A new method for generating distributions with an application to Weibull distribution

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Abstract

In the literature of probability theory, it has been noticed that the classical probability distributions do not furnish an ample fit and fail to model the real-life data with a non-monotonic hazard rate behaviour. To overcome this limitation, researchers are working in the refinement of these distributions. In this paper, a new method has been presented to add an extra parameter to a family of distributions for more flexibility and potentiality. We have specialized this method to two-parametric Weibull distribution. A comprehensive mathematical treatment of the new distribution is provided. We provide closed-form expressions for the density, cumulative distribution, reliability function, hazard rate function, the r-th moment, moment generating function, and also the order statistics. Moreover, we discussed mean residual life time, stress strength reliability and maximum likelihood estimation. The adequacy of the proposed distribution is supported by using two real lifetime data sets as well as simulated data.

Keywords: Weibull distribution, hazard rate function, survival function, mean residual life, Maximum likelihood estimation.

1. Introduction

Weibull distribution is a well known life time distribution in reliability engineering and failure analysis. The Weibull distribution is used in modelling the engineering, biological, weather forecasting and hydrogical data sets. It does not impart an admissible fit for some applications, espacially, when the hazard rates are bathtub, upside down bathtub, or bimodal shapes. To overcome these limitations, several researchers have developed various modifications and extensions of the Weibull distribution to model various types of data. Many extentions and generalizations of the Weibull distribution have accomplished the above purpose. Among these, Xie and Lai [1] introduced the additive Weibull distribution, Mudholkar et al. [2] proposed exponentiated Weibull (EW) distribution by adding an extra parameter to the Weibull distribution which provides bathtub shaped hazard rate function. Xie et al. [3] proposed the the extended Weibull distribution. Carrasco et al. [4] presented generalized modified Weibull (GMW) distribution. Modified Weibull by Lai et al. [5]; extended flexible Weibull by Bebbington et al. [6]. The exponential-Weibull distribution by Cordeiro et al. [7]. Lee et al. [8] and Alzaatreh et al. [9] proposed methods of generalized continuous and discrete distributions.

Mahdavi and Kundu [10] proposed a method called the Alpha Power Transformation (APT) and it is useful to assimilates skewness to a family of distributions. Let $F(x)$ be the cumulative distribution function (cdf) of a continuous random variable X, then they define the APT of $F(x)$ for $x \in \mathbb{R}$ as follows

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1} & ;\alpha \in \mathbb{R}^{+}, \quad \alpha \neq 1 \\ F(x) & ;\alpha = 1 \end{cases}$$

and the corresponding probability density function (pdf) as

$$f_{APT}(x) = \begin{cases} \frac{log\alpha}{\alpha-1} f(x)\alpha^{F(x)} & ;\alpha \in \mathbb{R}^{+}, \quad \alpha \neq 1 \\ f(x) & ;\alpha = 1 \end{cases}$$

They applied the proposed method to a one-parameter exponential distribution and generated a two-parameter Alpha Power Exponential distribution.

Recently, Ijaz et al. [11] proposed a new family of distributions named as New Alpha Power Trasformed family (NAPT) of distributions. They employed exponential distribution in NAPT family and derived a new distribution called New Alpha Power Trasformed exponential (NAPTE) distribution. Let $F(x)$ be the cdf of a continuous random variable X, then they define the NAPT of $F(x)$ for $x \in \mathbb{R}$ as follows

$$F_{NAPT}(x) = \alpha^{-log\left(\frac{1}{F(x)}\right)}; \qquad \alpha > 0$$

and the corresponding pdf as

$$f_{NAPT}(x) = \frac{log(\alpha)\alpha^{-log\left(\frac{1}{F(x)}\right)} f(x)}{F(x)}; \qquad \alpha > 0$$

The following are the primary motivations for disposing Ratio Transformation (RT) method in practise:

- A straightforward and efficient method for adding an extra parameter to an existing distributions.
- To enhance the characteristics and flexibility of existing distributions.
- It is quite easy to use, hence it can be used quite effectively for data analysis purposes.
- To present the extended version of the baseline distribution that includes closed forms of cdf, reliability function as well as hazard rate function.
- To provide better fits than the other modified models having the same or higher number of parameters.

The remainder of the paper is organized as follows: In section 2 a new family of probability distributions called RT has been highlighted and some general properties of this family have been discussed. In section 3, RTW distribution has been considered, some special cases are presented and its structural properties including moments, moment generatin function, mean residual life and mean waiting time, order statistic and stress-strength reliability have been discussed. In section 4, Maximum likelihood estimators of unknown parameter as well as simulation study have been carried out. In secton 5, Two real life data sets have been analyzed to illustrate the potency of the proposed model. Finally, the paper is concluded in section 6.

2. General properties of RT method

Let $F(x)$ be the cdf of a continuous random variable X, then the Ratio transformation of $F(x)$ for $x \in \mathbb{R}$, is defined as follows

$$F_{RT}(x) = \frac{F(x)}{1+\alpha-\alpha^{F(x)}} \; ; \qquad \alpha > 0 \tag{1}$$

Clearly, $F_{RT}(x)$ is a proper cdf. If $F(x)$ is an absolute continuous distribution function with the pdf $f(x)$, then $F_{RT}(x)$ is also an absolute continuous distribution function with the pdf

$$f_{RT}(x) = f(x)\frac{\left(1+\alpha-\alpha^{F(x)}\left(1-F(x)log\alpha\right)\right)}{\left(1+\alpha-\alpha^{F(x)}\right)^2} \; ; \qquad \alpha > 0 \tag{2}$$

A useful expansion for the cdf and pdf in (1) and (2) are respectively given by

$$F_{RT}(x) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}(F(x))^{k+1} \tag{3}$$

where,

$$a_{jk} = \frac{(j\ log\alpha)^k}{k!\ (1+\alpha)^{j+1}}$$

and

$$f_{RT}(x) = f(x)\left[1-\frac{\alpha^{F(x)}}{1+\alpha}\left(1-F(x)log\alpha\right)\right]\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{jk}F^k(x) \tag{4}$$

where,

$$b_{jk} = \frac{(j+1)(j\ log\alpha)^k}{(1+\alpha)^{j+1}k!}$$

The reliability function $R_{RT}(x)$ is given by

$$R_{RT}(x) = \frac{1+\alpha-\alpha^{F(x)}-F(x)}{1+\alpha-\alpha^{F(x)}} \; ; \qquad \alpha > 0 \tag{5}$$

