4349	1.1828
	25	2.0385	0.0385	0.0566	2.4285	1.6486	0.7799	2.0085	0.0085	0.0578	2.4347	1.5824	0.8523
	50	2.0078	0.0078	0.0257	2.2763	1.7392	0.5370	1.9905	-0.0095	0.0235	2.2740	1.7071	0.5669
	75	2.0117	0.0117	0.0185	2.2317	1.7916	0.4401	2.0021	0.0021	0.0181	2.2307	1.7736	0.4571
	100	2.0037	0.0037	0.0132	2.1932	1.8141	0.3790	1.9980	-0.0020	0.0135	2.1935	1.8024	0.3911
	125	2.0063	0.0063	0.0106	2.1761	1.8366	0.3395	2.0005	0.0005	0.0119	2.1750	1.8260	0.3490

Table 6: *MLE and MPS estimate for $\alpha = 0.3$ and $\beta = 5$*

	n	MLE						MPS					
		Est.	bias	MSE	CI			Est.	bias	MSE	CI		
					UL	LL	length				UL	LL	length
α=0.3	15	0.4282	0.1282	0.0589	0.8552	0.0012	0.8540	0.3231	0.0231	0.0375	0.8626	0.0000*	0.8626
	25	0.3846	0.0846	0.0383	0.7175	0.0517	0.6658	0.2996	-0.0004	0.0282	0.6955	0.0000*	0.6955
	50	0.3452	0.0452	0.0188	0.5815	0.1090	0.4725	0.2840	-0.0160	0.0169	0.5469	0.0211	0.5258
	75	0.3359	0.0359	0.0119	0.5284	0.1434	0.3850	0.2880	-0.0120	0.0117	0.4960	0.0801	0.4159
	100	0.3265	0.0265	0.0085	0.4933	0.1597	0.3336	0.2967	-0.0033	0.0089	0.4732	0.1202	0.3530
	125	0.3161	0.0161	0.0061	0.4657	0.1666	0.2991	0.2896	-0.0104	0.0070	0.4464	0.1328	0.3136
β=5	15	5.0498	0.0498	0.9296	6.7910	3.3085	3.4825	4.7917	-0.2083	0.8598	6.6624	2.9209	3.7415
	25	5.0434	0.0434	0.6235	6.4279	3.6589	2.7690	4.8389	-0.1611	0.5573	6.3059	3.3719	2.9340
	50	5.0169	0.0169	0.3575	6.0060	4.0279	1.9781	4.8789	-0.1211	0.3539	5.9168	3.8409	2.0759
	75	5.0488	0.0488	0.2548	5.8586	4.2390	1.6196	4.9499	-0.0501	0.2534	5.7944	4.1053	1.6891
	100	5.0276	0.0276	0.2150	5.7291	4.3261	1.4030	4.9380	-0.0620	0.2150	5.6565	4.2196	1.4369
	125	5.0530	0.0530	0.1917	5.6877	4.4183	1.2694	4.9720	-0.0280	0.1821	5.6200	4.3240	1.2960

Table 7: *MLE and MPS estimate for α = 0.3 and β = 10*

	n	MLE						MPS					
		Est.	bias	MSE	CI			Est.	bias	MSE	CI		
					UL	LL	length				UL	LL	length
α=0.3	15	0.4349	0.1349	0.0623	0.8606	0.0092	0.8514	0.3290	0.0290	0.0377	0.8659	0.0000*	0.8659
	25	0.3855	0.0855	0.0389	0.7182	0.0528	0.6654	0.3053	0.0053	0.0284	0.6994	0.0000*	0.6994
	50	0.3503	0.0503	0.0197	0.5860	0.1146	0.4714	0.2899	-0.0101	0.0173	0.5521	0.0277	0.5244
	75	0.3424	0.0424	0.0121	0.5344	0.1504	0.3840	0.2998	-0.0002	0.0120	0.5065	0.0931	0.4134
	100	0.3273	0.0273	0.0086	0.4941	0.1605	0.3336	0.3002	0.0002	0.0093	0.4765	0.1240	0.3525
	125	0.3237	0.0237	0.0070	0.4728	0.1745	0.2983	0.2916	-0.0084	0.0062	0.4482	0.1350	0.3132
β=10	15	9.9635	-0.0365	3.4750	13.3824	6.5447	6.8377	9.4550	-0.5450	3.4249	13.1285	5.7816	7.3469
	25	10.0247	0.0247	2.6036	12.7796	7.2697	5.5099	9.5618	-0.4382	2.4351	12.4525	6.6711	5.7814
	50	9.9386	-0.0614	1.5441	11.8934	7.9838	3.9096	9.6508	-0.3492	1.5991	11.6994	7.6022	4.0972
	75	10.0207	0.0207	1.1687	11.6192	8.4223	3.1969	9.7578	-0.2422	1.2321	11.4079	8.1076	3.3003
	100	10.0328	0.0328	1.0610	11.4341	8.6315	2.8026	9.7869	-0.2131	1.0286	11.2072	8.3667	2.8405
	125	10.0279	0.0279	0.8687	11.2806	8.7752	2.5054	9.8581	-0.1419	0.8287	11.1387	8.5774	2.5613

