Gumbel Marshall-Olkin Lomax: A new distribution for reliability modelling

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Abstract

A new distribution for modeling the two approaches (physical and actuarial) of reliability problems is introduced. The statistical properties including the moments, mode, quantile function are derived. Some reliability measures including the mean residual life and hazard rate are derived. An alternative measure for total time of test (TTT) for evaluation of the interfailure times is drived.The unknown parameters of the new distribution are estimated using the maximum likelihood approach. Furthermore, the asymptotic consistency of the estimated parameters is evaluated through a simulation study. Two real-life datasets were used to illustrate the applicability of the new distribution and comparison with already existing distributions.

Keywords: Lomax distribution, Reliability, Moment, Total time of test, Maximum likelihood

1. Introduction

There have been growing needs to provide solutions associated with reliability problems found in life testing, structural reliability, machine maintenance using probability distribution [1]. Many classical distributions including Weibull, Log-normal, Birnbaum-Saunders, Inverse normal, gamma, exponential, geometric, Poisson have been applied in reliability studies where interest is on nonrepairable system [2]. However, [1] noted that it may be difficult to differentiate among these distributions while fitting failure datasets but stated that the failure rate function provides distinguishing features for these distributions. [3] furthermore, pointed out that distributions with bathtub shape failure rate function describing the decreasing, normal or constant, and increasing failure rate of component would have wide applicability in reliability studies. Most of the classical distributions do not exhibit bathtub-shape hazard rate function [4]. However, a distribution to analyze business failure which is referred to as Lomax distribution was introduced by [5]. The application of Lomax distribution has been found in many other areas including income, size cities, reliability modeling [6], see [7] for more details. The Lomax distribution has been extended by introducing one or more additional parameter such as Marshall-Olkin Lomax due to [8], gamma Lomax by [9], exponential Lomax by [10], logistic-Lomax by [11] and McDonald Lomax distribution by [7]. The major aim of this paper is to introduce a new and more flexible extended Lomax distribution that will provide better fit and for modeling reliability datasets amongst other datasets from different areas of study. The reversed-J-shape, constant, and J-shape among many other shapes are the characterizations of the failure rate function shape of the new distribution. These shapes of failure rate function are suitable for modeling increasing failure rate (IFR), no-ware out and decreasing failure rate (DFR) datasets. Some statistical properties of this distribution are discussed and comparison with other existing distribution having Lomax distribution as baseline was made. The rest of the paper is organized as follows. The new distribution is derived in section two. In Section 3, the statistical properties of the distribution are derived and presented while the reliability measures are derived in Section 4. The Entropy and parameter estimation of the distribution are respectively considered in Sections 5 and 6. The asymptotic consistence of the maximum likelihood estimates is considered in Section 7 while the applications to real-life data sets are done in Section 8. The concluding remark is presented in Section 9.

2. The new distribution

A class of distribution having distribution function as defined by equation(1) was introduced by[12].

$$G(x) = e^{-Bp^{\frac{1}{\sigma}}\left[\frac{F(x;\xi)}{1-F(x;\xi)}\right]^{-\frac{1}{\sigma}}}; \tag{1}$$

where $B = e^{\frac{\mu}{\sigma}}$. Define $F(x;\xi) = 1 - \left(1+\frac{x}{\lambda}\right)^{-\alpha}$ in eq(1), where $\xi = (\alpha, \lambda)$ is the parameter vector, the cumulative density function (cdf) of the new distribution referred to as Gumbel Marshall-Olkin-Lomax (GMO-Lomax) is given by

$$G(x) = e^{-Bp^{\frac{1}{\sigma}}\left[\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right]^{-\frac{1}{\sigma}}}. \tag{2}$$

The density function corresponding to equation (2) is obtained as

$$g(x) = \frac{Bp^{\frac{1}{\sigma}}\alpha\left(1+\frac{x}{\lambda}\right)^{\alpha-1} e^{-Bp^{\frac{1}{\sigma}}\left[\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right]^{-\frac{1}{\sigma}}}}{\lambda\sigma\left[\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right]^{\frac{1}{\sigma}+1}}. \tag{3}$$

Furthermore, equation(3) can also be obtained using **Theorem 1**.

Theorem 1. Let X and Y be two random variables, if Y follows Gumbel distribution, then, $X = \lambda\left[\left(1+pe^{Y}\right)^{\frac{1}{\alpha}} - 1\right]$ follows GMO-Lomax distribution.

