Some Properties and Different Estimation Methods for Inverse $A(\alpha)$ Distribution with an Application to Tongue Cancer Data

Shreya Bhunia* and Proloy Banerjee

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Department of Mathematics and Statistics, Aliah University, Kolkata, India
shreyabhunia.stat@gmail.com and proloy.stat@gmail.com
*Corresponding author E-mail: shreyabhunia.stat@gmail.com

Abstract

The inverted distribution is the distribution of the reciprocal of a random variable that follows a specified distribution. Here, a new one parameter inverse $A(\alpha)$ distribution has been introduced, which is the reciprocal of the $A(\alpha)$ distribution. An account of mathematical and statistical properties of the new distribution such as survival characteristics, quantile functions, mode, order statistics, ageing intensity function and stochastic ordering have been derived and discussed. Furthermore, from the frequentist view point we discussed several estimation approaches including maximum likelihood method, method of maximum product of spacings, ordinary and weighted least square methods, Cramér-Von-Mises estimation and Anderson-Darling estimation methods. These methods are compared for both small and large samples by performing an extensive numerical simulation. The flexibility of the new lifetime distribution is demonstrated by modeling a tongue cancer data. The result indicates the superiority for proposed model compared to some popular competing ones.

Keywords: Inverse distribution, Estimation methods, Hazard rate function, Lifetime distribution

1. Introduction

In several applied fields of research such as engineering, medical sciences, economics, biological sciences etc., analyzing and modeling complex datasets are the most essential parts. Albeit in literature there exists many well known standard distributions, sometimes it may not always reflect the real world scenario. So, the researchers aspire to extend structures of the probability models. Recently, [1] introduced a new one parameter $A(\alpha)$ distribution and the applicability of the distribution is investigated by analyzing three datasets. A continuous random variable Y is said to follow an $A(\alpha)$ distribution if its probability density function (pdf) is of the form;

$$f_Y(y) = \frac{1}{y^2} exp\left[\frac{1}{\alpha}\left(1 - exp\left(\frac{\alpha}{y}\right)\right) + \frac{\alpha}{y}\right]; y > 0 \quad (1)$$

and is denoted by $Y \sim A(\alpha)$. The corresponding cumulative distribution function (cdf) of Y is given by,

$$F_Y(y) = exp\left[\frac{1}{\alpha}\left(1 - exp\left(\frac{\alpha}{y}\right)\right)\right]; y > 0 \quad (2)$$

with scale parameter $\alpha > 0$.

In statistical literature there are various methods for proposing new distributions by using baseline distributions. For example, [2] introduced a general method for obtaining more flexible distributions by adding a new parameter to an existing family of distributions, Quadratic rank transmutation map (QRTM) [3], DUS transformation [4], $\alpha-$power transformation method [5] etc. In this context, finding inversion of univariate probability distributions and their applicability under the inverse transformation method is one of the preferred areas of research in recent times. Sometimes it has been found that inverted version of the distributions are much more effective to explore additional aspects of the phenomena that non-inverted distribution cannot. For instances, inverse exponential distribution is studied by [6], inverse Weibull distribution is studied by [7], [8] studied inverse Lindley distribution, inverse Xgamma distribution is studied by [9], inverse power Lindley distribution by [10], inverted Gamma distribution by [11], inverse Kumaraswamy distribution is studied by [12].

In this present study, we have also introduced the inverted version of the $A(\alpha)$ distribution using the same technique and named it as the inverse $A(\alpha)$ distribution. The new distribution is flexible to model positive real datasets which possesses increasing hazard rate function. Another beauty of this distribution includes heavy-tail, unimodal, parsimonious in parameter and easy to use. The objectives of this article are: (i) to obtain some mathematical properties for inverse $A(\alpha)$ distribution and (ii) to estimate the unknown parameter of the model from frequentist perspectives. The maximum likelihood estimation (MLE), method of maximum product of spacings (MPS), ordinary least square estimation (OLS) and weighted least square estimation (WLS), Cramér-Von-Mises estimation (CVM) and the method of Anderson-Darling (AD) are considered as frequentist methods for parameter estimation. Also we compare these estimation procedures on the basis of root mean square error (RMSE) values for different sample sizes and different parameter values using Monte-Carlo simulation technique. Furthermore, to the best of our knowledge, no attempt has been made to compare all of these estimators for the inverse $A(\alpha)$ distribution along with mathematical and statistical properties. Additionally, to illustrate the flexibility of this distribution a tongue cancer patient data has been analyzed.

The remainder of this article is organized as follows. In Section 2, the new distribution has been introduced. Different statistical properties and associated measures of Inv-$A(\alpha)$ distribution have been discussed in Section 3. Different classical estimation procedures for the parameter of inverse $A(\alpha)$ distribution have been considered in Section 4. In Section 5, a simulation study is conducted to compare the various obtained estimators. Empirical application based on a real dataset is discussed in Section 6. Finally, concluding remarks are given in Section 7.

2. The inverted $A(\alpha)$ distribution

A new probability distribution, termed as inverted or inverse-$A(\alpha)$ distribution has been introduced in this section. By origin, this distribution is the reciprocal of the $A(\alpha)$ distribution and for simplicity throughout this study we use the notation Inv-$A(\alpha)$ for this new lifetime model. Here we consider the random variable Y having the density function (1), then the cdf of the inverted random variable $X = \frac{1}{Y}$ is defined as

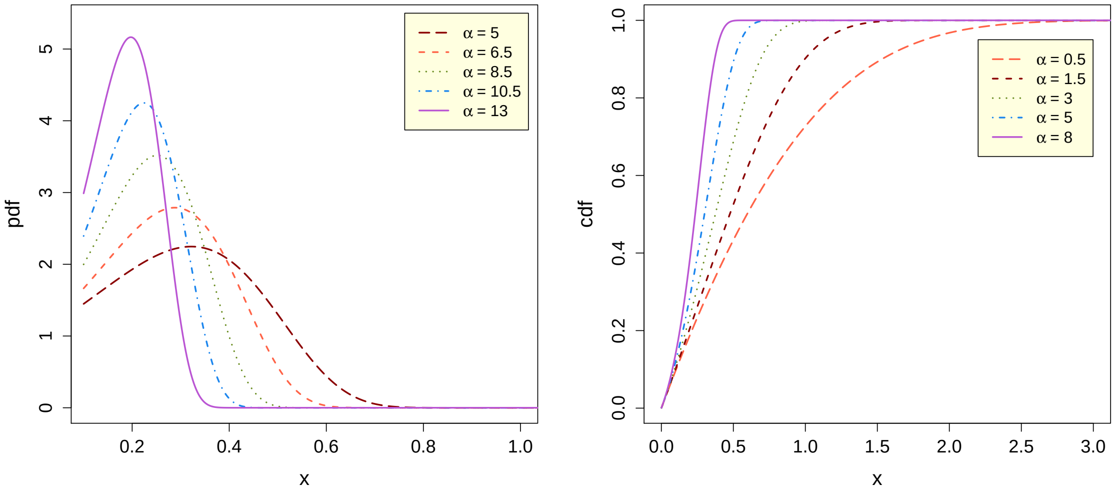
$$F_X(x) = \mathcal{P}[X \leq x] = 1 - F_Y\left(\frac{1}{X}\right) = 1 - exp\left[\frac{1}{\alpha}\left(1 - exp(\alpha x)\right)\right]; x > 0. \quad (3)$$