The hazard rate function $h_{RT}(x)$ is given by

$$h_{RT}(x) = f(x)\frac{\left(1+\alpha-\alpha^{F(x)}\left(1-F(x)log\alpha\right)\right)}{\left(1+\alpha-\alpha^{F(x)}\right)\left(1+\alpha-\alpha^{F(x)}-F(x)\right)} \; ; \qquad \alpha > 0 \tag{6}$$

If $R(x)$ and $h(x)$ are the reliability and hazard rate functions of f respectively, then the hazard rate $h_{RT}(x)$ can be written as

$$h_{RT}(x) = h(x)R(x)\frac{\left(1+\alpha-\alpha^{F(x)}\left(1-F(x)log\alpha\right)\right)}{\left(1+\alpha-\alpha^{F(x)}\right)\left(1+\alpha-\alpha^{F(x)}-F(x)\right)} \; ; \qquad \alpha > 0 \tag{7}$$

From (7), it is clear that

$$\lim_{x\to-\infty} h_{RT}(x) = \frac{1}{\alpha}\lim_{x\to-\infty} h(x)$$

and,

$$\lim_{x\to\infty} h_{RT}(x) = \lim_{x\to\infty} h(x)$$

3. RTW distribution and its properties

Let $\Theta = (\alpha, \lambda, \beta)^T$. From (2), The continuous randon variable X follows RTW distribution if its cdf, with scale parameter $\lambda > 0$ and shape parameters $\alpha > 0, \beta > 0$, for $x \in \mathbb{R}^+$ is given by

$$F_{RTW}(x,\Theta) = \frac{1 - e^{-\lambda x^\beta}}{1 + \alpha - \alpha^{1-e^{-\lambda x^\beta}}} \; ; \qquad \alpha > 0 \tag{8}$$

and the corresponding pdf is

$$f_{RTW}(x,\Theta) = \frac{\lambda \beta x^{\beta-1} e^{-\lambda x^\beta} \left(1 + \alpha - \alpha^{1-e^{-\lambda x^\beta}} \left(1 - (1 - e^{-\lambda x^\beta}) log\alpha\right)\right)}{\left(1 + \alpha - \alpha^{1-e^{-\lambda x^\beta}}\right)^2} \; ; \; \alpha > 0 \tag{9}$$

Using (3) and (4), the cdf and pdf in (8) and (9) can be respectively written as

$$F_{RTW}(x,\Theta) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k+1} a_{jkl} \; e^{-l\lambda x^\beta}$$

where,

$$a_{jkl} = \frac{(j \; log\alpha)^k \binom{k+1}{l} (-1)^l}{k! \; (1+\alpha)^{j+1}}$$

and

$$f_{RTW}(x,\Theta) = x^{\beta-1} \left[1 - \frac{\alpha^{\left(1-e^{-\lambda x^\beta}\right)}}{1+\alpha} (1 - log\alpha \; (1 - e^{-\lambda x^\beta})) \right] \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k} b_{jkl} e^{-\lambda (l+1) x^\beta}$$

where,

$$b_{jkl} = \frac{\lambda \beta (j+1) (j \; log\alpha)^k \binom{k}{l} (-1)^l}{(1+\alpha)^{j+1} k!}$$

The reliability function $R_{RTW}(x,\Theta)$ and the hazard rate function $h_{RTW}(x,\Theta)$ for $x \in \mathbb{R}^+$ are, respectively, given by

$$R_{RTW}(x,\Theta) = \frac{\alpha \left(1 - \alpha^{-e^{-\lambda x^\beta}}\right) + e^{-\lambda x^\beta}}{1 + \alpha - \alpha^{1-e^{-\lambda x^\beta}}} \; ; \qquad \alpha > 0 \tag{10}$$

$$h_{RTW}(x,\Theta) = \frac{\lambda \beta x^{\beta-1} e^{-\lambda x^\beta} \left(1 + \alpha - \alpha^{1-e^{-\lambda x^\beta}} \left(1 - (1 - e^{-\lambda x^\beta}) log\alpha\right)\right)}{\left(1 + \alpha - \alpha^{1-e^{-\lambda x^\beta}}\right) \left(\alpha \left(1 - \alpha^{-e^{-\lambda x^\beta}}\right) + e^{-\lambda x^\beta}\right)} \; ; \; \alpha > 0$$

The behaviour of the hazard rate function at extremes for different values of shape parameter β.

$$h(0) = \begin{cases} \infty & for \; 0 < \beta < 1, \\ \frac{\lambda}{\alpha} & for \; \beta = 1, \\ 0 & for \; \beta > 1, \end{cases} \qquad h(\infty) = \begin{cases} 0 & for \; 0 < \beta < 1, \\ \lambda & for \; \beta = 1, \\ \infty & for \; \beta > 1. \end{cases}$$

Remark: When $\alpha = 1$, the RTW distribution becomes the Weibull distribution. In that situation the shapes for hazard rate function are conspicuous in the literature. The seven important special cases of RTW distribution are presented in table 1

Figure 1 depicts some plots of the RTW density for selected parameter values. Plots of the hazard rate function of the RTW distribution for selected parameter values are displayed in Figure 2.

Table 1: *Sub-cases of the RTW Distribution*

α	λ	β	Reduced model
-	1	-	RT one-parameter Weibull distribution
1	-	-	Two-parameter Weibull distribution
1	1	-	One-parameter Weibull distribution
-	-	2	RT-Rayleigh distribution
1	-	2	Rayleigh distribution
-	-	1	RT-exponential distribution
1	-	1	Exponential distribution