Proof. Given that the random variable Y follows Gumbel distribution, its pdf is given as

$$h(y) = \frac{B}{\sigma}e^{-\frac{y}{\sigma}}e^{-Be^{-\frac{y}{\sigma}}}. \tag{4}$$

For $X = \lambda\left[\left(1+pe^{Y}\right)^{\frac{1}{\alpha}} - 1\right]$, the partial derivative w.r.t. x is obtained as

$$\frac{\partial y}{\partial x} = \frac{\alpha}{\lambda\left(1+\frac{x}{\lambda}\right)\left[1-\left(1-\frac{x}{\lambda}\right)^{-\alpha}\right]}.$$

The density function of X is defined as $g(x) = h(y)\left|\frac{\partial y}{\partial x}\right|$. Substituting the value of Y in $h(y)$ and $\left|\frac{\partial y}{\partial x}\right|$ and simplifying yields

$$g(x) = \frac{Bp^{\frac{1}{\sigma}}\alpha\left(1+\frac{x}{\lambda}\right)^{\alpha-1} e^{-Bp^{\frac{1}{\sigma}}\left[\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right]^{-\frac{1}{\sigma}}}}{\lambda\sigma\left[\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right]^{\frac{1}{\sigma}+1}}.$$

■

Some possible shapes of GMO-Lomax pdf, including monotone decreasing, monotone increasing, right-skewed, among other shapes are shown in Figure 1.

3. Statistical properties

Some of the GMO-Lomax statistical properties such as Quantile function, moments, moment generating function, mode are derived and presented in this section.

3.1. Quantile function

The quantile function is very important in probability distribution, θ^{th}, percentile and random number generation for a distribution can be obtained using the quantile function. Using the probability integral transform [13], the quantitle function of GMO-Lomax is obtained as

$$Q_X(u) = \lambda\left(\left\{1+B^{\sigma}p\left[log\left(u^{-1}\right)\right]^{-\sigma}\right\}^{\frac{1}{\alpha}}-1\right). \tag{5}$$

Using **Theorem 2**, the quantile function of GMO-Lomax can also be obtained.

Theorem 2. Given that a random variable, Y, follows Gumbel distribution, then the quantile function of GMO-Lomax is defined by $Q_X(u) = F^{-1}\left\{1+p^{-1}e^{-G^{-1}(u)}\right\}^{-1}$; where $G^{-1}(.)$ denotes the quantile function of Gumbel distribution and $F^{-1}(.)$ denotes the quantile function of Lomax distribution.

Proof. Equation(1) can also be re-written as

$$G(y) = \int_{-\infty}^{y}\frac{B}{\sigma}e^{-\frac{t}{\sigma}}e^{Be^{-\frac{t}{\sigma}}}dt, \tag{6}$$

where $y = log\left[\frac{F(x)}{p[1-F(x)]}\right]$.

By probability integral transform, the quantile function of a random variable, X, having a well-defined cdf, $F(x)$, is given by $x = F^{-1}(u)$, where $u = F(x)$. Then, the quantile function of Gumbel distribution is given by

$$y = G^{-1}(u) = log\left\{\left[B^{-1}log\left(u^{-1}\right)\right]^{-\sigma}\right\}. \tag{7}$$