Now, by differentiating $F_X(x)$ given in (3) the pdf of the Inv-$A(\alpha)$ distribution is obtained and expressed as follows;

$$f_X(x) = exp\left[\frac{1}{\alpha}\left(1 - exp(\alpha x)\right) + \alpha x\right]; x > 0 \quad (4)$$

and $\alpha > 0$ is the scale parameter. The Inv-$A(\alpha)$ distribution is an one parameter family of continuous probability distributions on the positive real line.

The plots of pdf and cdf function of Inv-$A(\alpha)$ distribution for different choices of scale parameter α are shown in Figures 1a and 1b respectively. The plots reveal that the Inv-$A(\alpha)$ density can be decreasing, unimodal and right skewed.



(a) **(b)**

Figure 1: *The pdf and cdf plots of Inv-$A(\alpha)$ distribution for different parameter choices.*

3. Some statistical properties

3.1. Reliability characteristics

The survival function (sf) and hazard rate function (hrf) are the basic characteristics of any lifetime distributions. Both the measures are commonly employed to describe and model the fundamental properties of a variety of survival datasets. The survival function $\mathbb{S}(t)$, which is defined as the probability that an individual or an item is survived at least t $(t \geq 0)$ unit of time and denoted as $\mathbb{S}(t) = P(X \geq t) = 1 - F(t)$.

Thus, the sf of the Inv-$A(\alpha)$ distribution is defined as,

$$\mathbb{S}(t) = e^{\frac{1}{\alpha}(1-e^{\alpha t})} \tag{5}$$

The hazard rate function, also known as the failure rate function, is another key feature to consider when measuring a real-life phenomenon with a lifetime distribution. It can be interpreted as the conditional probability of failure, given it has survived upto at least the time t $(t \geq 0)$ and is defined as $h(t) = \frac{f(t)}{1-F(t)} = \frac{f(t)}{\mathbb{S}(t)}$; where $f(t)$ is the pdf and $\mathbb{S}(t)$ is the sf of the corresponding distribution. Therefore, the hrf for the Inv-$A(\alpha)$ distribution is given by,

$$h(t) = \frac{e^{\frac{1}{\alpha}(1-e^{\alpha t})+\alpha t}}{e^{\frac{1}{\alpha}(1-e^{\alpha t})}} = e^{\alpha t} \tag{6}$$

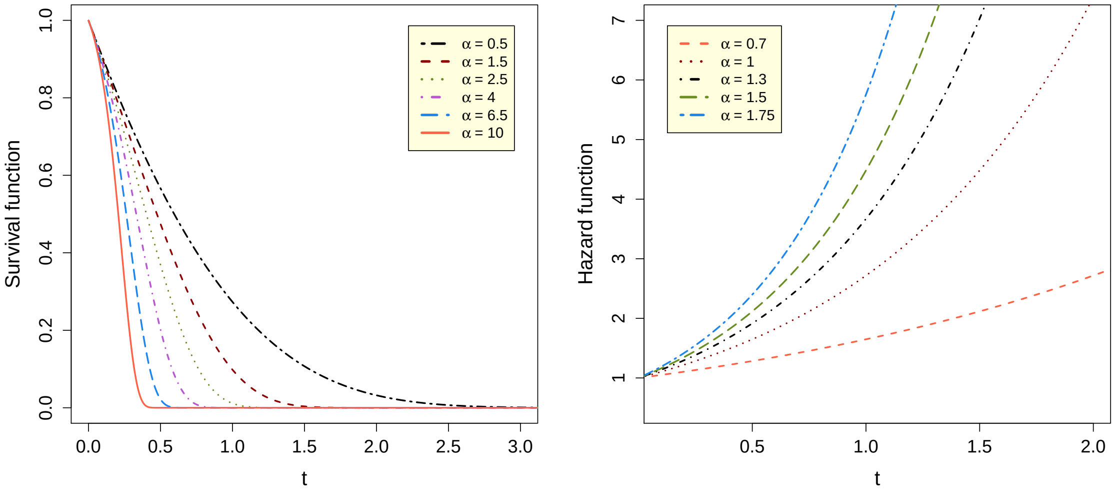
The ratio between the lifetime probability density and its distribution function is characterised as the reversed (or proportional) hazard rate function (rhrf) of a random life phenomena. For the Inv-$A(\alpha)$ distribution the rhrf is given as follows,

$$H(t) = \frac{f(t)}{F(t)} = \frac{e^{\frac{1}{\alpha}(1-e^{\alpha t})+\alpha t}}{1 - e^{\frac{1}{\alpha}(1-e^{\alpha t})}} \tag{7}$$

Another similar measure is cumulative hazard function and is defined as follows [13]:

$$\Lambda(t) = -log\ \mathbb{S}(t) = -log\left[e^{\frac{1}{\alpha}(1-e^{\alpha t})}\right] = \frac{1}{\alpha}\left(e^{\alpha t} - 1\right) \tag{8}$$

So, clearly from expression (6) we can see that the hazard rate function is increasing for $\alpha > 0$. The shape of the hazard rate is displayed in figure 2b for different choices of α, whereas figure 2a represents the shape of the survival function.



(a) **(b)**

Figure 2: *The survival and hazard plots of Inv-$A(\alpha)$ distribution for different parameter choices.*

The shape of the hazard rate function can also be derived mathematically by using the following lemma.

Lemma 1. Suppose $f(t)$, for $t > 0$ is the density function of a positive real valued continuous random variable. $f'(t)$ is the derivative of $f(t)$ and

$$\eta(t) = -\frac{f'(t)}{f(t)}.$$

Then if $\eta'(t) > 0$ for all $t > 0$ then the hazard rate function is an increasing function of t.