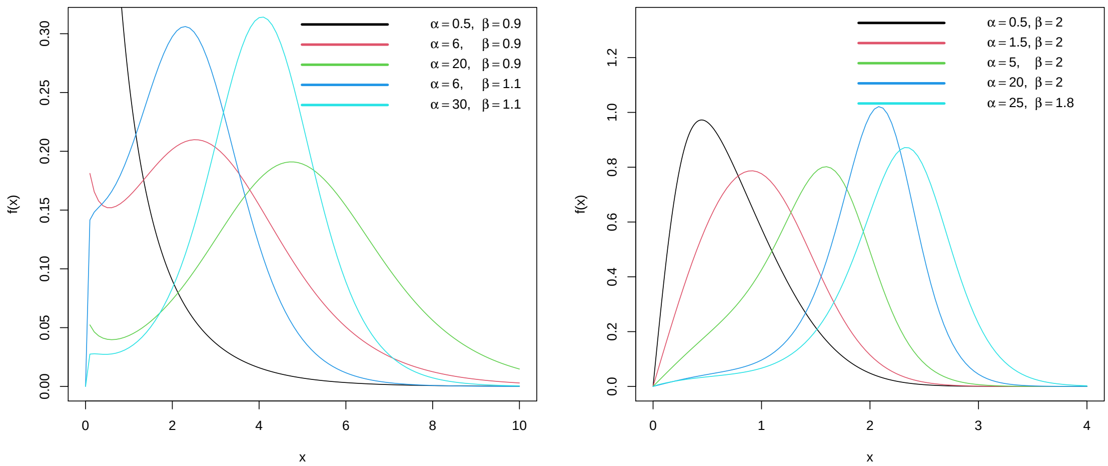

Figure 1: *Plots of the RTW density for $\lambda = 1$ and various values of α and β.*

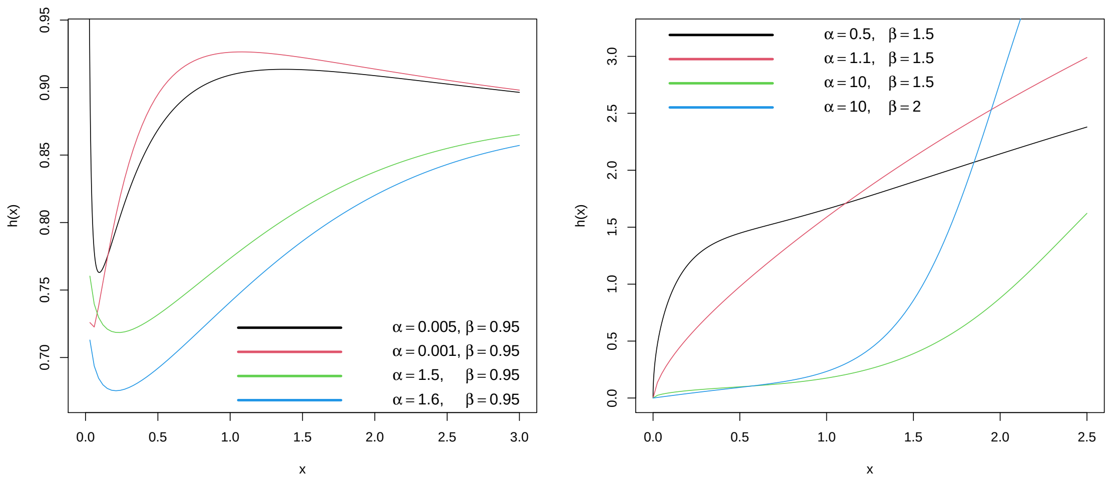

Figure 2: *Plots of the RTW hazard rate function for $\lambda = 1$ and various values of α and β.*

3.1. Moment and moment generating function

In this subsection, the rth moment and the moment generating function of the RTW distribution are obtained by using the following series representations.

$$\alpha^{-x} = \sum_{k=0}^{\infty} \frac{(-log\alpha)^k x^k}{k!} \tag{11}$$

$$(1-x)^{-2} = \sum_{k=0}^{\infty} (k+1)x^k \; ; \qquad |x| < 1, \tag{12}$$

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k \; ; \qquad |x| < 1, \tag{13}$$

The rth moment of X can be obtained as

$$\begin{aligned} E(X^r) &= \int_0^{\infty} x^r f(x) dx \\ &= \frac{1}{(1+\alpha)^2} \int_0^{\infty} x^r \lambda \beta x^{\beta-1} e^{-\lambda x^\beta} \left(1+\alpha-\alpha^{1-e^{-\lambda x^\beta}} \left(1-(1-e^{-\lambda x^\beta}) log\alpha\right)\right) \\ &\times \left(1-\frac{\alpha^{1-e-\lambda x^\beta}}{1+\alpha}\right)^{-2} dx \end{aligned} \tag{14}$$

By substituting $1-e^{-\lambda x^\beta} = y$ in (14), we get

$$E(X^r) = \sum_{j=0}^{\infty} \frac{1}{(1+\alpha)^{j+1}} \left(\int_0^1 \left(\frac{-1}{\lambda} log(1-y)\right)^{\frac{r}{\beta}} \left(\alpha^{jy} + \frac{\alpha^{(j+1)y}(j+1)log\alpha}{1+\alpha} y\right) dy \right) \tag{15}$$

Again, substituting $\frac{-1}{\lambda} log(1-y) = x$ in (15), we get the final expression as

$$E(X^r) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda \alpha^j (-log\alpha)^k}{(1+\alpha)^{j+1} k!} \Gamma(\frac{r}{\beta}+1) \{A+B\}$$

where,

$$A = \frac{j^k}{(\lambda(k+1))^{\frac{r}{\beta}+1}}$$

and

$$B = \frac{\alpha \; log\alpha \; (j+1)^{k+1}}{1+\alpha} \left(\frac{1}{(\lambda(k+1))^{\frac{r}{\beta}+1}} - \frac{1}{(\lambda(k+2))^{\frac{r}{\beta}+1}} \right)$$

and the moment generating function can be obtained as

$$M_X(t) = \int_0^{\infty} e^{tx} f(x) dx$$

by using the same procedure as above, we get the final expression for moment generating function as

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda t^i \alpha^j (-log\alpha)^k}{(1+\alpha)^{j+1} k! i!} \Gamma(\frac{i}{\beta}+1) \{C+D\}$$

where,

$$C = \frac{j^k}{(\lambda(k+1))^{\frac{i}{\beta}+1}}$$

and

$$D = \frac{\alpha\ log\alpha\ (j+1)^{k+1}}{1+\alpha}\left(\frac{1}{(\lambda(k+1))^{\frac{i}{\beta}+1}} - \frac{1}{(\lambda(k+2))^{\frac{i}{\beta}+1}}\right)$$