7. Real data analysis

The real data have been used to show the applicability of the SIM distribution. The results show this model is more appropriate than some other fitted model for this data. The data represent the active repair time (in hrs.) for airborne communication transceiver given in Jorgensen [24]. The data is given as below:

0.50	0.60	0.60	0.70	0.70	0.70	0.80	0.80
1.00	1.00	1.00	1.00	1.10	1.30	1.50	1.50
1.50	1.50	2.00	2.00	2.20	2.50	2.70	3.00
3.00	3.30	4.00	4.00	4.50	4.70	5.00	5.40
5.40	7.00	7.50	8.80	9.00	10.20	22.00	24.50

For the fitting of above real data to the proposed model we used Kolmogorov–Smirnov test (K–S test). In order to compare the models we used negative log-likelihood function define as $-logL(\hat{\alpha}, \hat{\beta})$ values, Akaike information criteria (AIC) values defined by $AIC = -2log(L) + 2q$ and Bayesian information criterion (BIC) values defined $BIC = -2log(L) + q \cdot log(n)$ by BIC where, $\hat{\alpha}_{ml}$, $\hat{\beta}_{ml}$ are the estimates of parameter α and β by using MLE method, q is the number of parameters and n is the sample size. The best fitted distribution is that distribution which gives the lower values of $-log(L)$, AIC and BIC.
From the Table 8 it is obtained that SIM distribution give best fit among some other popular distributions. And the MLE of parameters of SIM and some other distributions given in Table 9. Figure 3 shows that empirical cdf and fitted cdf plot for SIM and some other distributions.

Table 8: *Comparison criterion values for different distribution.*

Model	AIC	BIC	-log(L)	k-s statistic	p-value
SIMD $(x; \alpha, \beta)$	182.6664	182.3504	89.3332	0.0869	0.9231
EPLD $(x; \alpha, \beta, \theta)$	186.5721	191.6387	90.2861	0.0909	0.8627
PLD $(x; \beta, \theta)$	195.8854	199.2631	95.9427	0.1346	0.4637
GLD $(x; \alpha, \theta)$	199.8218	203.1995	97.9107	0.1660	0.2201

SIMD: Scaled inverse Muth distribution; **EPLD:** Exponentiated power Lindley distribution; **PLD:** Power Lindley distribution; **GLD:** Generalized Lindley distribution.

Table 9: *MLE for the parameters of different distributions.*

Model	θ	β	α
SIMD $(x;\alpha,\beta)$	-	1.5464	0.2630
EPLD $(x;\alpha,\beta,\theta)$	3.5472	0.2901	30.8299
PLD $(x;\beta,\theta)$	0.5867	0.7988	-
GLD $(x;\alpha,\theta)$	0.3588	-	0.7460

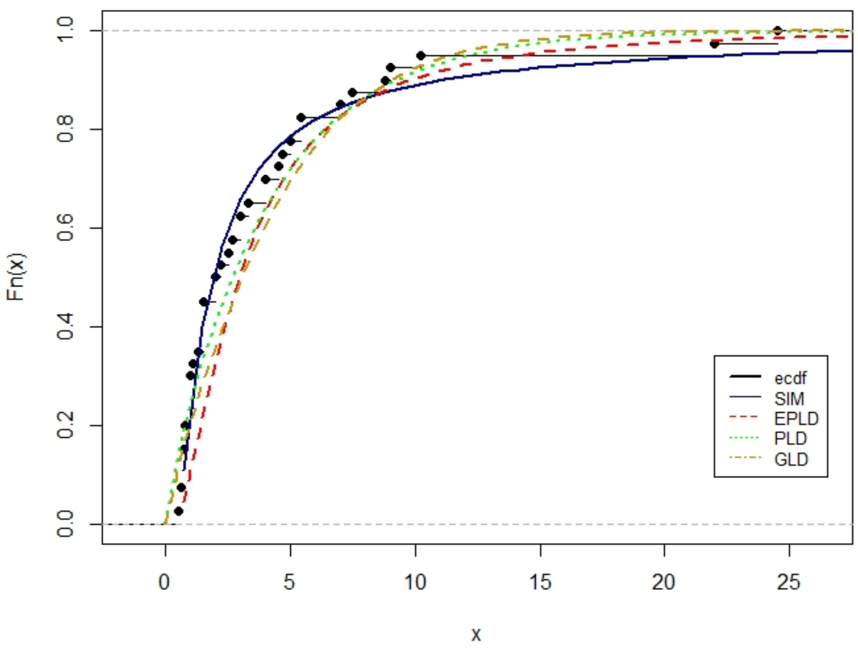


Figure 3: *Empirical cdf and fitted cdf plot.*

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