Proof. [14].
Here,

$$\eta(x) = e^{\alpha x} - \alpha.$$

After differentiating with respect to x, we have

$$\eta'(x) = \alpha e^{\alpha x}; \quad \alpha > 0.$$

It is clearly seen that $\eta'(x) > 0$ for $x > 0$. Therefore, the distribution has increasing hazard rate function.

3.2. Quantile functions, median and mode

The qth quantile function x_q of Inv-$A(\alpha)$ distribution will be obtained by solving the following equation

$$F(x_q) = q \quad \text{where, } 0 < q < 1.$$

$$\text{Therefore,} \qquad 1 - e^{\frac{1}{\alpha}(1-e^{\alpha x_q})} = q$$

$$x_q = \frac{1}{\alpha} log\left[1 - \alpha\, log(1-q)\right]. \tag{9}$$

Thus, the median (or 2nd quartile) of the proposed distribution is obtained by substituting $q = \frac{1}{2}$ in (9).

$$\text{i.e.,} \qquad X_{\frac{1}{2}} = Q_2 = \frac{1}{\alpha} log\left[1 - \alpha\, log\left(\frac{1}{2}\right)\right] = \frac{1}{\alpha} log\left[1 + \alpha\, log\, 2\right]. \tag{10}$$

Similarly, the 1^{st} and 3^{rd} quartiles are obtained by replacing $q = \frac{1}{4}$ and $q = \frac{3}{4}$ respectively. Thus expressions for the 1^{st} and 3^{rd} quartiles are as follows.

$$Q_1 = \frac{1}{\alpha} log \left[1 - \alpha\, log\left(\frac{3}{4}\right)\right] \qquad \text{and} \qquad Q_3 = \frac{1}{\alpha} log\left[1 + \alpha\, log\, 4\right].$$

Now, the mode of the inverse $A(\alpha)$ distribution denoted by x_m will be derived by solving the equation $f'(x;\alpha) = 0$, for which $f''(x) < 0$. The solution x_m will be the mode of the distribution for which $f(x)$ attains the maximum value. Therefore, the mode of the proposed distribution is obtained by solving the differentiation expressed as

$$\frac{\partial}{\partial x} e^{\left[\frac{1}{\alpha}(1-e^{\alpha x})+\alpha x\right]} = 0 \tag{11}$$

After simplification, we get

$$x_m = \frac{log\, \alpha}{\alpha} \tag{12}$$

It is noted that, though the mode of $A(\alpha)$ distribution exists but it cannot be expressed in a closed form. However, in our study of Inv-$A(\alpha)$ distribution, we see that the mode exists with an explicit form.

3.3. Order statistics

In nonparametric statistics and inference, order statistics are one of the useful techniques. In life testing and reliability analysis order statistics have a wide range of applications. Let us assume that $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \cdots, X_n$ drawn from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$. Therefore under these assumptions pdf and cdf of the order statistics $X_{(r)}$, r = 1, 2, $\cdots$, n is expressed as

$$f_r(x) = \frac{n!}{(r-1)!\,(n-r)!} f(x) F^{(r-1)}(x)[1 - F(x)]^{(n-r)}; \quad r = 1, 2, \cdots, n \tag{13}$$

and

$$F_r(x) = \sum_{i=r}^{n} \binom{n}{i} F_X^i(x)[1 - F_X(x)]^{(n-i)}. \tag{14}$$

Now, by using the pdf (4) and cdf (3) in equation (13), we can easily derive the pdf of r^{th} order statistic for the Inv-$A(\alpha)$ distribution as in the following expression

$$\begin{aligned} f_{X_{(r)}}(x) &= \frac{n!}{(r-1)!\,(n-r)!} \left\{1 - e^{\frac{1}{\alpha}(1-e^{\alpha x})}\right\}^{(r-1)} \left\{e^{\frac{1}{\alpha}(1-e^{\alpha x})}\right\}^{(n-r+1)} e^{\alpha x} \\ &= \frac{n!}{(r-1)!\,(n-r)!} \sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^k e^{\alpha x} \left[e^{\frac{1}{\alpha}(1-e^{\alpha x})}\right]^{(n-r+k+1)}. \end{aligned} \tag{15}$$

While using equation (3) in (14) the cdf of r^{th} order statistic becomes,

$$F_{X_{(r)}}(x) = \sum_{i=r}^{n} \sum_{k=0}^{i} \binom{n}{i} \binom{i}{k} (-1)^k e^{\frac{n}{\alpha}(1-e^{\alpha x})}. \tag{16}$$

In particular, the densities of the smallest and largest order statistics of the Inv-$A(\alpha)$ distribution are obtained by substituting $r = 1$ and n simultaneously in the expression (15). Hence the pdf of the smallest order statistic $X_{(1)}$ is expressed as,

$$f_{X_{(1)}}(x) = n e^{\left\{\frac{n}{\alpha}(1-e^{\alpha x})+\alpha x\right\}}$$

and the pdf of the largest order statistic $X_{(n)}$ is as follows,

$$f_{X_{(n)}}(x) = n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k e^{\left\{\frac{(1-e^{\alpha x})(k+1)}{\alpha}+\alpha x\right\}}.$$

3.4. Ageing intensity function

Ageing intensity (AI) function has been developed by [15] and according to him, a unimodal failure rate can be represented as either approximately decreasing or approximately increasing or approximately constant. [16] investigated various features of AI functions, whereas [17] discussed AI function in the field of reliability theory. The AI function for a positive random variable X, denoted by $L_X(t)$, for any $t > 0$. The ratio of the instantaneous failure rate to a baseline failure rate is used to calculate the AI function. It is defined as

$$L_X(t) = \frac{h(t)}{H(t)}, \\ = \frac{-tf(t)}{S(t)lnS(t)}, t > 0.$$

Where $f(.)$ and $S(.)$ are the probability density function and survival function of the random variable X respectively. $H(t)$ is failure rate average and it can be written as $H(t) = (\int_0^t h(u)du)/t$. Now, if $X \sim$ Inv-$A(\alpha)$ then the expression for the AI function is obtained as

$$L_X(t) = -\frac{\alpha t e^{\alpha t}}{1 - e^{\alpha t}}. \tag{17}$$

AI function is uniquely determined by the failure rate function, however the converse is not true. The stronger the ageing tendency of the related random variable, the higher the value of the AI function. If the failure rate is a constant then the $AI = 1$, If the failure rate is increasing then the $AI > 1$, if the failure rate is decreasing then the $AI < 1$.