3.2. Mean residual life and mean waiting time

Suppose that X is a continuous random variable with reliability function $R(x)$, the mean residual life is the expected additional lifetime given that a component has survived until time t. The mean residual life function, say $\mu(t)$, is given by

$$\mu(t) = \frac{1}{R(t)}\left(E(t) - \int_0^t xf(x)dx\right) - t \tag{16}$$

where

$$E(t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\lambda\alpha^j(-log\alpha)^k}{(1+\alpha)^{j+1}k!}\Gamma(\frac{1}{\beta}+1)\left\{\frac{j^k}{(\lambda(k+1))^{\frac{1}{\beta}+1}} + \frac{\alpha\ log\alpha\ (j+1)^{k+1}}{1+\alpha}\right.$$
$$\left.\times\left(\frac{1}{(\lambda(k+1))^{\frac{1}{\beta}+1}} - \frac{1}{(\lambda(k+2))^{\frac{1}{\beta}+1}}\right)\right\} \tag{17}$$

and

$$\int_0^t xf(x)dx = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\alpha^j j^k(-log\alpha)^k}{(1+\alpha)^{j+1}k!}\left\{\frac{1}{\lambda^{\frac{1}{\beta}}(k+1)^{\frac{1}{\beta}+1}}\gamma\left(\lambda(k+1)t^\beta, \frac{1}{\beta}+1\right)\right.$$
$$\left.+\frac{(j+1)log\alpha}{1+\alpha}\left[\frac{\gamma\left(\lambda(k+1)t^\beta, \frac{1}{\beta}+1\right)}{\lambda^{\frac{1}{\beta}}(k+1)^{\frac{1}{\beta}+1}} - \frac{\gamma\left(\lambda(k+1)t^\beta, \frac{1}{\beta}+1\right)}{\lambda^{\frac{1}{\beta}}(k+2)^{\frac{1}{\beta}+1}}\right]\right\} \tag{18}$$

Substituting (10), (17) and (18) in (16), $\mu(t)$ can be written as

$$\mu(t) = \frac{1+\alpha-\alpha^{1-e^{-\lambda t^\beta}}}{\alpha\left(1-\alpha^{-e^{-\lambda t^\beta}}\right)+e^{-\lambda t^\beta}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\alpha^j(-log\alpha)^k}{(1+\alpha)^{j+1}k!}$$
$$\times\left(A' + B'\frac{-j^k}{\Gamma(\frac{1}{\beta}+1)}(C'+D')\right) - t$$

where,

$$A' = \frac{j^k}{(\lambda(k+1))^{\frac{1}{\beta}+1}},$$
$$B' = \frac{\alpha\ log\alpha\ (j+1)^{k+1}}{1+\alpha}\left(\frac{1}{(\lambda(k+1))^{\frac{1}{\beta}+1}} - \frac{1}{(\lambda(k+2))^{\frac{1}{\beta}+1}}\right),$$
$$C' = \frac{\gamma\left(\lambda(k+1)t^\beta, \frac{1}{\beta}+1\right)}{\lambda^{\frac{1}{\beta}}(k+1)^{\frac{1}{\beta}+1}}$$

and

$$D' = \frac{(j+1)log\alpha}{1+\alpha}\left[\frac{\gamma\left(\lambda(k+1)t^{\beta},\frac{1}{\beta}+1\right)}{\lambda^{\frac{1}{\beta}}(k+1)^{\frac{1}{\beta}+1}} - \frac{\gamma\left(\lambda(k+1)t^{\beta},\frac{1}{\beta}+1\right)}{\lambda^{\frac{1}{\beta}}(k+2)^{\frac{1}{\beta}+1}}\right]$$

where $\gamma(a,b) = \int_0^a x^{b-1}e^{-x}dx$ is the lower incomplete gamma function.

The mean waiting time represents the waiting time elapsed since the failure of an object on condition that this failure had occurred in the interval [0, t]. The mean waiting time of X, say $\bar{\mu}(t)$, is defined by

$$\bar{\mu}(t) = t - \frac{1}{F(t)}\int_0^t xf(x)dx. \tag{19}$$

Substituting (8) and (18) in (19), we get

$$\begin{aligned}\bar{\mu}(t) = t - \frac{1+\alpha-\alpha^{1-e^{-\lambda t^{\beta}}}}{1-e^{-\lambda t^{\beta}}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\alpha^j j^k(-log\alpha)^k}{(1+\alpha)^{j+1}k!}\Bigg\{\frac{1}{\lambda^{\frac{1}{\beta}}(k+1)^{\frac{1}{\beta}+1}} \\ \times\ \gamma\left(\lambda(k+1)t^{\beta},\frac{1}{\beta}+1\right) + \frac{(j+1)log\alpha}{1+\alpha}\left[\frac{\gamma\left(\lambda(k+1)t^{\beta},\frac{1}{\beta}+1\right)}{\lambda^{\frac{1}{\beta}}(k+1)^{\frac{1}{\beta}+1}}\right. \\ \left. - \frac{\gamma\left(\lambda(k+1)t^{\beta},\frac{1}{\beta}+1\right)}{\lambda^{\frac{1}{\beta}}(k+2)^{\frac{1}{\beta}+1}}\right]\Bigg\}\end{aligned}$$

3.3. Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n, and let $X_{r:n}$ denote the rth order statistic, then, the pdf of $X_{r:n}$, say $f_{r:n}(x)$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}F(x)^{r-1}f(x)(1-F(x))^{n-r}. \tag{20}$$

Substituting (8) and (9) in (20), we get

$$\begin{aligned}f_{r:n}(x) = \frac{\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}\left(1+\alpha-\alpha^{1-e^{-\lambda x^{\beta}}}\left(1-(1-e^{-\lambda x^{\beta}})log\alpha\right)\right)}{B(r,n-r+1)\left(1+\alpha-\alpha^{1-e^{-\lambda x^{\beta}}}\right)^{n+1}} \\ \times\left(1-e^{-\lambda x^{\beta}}\right)^{r-1}\left(\alpha\left(1-\alpha^{-e^{-\lambda x^{\beta}}}\right)+e^{-\lambda x^{\beta}}\right)^{n-r}\end{aligned}$$

where $B(a,b)$ is the beta function.