3.5. Stochastic ordering

The notion of stochastic ordering was first suggested by [18] and used to demonstrate the comparative behaviour of two positive continuous random variables. Suppose X and Y are the two random variables with respective cdfs F_X and F_Y, then X is said to be smaller than Y in the following cases

- Stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x;
- Hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x;
- Mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \leq m_Y(x)$ for all x;
- Likelihood ratio order $(X \leq_{lr} Y)$ if $\frac{f_X(x)}{fY(x)}$ decreases in x.

The aforementioned relationship are well known for establishing stochastic ordering of distributions.

$$\begin{array}{ccc} X \leq_{lr} Y \Longrightarrow X & \leq_{hr} & Y \Longrightarrow X \leq_{mrl} Y \\ & \Downarrow & \\ & X \leq_{st} Y & \end{array}$$

When the required conditions are met, the Inv-$A(\alpha)$ distribution is ordered with regard to the strongest likelihood ratio ordering, as shown by the following theorem.

Theorem 1. let, X $\sim$ Inv-$A(\alpha_1)$ and Y $\sim$ Inv-$A(\alpha_2)$. If $\alpha_1 < \alpha_2$, then X $\leq_{lr}$ Y and hence it implies other orderings.

Proof. According to the definition, the Likelihood ratio is defined as

$$\xi(x) = \frac{f_X(x)}{f_Y(x)} = \frac{exp\left[\frac{1}{\alpha_1}\left(1 - exp(\alpha_1 x)\right) + \alpha_1 x\right]}{exp\left[\frac{1}{\alpha_2}\left(1 - exp(\alpha_2 x)\right) + \alpha_2 x\right]}$$

$$\Rightarrow log\xi(x) = \frac{1}{\alpha_1} - \frac{e^{\alpha_1 x}}{\alpha_1} - \frac{1}{\alpha_2} + \frac{e^{\alpha_2 x}}{\alpha_2} + (\alpha_1 - \alpha_2)x$$

Now differentiating with respect to x, we get

$$\frac{\xi'(x)}{\xi(x)} = (\alpha_1 - \alpha_2) + e^{\alpha_2 x} - e^{\alpha_1 x}$$

$$\Rightarrow \xi'(x) = \xi(x)\left\{(\alpha_1 - \alpha_2) + e^{\alpha_2 x} - e^{\alpha_1 x}\right\}$$

$$\Rightarrow \xi'(x) < 0 \qquad if\ \alpha_1\ <\ \alpha_2.$$

Therefore, $\xi(x)$ is decreasing function in x if $\alpha_1 < \alpha_2$ and hence $X \leq_{lr} Y$. The remaining orderings can also be established in similar manner.

4. Methods of Estimation

In this section, we describe some parameter estimation techniques for Inv-$A(\alpha)$ distribution under frequentist view point. In particular, the methods which we have discussed here, those are: maximum likelihood estimation (MLE), maximum product of spacings (MPS), ordinary least square (OLS) and weighted least square estimation (WLS), Cramér-Von-Mises estimation (CVM) and Anderson-Darling (AD) estimation.

4.1. Method of Maximum Likelihood

Here, we discuss the maximum likelihood estimation (MLE) method for estimating the unknown scale parameter α. Several desirable properties like consistency, asymptotic efficiency and invariance make this estimation technique most popular among others [19]. Let $X_1, X_2, \cdots, X_n$ be an observed random sample from Inv-$A(\alpha)$ distribution with pdf (4) and the MLE for the unknown parameter is derived as follows. The likelihood function is defined as

$$L(x) = \prod_{i=i}^{n} f(x_i;\alpha) = \prod_{i=i}^{n} exp\left[\frac{1}{\alpha}\left(1 - exp(\alpha x_i)\right) + \alpha x_i\right].$$

So, the log-likelihood function becomes

$$logL = \frac{n}{\alpha} - \frac{1}{\alpha}\sum_{i=1}^{n} e^{\alpha x_i} + \alpha\sum_{i=1}^{n} x_i. \tag{18}$$

After differentiating $\log \mathrm{L}$ in (18) with respect to α and equating to zero, we get the system of non-linear equation as,

$$\frac{\delta logL}{\delta\alpha} = \frac{1}{\alpha^2}\sum_{i=1}^{n} e^{\alpha x_i} - \frac{1}{\alpha}\sum_{i=1}^{n} x_i e^{\alpha x_i} - \frac{n}{\alpha^2} + \sum_{1=1}^{n} x_i = 0.$$

$$\Longrightarrow \sum_{i=1}^{n} e^{\alpha x_i} - \alpha\sum_{i=1}^{n} x_i e^{\alpha x_i} + \alpha^2\sum_{i=1}^{n} x_i = n. \tag{19}$$

The solution of the above non-linear equation (19) gives the MLE for the parameter α. As the equation cannot be solved analytically, therefore some iteration techniques like Newton-Raphson method may be adopted to obtain the MLE.

4.2. Method of maximum product of spacings

For the estimation of unknown parameters of continuous univariate distributions, the maximum product of spacings (MPS) approach provides a strong alternative to MLE. [20] initially discussed the use of MPS estimation method whereas [22] demonstrated that this technique as efficient as the MLE and consistent across a wider range of situations. [21] developed the MPS approach as an approximation to the Kullback–Leibler information measure independently. Recently, [23], [24], [25], [26] etc. applied this approach in parameter estimation problem.

According to the procedure, the uniform spacings of a random sample drawn from the Inv-$A(\alpha)$ distribution are defined as

$$\mathbb{D}_i(\alpha) = F(x_{i:n};\alpha) - F(x_{i-1:n};\alpha), \quad i = 1, 2, \cdots, n+1.$$

Where, $F(x_{0:n};\alpha) = 0$ and $F(x_{n+1:n};\alpha) = 1$. Clearly, $\sum_{i=1}^{n+1} \mathbb{D}_i(\alpha) = 1$.

The maximum product of spacings estimate $\hat{\alpha}_{MPS}$ is obtained by maximizing the geometric mean of the spacings,

$$\mathbb{G}(\alpha) = \left[\prod_{i=1}^{n+1} \mathbb{D}_i(\alpha)\right]^{1/(n+1)}$$

with respect to α, or equivalently, by maximizing the logarithm of the geometric mean of sample spacings:

$$\eta(\alpha) = \frac{1}{n+1}\sum_{i=1}^{n+1} log\mathbb{D}_i(\alpha). \tag{20}$$

The estimate $\hat{\alpha}_{MPS}$ of the parameter α can be obtained by solving the following non-linear equation.

$$\frac{e^{\frac{1}{\alpha}(1-e^{\alpha x_1})}\left\{\frac{1}{\alpha}x_1 e^{\alpha x_1} - \frac{1}{\alpha^2}\left(e^{\alpha x_1}-1\right)\right\}}{1-e^{\frac{1}{\alpha}(1-e^{\alpha x_1})}} + \sum_{i=2}^{n} \frac{\left[e^{\frac{1}{\alpha}(1-e^{\alpha x_{i-1}})}\left\{\frac{1}{\alpha^2}\left(e^{\alpha x_{i-1}}-1\right) - \frac{1}{\alpha}x_{i-1}e^{\alpha x_{i-1}}\right\} - e^{\frac{1}{\alpha}(1-e^{\alpha x_i})}\left\{\frac{1}{\alpha^2}\left(e^{\alpha x_i}-1\right) - \frac{1}{\alpha}x_i e^{\alpha x_i}\right\}\right]}{e^{\frac{1}{\alpha}(1-e^{\alpha x_{i-1}})} - e^{\frac{1}{\alpha}(1-e^{\alpha x_i})}} + \frac{1}{\alpha^2}\left(e^{\alpha x_n}-1\right) - \frac{1}{\alpha}x_n e^{\alpha x_n} = 0 \tag{21}$$

Since the above non-linear equation is not having a closed form solution, it cannot be solved analytically. Therefore, we derived it numerically in next section by using some iteration technique.

4.3. Ordinary and weighted least square estimation

The ordinary least square and the weighted least square are the two conventional estimation procedures were developed by [27] in context of the parameters estimation of the Beta distribution. Let, $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$ be the ordered sample of size n from a distribution function $F(x_{i:n};\alpha)$. Then the ordinary least square estimator $\hat{\alpha}_{OLS}$ can be obtained by minimizing

$$OLS = \sum_{i=1}^{n}\left[F(x_{i:n};\alpha) - \frac{i}{n+1}\right]^2,$$

with respect to α. Now, the OLS estimator for the parameter of Inv-$A(\alpha)$ distribution can be obtained by solving the following non-linear equation

$$\sum_{i=1}^{n}\left(1 - e^{\frac{1}{\alpha}(1-e^{\alpha x_i})} - \frac{i}{n+1}\right) e^{\frac{1}{\alpha}(1-e^{\alpha x_i})}\left\{\frac{1}{\alpha^2}\left(e^{\alpha x_i}-1\right) - \frac{1}{\alpha}x_i e^{\alpha x_i}\right\} = 0 \tag{22}$$

Similarly, the weighted least square estimate (WLS) of the unknown parameter can be obtained by minimizing the following expression

$$WLS = \sum_{i=1}^{n}\omega_i\left[F(x_{i:n};\alpha) - \frac{i}{n+1}\right]^2,$$

with respect to α and $\omega_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$ be the weight function at the i^{th} point.

Using equation (3) in the above expression and differentiating with respect to α we obtained $\hat{\alpha}_{WLS}$ by solving the following non-linear equation

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(1 - e^{\frac{1}{\alpha}(1-e^{\alpha x_i})} - \frac{i}{n+1}\right) e^{\frac{1}{\alpha}(1-e^{\alpha x_i})}\left\{\frac{1}{\alpha^2}\left(e^{\alpha x_i}-1\right) - \frac{1}{\alpha}x_i e^{\alpha x_i}\right\} = 0. \tag{23}$$

4.4. Minimum distance estimators

In this subsection, we briefly present two estimation approaches for the unknown parameter α of the proposed lifetime distribution based on the minimization, with respect to α, of the goodness-of-fit statistics. This class of statistics is defined based on the discrepancies between the estimate of the cdf and the empirical distribution function [28], [29].

4.4.1 Cramér-Von-Mises estimation

The Cramér-Von-Mises (CVM) estimator is a sort of minimal distance estimator, computed based on the discrepancies between the estimate of the cumulative distribution function and the empirical distribution function. This estimator is also known as maximum goodness of fit estimator. For more details about this method we refer [28, 29, 30] etc. [31] justified the use of Cramér-Von-Mises type minimal distance estimators by demonstrating that their bias is lower than that of other minimum distance estimators.

Let $x_1 < x_2 < \cdots < x_n$ be the ordered samples from the pdf (4). Then the Cramér-Von-Mises estimator $\hat{\alpha}_{CVM}$ can be obtained by minimizing ζ with respect to α, where

$$\begin{aligned}\zeta &= \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_i) - \frac{2i-1}{2n}\right)^2 \\ &= \frac{1}{12n} + \sum_{i=1}^{n}\left(1 - e^{\frac{1}{\alpha}(1-e^{\alpha x_i})} - \frac{2i-1}{2n}\right)^2\end{aligned}$$

Thus, CVM estimator can be obtained by solving the following non-linear equation

$$\sum_{i=1}^{n}\left(1 - e^{\frac{1}{\alpha}(1-e^{\alpha x_i})} - \frac{2i-1}{2n}\right) e^{\frac{1}{\alpha}(1-e^{\alpha x_i})}\left\{\frac{1}{\alpha^2}\left(e^{\alpha x_i}-1\right) - \frac{1}{\alpha}x_i e^{\alpha x_i}\right\} = 0. \quad (24)$$

4.4.2 Anderson-Darling estimation

The Anderson-Darling (AD) estimator is another type of minimal distance estimator that is based on the Anderson-Darling statistic, is an alternative to traditional statistical tests for detecting sample distributions departure from normality [32]. The AD estimate, $\hat{\alpha}_{AD}$ of the parameter is obtained by minimizing the following expression with respect to α,

$$\mathbb{A} = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[logF(x_{i:n}) + log(1 - F(x_{n+1-i:n}))\right].$$

After minimizing the above expression with respect to α, we have the following nonlinear equation which will be solved numerically to obtain the AD estimator.