3.4. Stress Strength Reliability

Suppose X_1 and X_2 be independent strength and stress random variables respectively, where $X_1 \sim RTW(\alpha_1, \lambda_1, \beta)$ and $X_2 \sim RTW(\alpha_2, \lambda_2, \beta)$, then the stress strength reliability $\mathbb{P}(X_1 > X_2)$, say SSR, is defined as

$$SSR = \int_{-\infty}^{\infty} f_1(x)F_2(x)dx$$

Table 2: *Average values of MLEs and the corresponding MSEs(n=50).*

Parameter			MLE			MSE		
λ	α	β	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
1	0.5	1	1.18927	0.75566	0.74107	0.27647	0.71977	0.04782
		1.5	1.22622	0.87384	1.51407	0.40466	0.58467	0.07297
		2	1.18561	0.77796	1.98113	0.24700	0.61990	0.11475
	1	1	1.08698	1.18120	1.04943	0.23511	1.10633	0.03877
		1.5	1.12510	1.27644	1.55446	0.24403	1.12121	0.09897
		2	1.13204	1.28600	2.04414	0.29381	1.45360	0.17665
	1.5	1	1.04126	1.72249	1.05876	0.27080	2.20896	0.05698
		1.5	1.08026	1.78405	1.54105	0.28238	2.09977	0.10365
		2	1.07287	1.81606	2.12707	0.27423	1.89261	0.24845
	2	1	0.98987	2.07177	1.07668	0.24656	2.17639	0.07918
		1.5	0.98794	2.19104	1.60992	0.22690	2.61831	0.17018
		2	0.98397	2.19145	2.15508	0.22988	2.63456	0.31296
2	0.5	1	2.28613	0.68043	1.02709	0.30981	0.33733	0.03757
		1.5	2.16929	0.57581	1.57420	0.19697	0.27864	0.07851
		2	2.23597	0.60438	2.09259	0.30316	0.31276	0.11952
	1	1	2.17700	1.19918	1.04533	0.51708	1.12488	0.03843
		1.5	2.21341	1.27536	1.55452	0.49046	1.11322	0.09883
		2	2.15140	1.33614	2.09166	0.55797	1.57897	0.15442
	1.5	1	2.05912	1.74110	1.05653	0.55367	2.22683	0.05775
		1.5	2.00179	1.58862	1.61057	0.45244	1.48239	0.15847
		2	1.95160	1.56072	2.18486	0.52078	2.38803	0.27118
	2	1	1.99288	2.15485	1.09058	0.58386	2.18773	0.10266
		1.5	2.01983	2.26075	1.58730	0.53255	2.56440	0.15576
		2	1.99596	2.23927	2.14504	0.58591	3.79841	0.27387

The stress strength reliability SSR, is obtained by using (8) , (9), (11), (12) and (13) and is given by

$$\begin{aligned} SSR =& \lambda_1 \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha_1^j \alpha_2^k k^m (-log\alpha_1)^l (-log\alpha_2)^m}{(1+\alpha_1)^{j+1}(1+\alpha_2)^{k+1} l! m!} \left\{ \left(j^l + \frac{\alpha_1 log\alpha_1 (j+1)^{l+1}}{1+\alpha_1} \right) \right. \\ & \times \frac{\lambda_2}{[(l+1)\lambda_1 + m\lambda_2][(l+1)\lambda_1 + (m+1)\lambda_2]} \\ & \left. - \frac{\lambda_2 \alpha_1 log\alpha_1 (j+1)^{l+1}}{(1+\alpha_1)[(l+2)\lambda_1 + (m+1)\lambda_2][(l+2)\lambda_1 + m\lambda_2]} \right\} \end{aligned}$$

Table 3: *Average values of MLEs and the corresponding MSEs(n=100).*

Parameter			MLE			MSE		
λ	α	β	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
1	0.5	1	1.12427	0.85980	1.01394	0.16520	0.36686	0.02039
		1.5	1.10945	0.64163	1.49813	0.08311	0.15675	0.04790
		2	1.03860	0.51790	2.00760	0.04648	0.11094	0.08922
	1	1	1.09751	1.20162	1.00568	0.13615	0.44997	0.02630
		1.5	1.09006	1.14954	1.52710	0.14385	0.47309	0.05824
		2	1.07148	1.14295	2.07225	0.17511	0.69167	0.12353
	1.5	1	1.04769	1.66319	1.01968	0.17384	1.08369	0.02691
		1.5	1.05555	1.68631	1.54287	0.19368	1.20057	0.07301
		2	1.05153	1.64619	2.06356	0.17319	1.02013	0.14828
	2	1	0.96890	2.04959	1.05881	0.19612	1.91723	0.04225
		1.5	1.02776	2.05201	1.52135	0.15814	1.90158	0.04479
		2	1.02772	2.04339	2.02851	0.15842	1.92072	0.07969
2	0.5	1	2.20157	0.64279	1.00275	0.20282	0.21313	0.01958
		1.5	2.15936	0.60787	1.50924	0.18855	0.18872	0.03718
		2	2.20967	0.64716	2.00121	0.25218	0.27663	0.07502
	1	1	2.14197	1.15711	1.03218	0.35039	0.70753	0.03321
		1.5	2.12690	1.14278	1.55439	0.36045	0.69360	0.06954
		2	2.22353	1.21377	1.98709	0.51312	1.10397	0.09771
	1.5	1	2.09963	1.74911	1.03669	0.50675	1.89731	0.03895
		1.5	2.05345	1.67112	1.55151	0.33247	0.93642	0.06657
		2	2.06760	1.68920	2.06563	0.37754	1.67294	0.15132
	2	1	1.92617	2.06013	1.05836	0.42906	1.98756	0.04258
		1.5	2.01885	2.20664	1.55748	0.40409	1.79326	0.08481
		2	1.99881	2.13749	2.07929	0.41542	2.86112	0.14579

4. Statistical Inference

4.1. Maximum Likelihood Estimators

Let $x_1, x_2, ..., x_n$ be a random sample from RTW distribution, then the logarithm of the likelihood function is

$$l = nlog\alpha + nlog\beta + (\beta - 1)\sum_{i=1}^{n} logx_i - \lambda\beta\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} log\left(1 + \alpha - \alpha^{1-e^{-\lambda x_i^{\beta}}}\right)$$
$$+ \sum_{i=1}^{n} log\left[1 + \alpha - \alpha^{1-e^{-\lambda x_i^{\beta}}}\left(1 - log\alpha\ (1 - e^{-\lambda x_i^{\beta}})\right)\right] \quad (21)$$