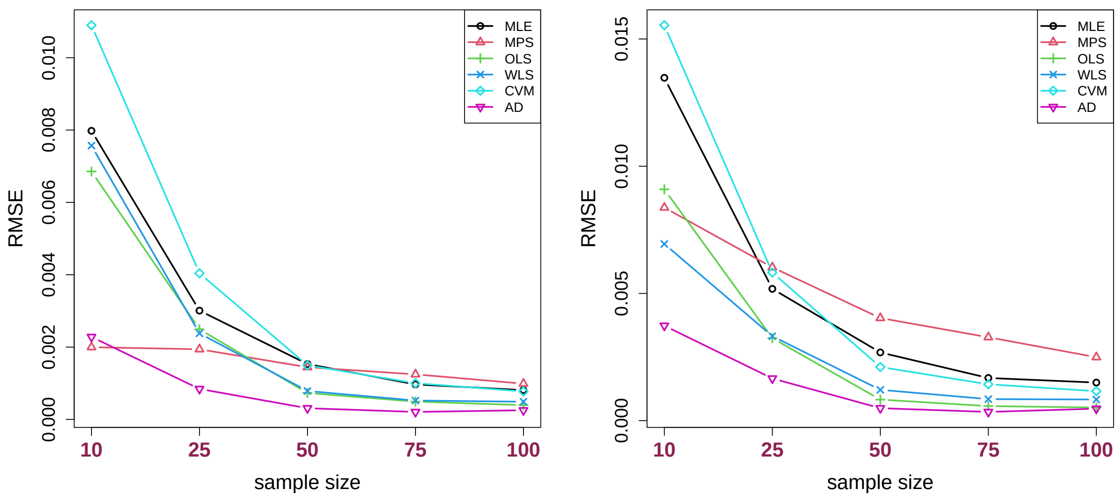
$$\sum_{i=1}^{n}(2i-1)\left[2e^{\frac{1}{\alpha}(1-e^{\alpha x_i})} - 1\right]\left\{\frac{1}{\alpha}x_i e^{\alpha x_i} - \frac{1}{\alpha^2}(e^{\alpha x_i}-1)\right\} = 0. \quad (25)$$

5. Simulation study for different estimation methods

In this section, a Monte Carlo simulation study has been performed to investigate the behaviour of the proposed estimators. The performance is evaluated based on the root mean square error (RMSE) values of the following six estimates namely, maximum likelihood estimate (MLE), maximum product spacing (MPS), ordinary least square (OLS), weighted least square (WLS), Cramér-Von-Mises (CVM) and Anderson-Darling (AD) estimate. We generate K=1000 random samples $X_1, X_2, ..., X_n$ of sizes n = 10, 25, 50, 75, 100 from Inv-$A(\alpha)$ distribution by using inverse transformation method. The initial choices of parameter are taken as $\alpha = 0.1, 0.5, 1.0, 3.0, 5.0$. We calculate the ML, MPS, OLS, WLS, CVM, AD estimates for all choices of the scale parameter. Numerical outcomes are constructed in Table 1 where the average estimates and corresponding RMSE values are displayed.

Table 1: *Average estimate values and the associated RMSEs for Inv-$A(\alpha)$ distribution*

Parameter choice	Sample sizes (n)	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{MPS}$	$\hat{\alpha}_{OLS}$	$\hat{\alpha}_{WLS}$	$\hat{\alpha}_{CVM}$	$\hat{\alpha}_{AD}$
	10	0.334841	0.071176	0.322474	0.353796	0.426889	0.169213
		0.007426	0.000912	0.007035	0.008026	0.010337	0.002189
	25	0.189182	0.061142	0.180246	0.174847	0.224161	0.123333
		0.002820	0.001229	0.002538	0.002367	0.003926	0.000738
$\alpha = 0.1$	50	0.145191	0.068292	0.124540	0.124579	0.146387	0.108763
		0.001429	0.001002	0.000776	0.000777	0.001467	0.000277
	75	0.128844	0.071494	0.116508	0.116079	0.130995	0.105799
		0.000912	0.000901	0.000522	0.000508	0.000980	0.000183
	100	0.123515	0.076731	0.112897	0.114688	0.123671	0.106887
		0.000744	0.000736	0.000408	0.000464	0.000749	0.000218
	10	0.752290	0.436867	0.716765	0.739353	0.844578	0.572157
		0.007978	0.001996	0.006855	0.007569	0.010897	0.002282
	25	0.595096	0.438685	0.578822	0.575220	0.627734	0.526648
		0.003007	0.001939	0.002493	0.002379	0.004040	0.000843
$\alpha = 0.5$	50	0.548418	0.454349	0.523074	0.524836	0.547285	0.509773
		0.001531	0.001444	0.000730	0.000785	0.001495	0.000309
	75	0.530561	0.460606	0.515562	0.516496	0.531643	0.506529
		0.000966	0.001246	0.000492	0.000522	0.001000	0.000206
	100	0.525595	0.468765	0.512467	0.515438	0.524463	0.508000
		0.000809	0.000988	0.000394	0.000488	0.000774	0.000253
	10	1.275037	0.904548	1.226252	1.223553	1.425039	1.077413
		0.008697	0.003018	0.007155	0.007069	0.013441	0.002448
	25	1.103888	0.918020	1.079842	1.076663	1.133836	1.030476
		0.003285	0.002592	0.002525	0.002424	0.004232	0.000964
$\alpha = 1.0$	50	1.053224	0.941482	1.022732	1.026182	1.049406	1.010681
		0.001683	0.001850	0.000719	0.000828	0.001562	0.000338
	75	1.033451	0.950529	1.015402	1.017632	1.033153	1.007198
		0.001058	0.001564	0.000487	0.000558	0.001048	0.000228
	100	1.028529	0.961345	1.012633	1.016743	1.025899	1.00906
		0.000902	0.001222	0.000399	0.000529	0.000819	0.000286
	10	3.356249	2.809940	3.258877	3.224152	3.430144	3.098857
		0.011266	0.006010	0.008186	0.007088	0.013602	0.003126
	25	3.136043	2.857535	3.090781	3.091121	3.160051	3.042742
		0.004302	0.004505	0.002871	0.002882	0.005062	0.001352
$\alpha = 3.0$	50	3.070314	2.903247	3.024113	3.032452	3.058492	3.013378
		0.002224	0.003060	0.000763	0.001026	0.001850	0.000423
	75	3.044011	2.920498	3.016559	3.022434	3.039508	3.009221
		0.001392	0.002514	0.000524	0.000709	0.001249	0.000292
	100	3.038722	2.939091	3.014403	3.021810	3.031602	3.012286
		0.001224	0.001926	0.000455	0.000690	0.000999	0.000389
	10	5.426122	4.735217	5.287511	5.219631	5.491406	5.11804
		0.013475	0.008373	0.009092	0.006945	0.01554	0.003733
	25	5.16381	4.809465	5.102561	5.104967	5.184119	5.052456
$\alpha = 5.0$		0.005180	0.006025	0.003243	0.003319	0.005822	0.001659
	50	5.084771	4.872474	5.026128	5.038306	5.06668	5.015507
		0.002681	0.004033	0.000826	0.001211	0.002109	0.00049
	75	5.053036	4.896338	5.018092	5.026767	5.045195	5.010834
		0.001677	0.003278	0.000572	0.000846	0.001429	0.000343
	100	5.047319	4.92115	5.016312	5.026294	5.036646	5.014891
		0.001496	0.002493	0.000516	0.000831	0.001159	0.000471