The MLEs of α, λ and β are obtained by partially differentiating (21) with respect to the corresponding parameters and equating to zero, we have

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{n} \frac{1 + (1 - e^{-\lambda x_i^{\beta}})^2 \alpha^{-e^{-\lambda x_i^{\beta}}} log\alpha}{1 + \alpha - \alpha^{1-e^{-\lambda x_i^{\beta}}} \left(1 - (1 - e^{-\lambda x_i^{\beta}}) log\alpha\right)} - 2\sum_{i=1}^{n} \frac{1 - (1 - e^{-\lambda x_i^{\beta}})\alpha^{-e^{-\lambda x_i^{\beta}}}}{1 + \alpha - \alpha^{1-e^{-\lambda x_i^{\beta}}}} \tag{22}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + (1-\lambda)\sum_{i=1}^{n} x_i + \alpha\lambda\beta\, log\alpha \sum_{i=1}^{n} x_i^{\beta-1} e^{-\lambda x_i^{\beta}} \alpha^{-e^{-\lambda x_i^{\beta}}} \times \left[\frac{\alpha}{1 + \alpha - \alpha^{1-e^{-\lambda x_i^{\beta}}}} - \frac{(1 - e^{-\lambda x_i^{\beta}}) log\alpha}{1 + \alpha - \alpha^{1-e^{-\lambda x_i^{\beta}}} \left(1 - (1 - e^{-\lambda x_i^{\beta}}) log\alpha\right)}\right] \tag{23}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + \beta\sum_{i=1}^{n} x_i - \alpha log\alpha \sum_{i=1}^{n} x_i^{\beta} e^{-\lambda x_i^{\beta}} \alpha^{-e^{-\lambda x_i^{\beta}}} \left[\frac{2}{1 + \alpha - \alpha^{1-e^{-\lambda x_i^{\beta}}}} - \frac{(1 - e^{-\lambda x_i^{\beta}}) log\alpha}{1 + \alpha - \alpha^{1-e^{-\lambda x_i^{\beta}}} \left(1 - (1 - e^{-\lambda x_i^{\beta}}) log\alpha\right)}\right] \tag{24}$$

The above three equations (22),(23) and (24) are not in closed form. thus, it is difficult to calculate the values of the parameters α, β and λ. However, R software can be used to get the MLE.

4.2. Simulation study

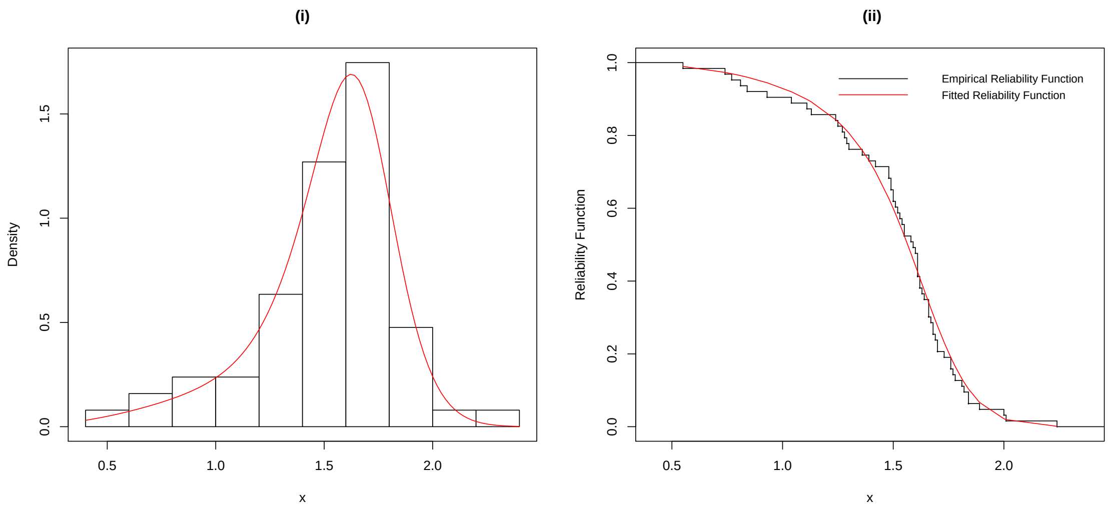

Figure 3: *(i) The relative histogram and the fitted RTW distribution. (ii) The fitted RTW reliability function and empirical reliability function for first data set.*

The simulation study has been performed using R Software to show the behaviour of the MLEs in terms of the sample size n. Two sets of sample (n=50, n=100) each replicated 100 times with different values of parameters $\lambda = (1, 2)$, $\alpha = (0.5, 1, 1.5, 2)$ and $\beta = (1, 1.5, 2)$ were

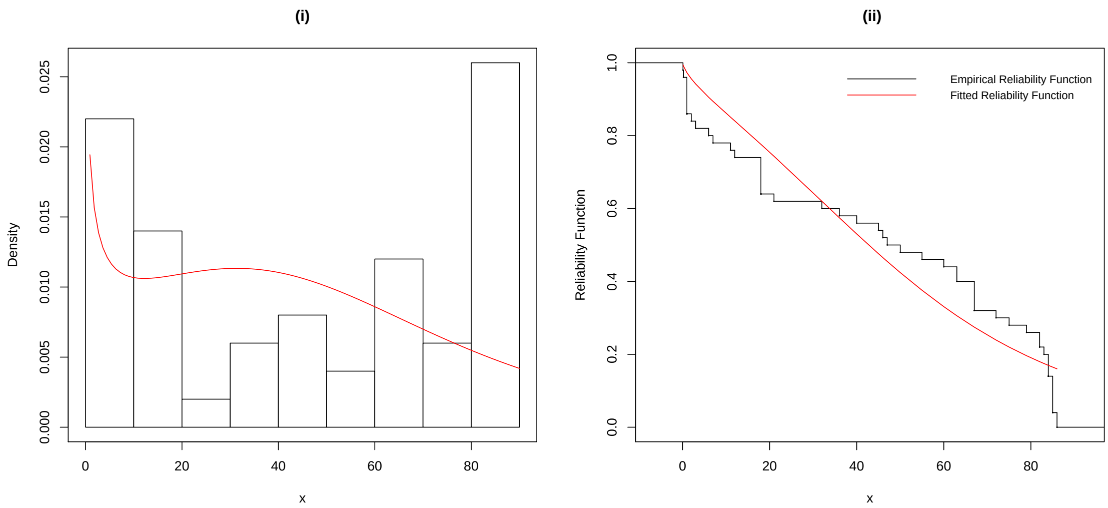

Figure 4: *(i) The relative histogram and the fitted RTW distribution. (ii) The fitted RTW reliability function and empirical reliability function for second data set.*

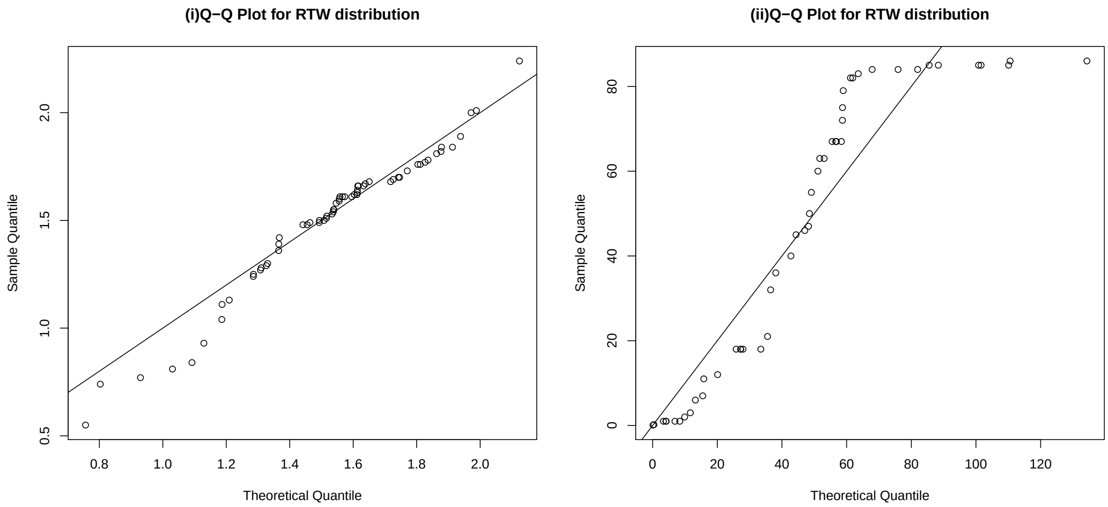

Figure 5: *Q-Q plot for the RTW distribution for data set first and data set second, respectively.*

generated from RTW. In each setting, the average values of MLEs and the corresponding empirical mean squared errors (MSEs) were obtained. The simulation results are presented in table 2 and table 3. From tables 2 and 3, it can be seen that the estimates are stable and quite close to the true parameter values. As the sample size increases the MSE decreases in all the cases.

5. Applications

In this section, we analyse two data sets to describe the significance and flexibility of the RTW distribution. The data set first reported by Nassar et al. [12], orginally published by Smith and Naylor [13], corresponding to strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. The data are as follows: 0.55, 0.93,1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53,1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28,

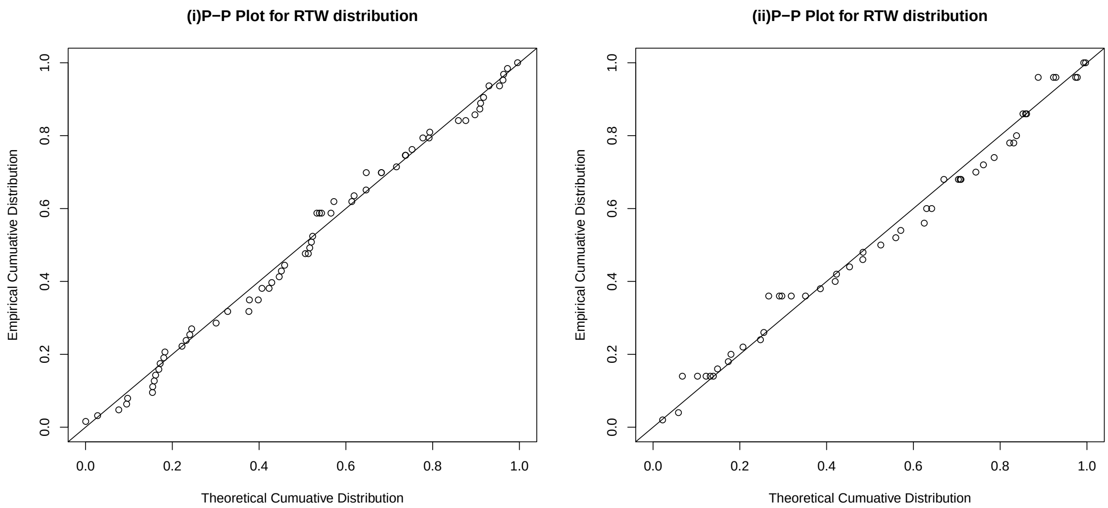

Figure 6: *P-P plot for the RTW distribution for data set first and data set second, respectively.*

Table 4: *MLEs (standard errors in parentheses), K-S Statistic, and p-values for the first data set.*

	Estimates			Statistics	
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	K-S	p-value
RTW	9.49959 (6.00647)	3.261905 (0.69075)	0.72053 (0.40517)	0.08745	0.72090
APW	10.86178 (12.72527)	4.48322 (0.76269)	0.19483 (0.10826)	0.12249	0.30090
APIW	193.05946 (267.40709)	3.87688 (0.30960)	0.63654 (0.1823435)	0.21627	0.00551
MW	0.03088 (0.04349)	6.37442 (0.96544)	0.04087 (0.02476)	0.13341	0.21210
TW	0.92496 (0.21931)	5.97478 (0.74495)	1.80960 (0.07553)	0.15191	0.10920
LW	0.53504 (0.48673)	4.94433 (0.65927)	0.77920 (0.18296)	0.13673	0.18950
ZBLL	0.25140 (0.06121)	18.41002 (3.05420)	1.82436 (0.04629)	0.13053	0.23330
APE	145351 (23726.57)	-	2.15458 (0.09901)	0.22099	0.00425
W	-	5.77962 (0.57515)	0.05978 (0.02047)	0.15232	0.10750

1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69,1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

The second data set was reported by Elbatal et al. [14], orginally published by Aarset [15], which represents the failure times of 50 devices. The data are as follows: 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3,