(a) *RMSE for* $\alpha = 0.5$ **(b)** *RMSE for* $\alpha = 5$

Figure 3: *RMSE of* $\hat{\alpha}$ *under the six different estimation methods with the variation of sample size n*

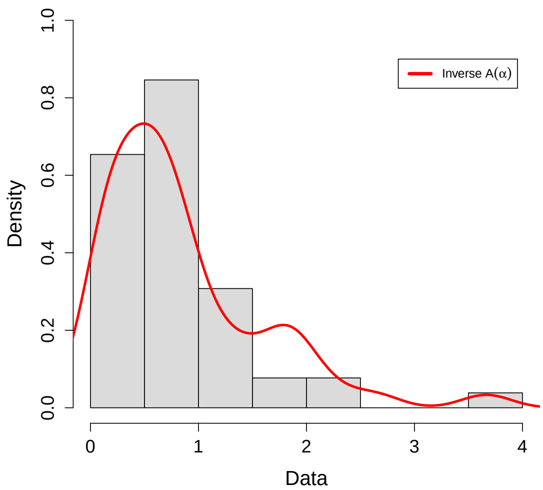
According to Table 1, as the sample size increases, the RMSE of ML, MPS, OLS, WLS, CVM and AD estimates of the scale parameter decrease. Hence, all the estimators hold the property of consistency. Also, it has been observed that the RMSE of the ML, MPS, WLS, CVM and AD estimates increase with the increment of the scale parameter. For small size of sample n=10, the performance of MPS estimate is effective when $\alpha < 1$. Overall, the AD estimate is most effective among all the estimates as it produces the least RMSE value for most of the cases we have considered in our study. The results are also verified from the Figure 3.

6. Real data application of the Inv-$A(\alpha)$ distribution

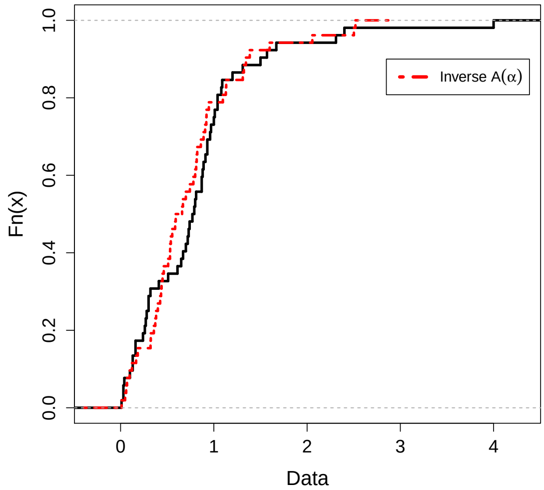
A real dataset has been considered with the goal of evaluating the potentiality of the Inv-$A(\alpha)$ distribution by comparing it with some other well known distributions already available in literature. Inverse exponential (IE) [6], inverse Xgamma (IXg) [9], inverse Lindley (IL) [8], inverse Gamma (IG) [11], inverse Kumaraswamy (IK) [12], inverse Weibull (IW) [7], inverted Nadarajah–Haghighi (INH) [33], Exponentiated inverse Rayleigh (EIR) [34], Inverse power Lindley (IPL) [10] are the few distributions belong to the inverse family have been selected as the competitive models. The parameters of the considered models have been estimated through the MLE approach.

The data consists of death times (in weeks) of 52 patients having tongue cancer with an aneuploid DNA profile discussed by [35] and given by [36]. Recently, [37] used this dataset in their study. Patients with sexually transmitted illnesses who had a paraffin-embedded sample of malignant tissue obtained were chosen and the time frames to reinfection were estimated. Using a flow cytometer, the tissue samples were evaluated to see if the tumour had an aneuploid (abnormal) or diploid (normal) DNA profile, as described by [35].

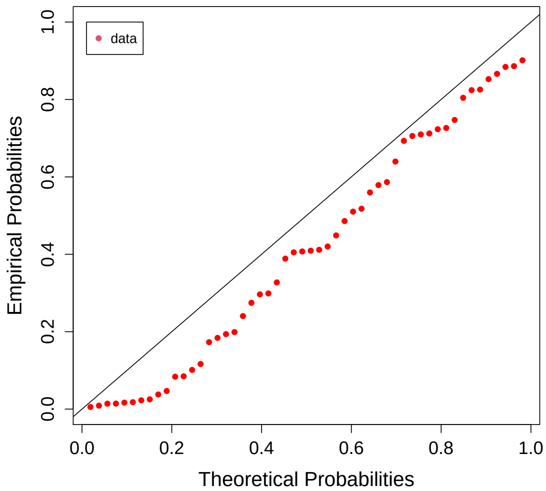
In ordered to make comparison among the considered models some criterion includes 2×negative log-likelihood ($-2\,ln\,L$), Akaike Information Criterion (AIC), Corrected AIC (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) are utilized. A model with minimum values of these statistics are considered to be the best model. Further, we also use goodness of fit tests such as Kolmogorov-Smirnov (K-S), Cramér-Von-Mises (CVM) and Anderson-Darling (AD) tests along with their corresponding P Values. The MLE with respective standard error (in parentheses) of the parameters and values of $-2\,ln\,L$, AIC, BIC, CAIC and HQIC and the numerical values of K-S, CVM and AD statistics along with their corresponding P values are displayed in Table 2 and 3 respectively. It has been observed that the Inv-$A(\alpha)$ distribution



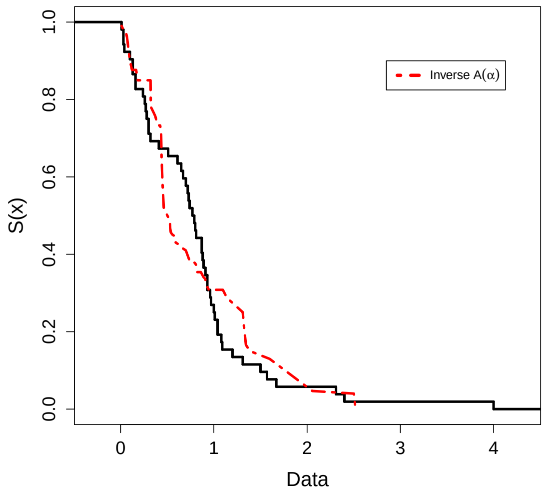
(a) *Histogram*



(b) *Empirical cdf*



(c) *PP Plot*



(d) *Empirical Survival Plot*

Figure 4: *Empirical pdf, cdf, pp and sf plots for the tongue cancer data*

have the lowest values for all goodness-of-fit statistics and the largest P value among all other competitive models. As a result, our proposed model outperformed the other models for the tongue cancer data.