Table 5: *$-2l(\hat{\theta})$, AIC, AICC, BIC for the first data set.*

Model	$-2l(\hat{\theta})$	AIC	AICC	BIC
RTW	22.16977	28.16977	28.57655	34.59917
APW	26.94826	32.94826	33.35504	39.37766
APIW	75.77237	81.77237	82.17915	88.20177
MW	29.78938	35.78938	36.19616	42.21878
TW	30.28635	36.28635	36.69313	42.71576
LW	28.42141	34.42141	34.82819	40.85081
ZBLL	24.23729	30.23729	30.64407	36.66669
APE	67.56511	71.56511	71.76511	75.85138
W	30.41369	34.41369	34.61369	38.69995

Table 6: *MLEs (standard errors in parentheses), K-S Statistic, and p-values for the second data set.*

	Estimates			Statistics	
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	K-S	p-value
RTW	6.28982 (2.80293)	0.71267 (0.12226)	0.17523 (0.10967)	0.16014	0.15390
APW	4.51340 (4.01925)	0.83571 (0.13558)	0.05854 (0.03910)	0.17492	0.09379
APIW	62.22037 (86.31937)	0.59918 (0.05672)	1.14499 (0.39802)	0.27478	0.00105
MW	0.01863 (0.00375)	0.37305 (0.18838)	0.04043 (0.03113)	0.19432	0.04583
TW	0.00010 (0.42067)	0.94905 (0.12873)	44.91508 (12.90900)	0.1928	0.04860
LW	0.91774 (0.69388)	0.88097 (0.12668)	0.04050 (0.04259)	0.18488	0.06555
ZBLL	20.23812 (4.33771)	2.25295 (0.46228)	0.00273 (0.00091)	0.23307	0.00874
APE	2.64622 (1.90895)	-	0.02687 (0.00474)	0.17657	0.08851
W	-	0.94770 (0.11778)	0.02719 (0.01375)	0.19313	0.04800

6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

We compare the fit of the proposed RTW distribution with its sub-model Weibull (W) distribution and with several other competitive models, namely Alpha Power Weibull (APW) (see [12]), Alpha Power Inverse Weibull (APIW) (see [16]), Modified Weibull (MW) (see [17]), Transmuted Weibull (TW) (see [18]), Lindley Weibull (LW) (see [19]), Zografos–Balakrishnan log-logistic

Table 7: *$-2l(\hat{\theta})$, AIC, AICC, BIC for the second data set.*

Model	$-2l(\hat{\theta})$	AIC	AICC	BIC
RTW	470.2143	476.2143	476.7360	481.9504
APW	479.2431	485.2431	485.7648	490.9791
APIW	519.9063	525.9063	526.4280	531.6423
MW	478.9685	484.9685	485.4902	490.7045
TW	482.0043	488.0043	488.5261	493.7404
LW	479.5173	485.5173	486.0390	491.2534
ZBLL	517.3178	523.3178	523.8396	529.0539
APE	480.5838	484.5838	484.8391	488.4078
W	482.0038	486.0038	486.2591	489.8278

(ZBLL) (see [20]), and Alpha Power Exponential (APE) (see [10]), their correspinding density functions for $x > 0$ are as follows

$$\text{APW} \quad f(x) = \frac{log\alpha}{\alpha - 1}\lambda\beta\alpha^{1-e^{-\lambda x^{\beta}}} x^{\beta-1}e^{-\lambda x^{\beta}}$$

$$\begin{aligned}
\text{APIW} \quad f(x) &= \frac{log\alpha}{\alpha-1}\lambda\beta x^{-(\beta+\alpha)}e^{-\lambda x^{-\beta}}\alpha^{e^{-\lambda x^{-\beta}}} \\
\text{MW} \quad f(x) &= (\alpha + \lambda\beta x^{\beta-1})e^{-\alpha x - \lambda x^{\beta}} \\
\text{TW} \quad f(x) &= \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1} e^{-\left(\frac{x}{\lambda}\right)^{\beta}}\left(1 - \alpha + 2\alpha e^{-\left(\frac{x}{\lambda}\right)^{\beta}}\right) \\
\text{LW} \quad f(x) &= \frac{\beta\alpha^2}{\alpha+1}\lambda^{\beta}x^{\beta-1} + \lambda^{2\beta}x^{2\beta-1}e^{-\alpha(\lambda x)^{\beta}} \\
\text{ZBLL} \quad f(x) &= \frac{\beta}{\lambda^{\beta}\Gamma(\alpha)}x^{\beta-1}\left(1+\left(\frac{x}{\lambda}\right)^{\beta}\right)^{-2}\left(log\left(1+\left(\frac{x}{\lambda}\right)^{\beta}\right)\right)^{\alpha-1} \\
\text{APE} \quad f(x) &= \frac{log\alpha}{\alpha-1}\lambda e^{-\lambda x}\alpha^{1-e^{-\lambda x}}
\end{aligned}$$

where $\alpha,\beta,\lambda > 0$ and $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}dx$ is the gamma function.

From Table 4, Table 5, Table 6 and Table 7, it is evident that RTW distribution has lowest $-2l(\hat{\theta})$, AIC, AICC, BIC, K-S values and highest p-value among all the other competitive models. Hence the proposed model yeilds the better fit than the other models for both data sets.

The relative histogram and the fitted RTW distribution of the data set first and second are shown in Figures 3(i) and 4(i), respectively. The plots of the fitted RTW reliability function and empirical reliability function of the data set first and second are shown in Figures 3(ii) and 4(ii), respectively. The Q-Q plots for data set first and second are shown in Figure 5(i) and 5(ii) respectively. Also, The P-P plots for data set first and second are shown in Figure 6(i) and 6(ii) respectively that allows us to differentiate between the empirical distribution of the data with the RTW distribution. These graphical goodness of fit measures clearly support the results in

Tables 4, Table 5, Table 6 and Table 7.

6. Conclusion

A new family of distributions has been introduced called RT method. RT method has been specialized on the two-parameter Weibull distribution and a new three-parameter RTW distribution has been introduced. We have discussed various properties of RTW distribution. It has been realized that the three-parameter RTW distribution has more flexibility in terms of the hazard rate function and the density function. The effectiveness of the proposed model is compared with other existing models by using goodness of fit measures. The model has been fitted to two different real life data sets, the figures show that the proposed model provides better fit for both data sets in comparison to all other competitive models.

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