Figure 4 shows a plot of estimated histogram, empirical cdf, PP-plot and a plot of the survival functions modified by the suggested theoretical models onto the empirical survival function (Kaplan-Meier estimate), which may be used to verify the goodness of fit for the proposed model. A graphical technique based on total time on test (TTT) plot is also used here to identify the shapes of the data. According to [38], the hrf is constant if the TTT plot is visually portrayed as a straight diagonal, the hrf is increasing (or decreasing) if the TTT plot is concave (or convex). The hrf is U-shaped (bathtub) if the TTT plot is firstly convex and then concave, if not, the hrf is unimodal. The TTT plot in Figure 5 indicates that the empirical hrf of the tongue cancer dataset is 'monotonically increasing'. Hence the proposed lifetime model Inv-$A(\alpha)$ might be a good fit for the cancer data theoretically.

Table 2: *Analytical results of the Inv-$A(\alpha)$ distribution and the other competing models for the tongue cancer data*

Model	Estimates (SE)	$-2\log L$	AIC	CAIC	BIC	HQIC
Inv-$A(\alpha)$	0.2333 (0.1187)	80.9634	82.9634	83.0434	84.9146	83.7114
IE (θ)	0.1766 (0.0245)	146.0563	148.0563	148.1363	150.0075	148.8044
IXg (θ)	0.3741 (0.0351)	216.1880	218.1880	220.1393	218.2680	218.9361
IL (θ)	0.3112 (0.0310)	191.1509	193.1509	193.2309	195.1021	193.8989
IG (α, β)	0.5805 (0.0953) 0.1025 (2.3912)	133.2271	137.2271	137.4720	141.1295	138.7232
IK (α, β)	2.5056 (0.3860) 1.6592 (0.3184)	88.4087	92.4087	92.6536	96.3112	93.9048
IW (α, β)	0.4113 (0.0790) 0.6745 (0.0619)	121.5629	125.5629	125.8078	129.4654	127.0590
INH (α, β)	1.2836 (0.4463) 0.4228 (0.0580)	107.3259	111.3259	111.5708	115.2284	112.8220
EIR (α, σ)	0.1706 (0.0256) 0.0283 (0.0049)	161.3631	165.3631	165.6080	169.2656	166.8592
IPL (α, β)	0.5790 (0.0497) 0.7788 (0.1020)	122.8562	126.8562	127.1011	130.7587	128.3524

Table 3: *Goodness of fit measures of the Inv-$A(\alpha)$ distribution and the other competing models for the tongue cancer data*

Model	K-S	P value	CVM	P value	AD	P value
Inv-$A(\alpha)$	0.13896	0.26783	0.22006	0.23203	1.13202	0.29461
IE (θ)	0.40253	9.606×10^{-8}	2.52648	4.603×10^{-7}	12.57664	1.153×10^{-5}
IXg (θ)	0.51125	3.130×10^{-12}	4.93455	0.00000	28.18699	1.153×10^{-5}
IL (θ)	0.48784	3.563×10^{-11}	4.04569	0.00000	23.22221	1.153×10^{-5}
IG (α, β)	0.27891	6.130×10^{-4}	1.21856	6.903×10^{-4}	6.15950	8.319×10^{-4}
IK (α, β)	0.20757	2.265×10^{-2}	0.41892	6.405×10^{-2}	2.08487	8.278×10^{-2}
IW (α, β)	0.21708	1.488×10^{-2}	0.81699	6.410×10^{-3}	4.48490	5.123×10^{-3}
INH (α, β)	0.19485	3.857×10^{-2}	0.63488	1.800×10^{-2}	3.51034	1.531×10^{-2}
EIR (α, σ)	0.34511	8.350×10^{-6}	2.08864	5.598×10^{-6}	9.97822	1.867×10^{-5}
IPL (α, β)	0.21509	1.627×10^{-2}	0.81766	6.386×10^{-3}	4.5175	4.941×10^{-3}

7. Conclusion

In medical science and reliability engineering, development of a distribution with an increasing hazard rate function constitutes a considerable practical interest. In this article, we have presented an Inv-$A(\alpha)$ distribution with some properties such as quantile function, median, reliability function, hazard rate function, order statistics, ageing intensity function etc. The flexibility of this distribution primarily depends on the reliability behaviour as the distribution has an increasing hazard rate function. This feature of such distribution enhances the applicability to the real world. For instance, it describes the real scenarios which are more likely to fail with age, either of a human being or a machine whose parts wear out.

The model parameter is estimated through the ML estimation, maximum product of spacings estimation, ordinary and weighted least square estimation, CVM and AD estimation respectively.

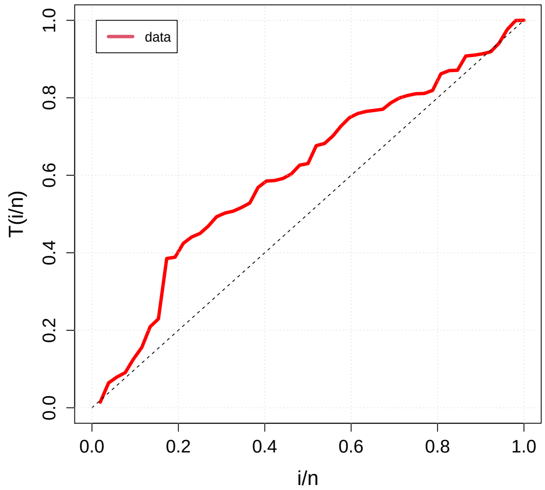


Figure 5: *TTT plot for the Tongue cancer data*

The Monte Carlo simulation study has been performed to investigate the performance of the obtained estimators and it is noticed that all the estimators are asymptotically unbiased and consistent. Among all the traditional estimation methods, Anderson-Darling method outperforms the others. Furthermore, we consider tongue cancer data to exhibit the applicability of the Inv-$A(\alpha)$ distribution in the field of bio-medical science. To examine the superiority of the proposed model, we compared it with some competitive models and found that our model has the best fittings amongst them based on the goodness of fit measures. Therefore, we hope that our new proposed model from the family of inverse probability distribution might be taken as a viable choice to analyze several medical science data.

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