Furthermore, the quantile function of Lomax distribution is given by

$$x = F^{-1}(u) = \lambda\left[(1-u)^{-\frac{1}{\alpha}}-1\right]. \tag{8}$$

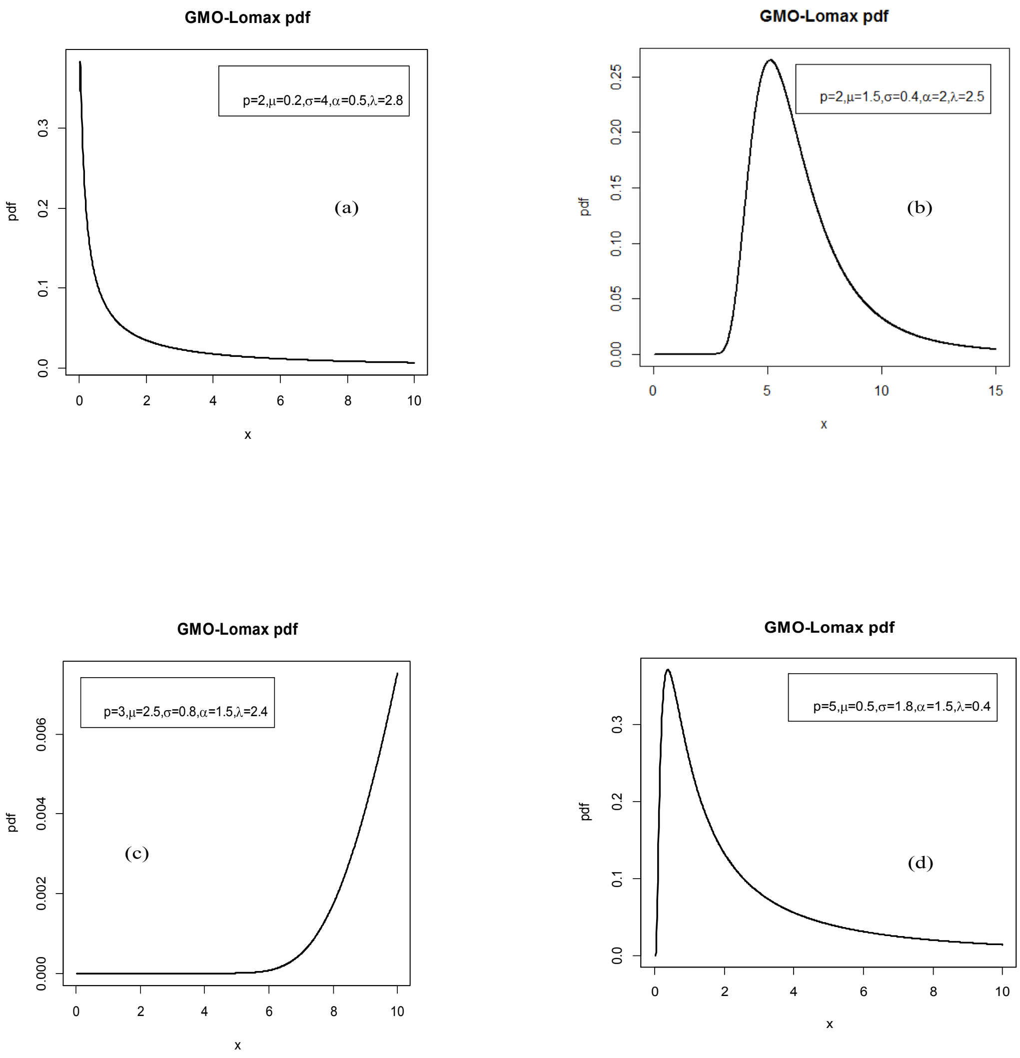


Figure 1: *Some possible shapes of GMO-Lomax pdf: a) monotone decreasing b) unimodal c) monotone increasing d) right-skewed.*

From equation(6)

$$\begin{aligned} x &= F^{-1}\left[1+p^{-1}e^{-y}\right]^{-1} \\ &= F^{-1}\left[1+p^{-1}e^{-G^{-1}(u)}\right]^{-1} \\ &= F^{-1}\left(\left\{1+pB^{\sigma}\left[log\left(u^{-1}\right)\right]^{-\sigma}\right\}^{-1}\right). \end{aligned} \tag{9}$$

Substituting the value of $u = \left\{1+pB^{\sigma}\left[log\left(u^{-1}\right)\right]^{-\sigma}\right\}^{-1}$ in equation(9) and simplifying yields

$$Q_X(u) = \lambda \left(\left\{ 1 + B^{\sigma} p \left[log \left(u^{-1} \right) \right]^{-\sigma} \right\}^{\frac{1}{\alpha}} - 1 \right).$$

■

3.2. Moments

Corollary 1. The n^{th} non-central moment of GMO-Lomax random variable, X denoted by $E(X^n)$ is obtained as $E(X^n) = \lambda^n \sum_{j=0}^{\infty} \psi_j \Gamma(1 - j\sigma)$

Proof.

$$\begin{aligned} E\left(X^n\right) &= \int_0^{\infty} \left\{ \lambda \left[(1 + pe^y)^{\frac{1}{\alpha}} - 1 \right] \right\}^n \frac{B}{\sigma} e^{-\frac{y}{\sigma}} e^{-Be^{-\frac{y}{\sigma}}} dy \\ &= \frac{B\lambda^n}{\sigma} \int_0^{\infty} (1 + pe^y)^{\frac{n}{\alpha}} \left[1 - (1 + pe^y)^{-\frac{1}{\alpha}} \right]^n e^{-\frac{y}{\sigma}} e^{-Be^{-\frac{y}{\sigma}}} dy \\ &= \frac{B\lambda^n}{\sigma} \sum_{i,j=0}^{\infty} (-1)^i \binom{n}{i} \binom{\frac{n-i}{\alpha}}{j} p^j \int_0^{\infty} e^{jy} e^{-\frac{y}{\sigma}} e^{-Be^{-\frac{y}{\sigma}}} dy \\ &= \frac{B\lambda^n}{\sigma} \sum_{i,j=0}^{\infty} (-1)^i \binom{n}{i} \binom{\frac{n-i}{\alpha}}{j} p^j \int_0^{\infty} e^{-\frac{y}{\sigma}(1-j\sigma)} e^{-Be^{-\frac{y}{\sigma}}} dy \end{aligned} \tag{10}$$

Letting $x = Be^{-\frac{y}{\sigma}}$ implies that $dy = -\frac{\sigma}{x} dx$ and equation(10) becomes

$$\begin{aligned} E\left(X^n\right) &= B\lambda^n \sum_{i,j=0}^{\infty} (-1)^i \binom{n}{i} \binom{\frac{n-i}{\alpha}}{j} p^j B^{j\sigma - 1} \int_0^{\infty} x^{-j\sigma} e^{-x} dx \\ &= \lambda^n \sum_{j=0}^{\infty} \psi_j \Gamma\left(1 - j\sigma\right), \end{aligned}$$

where
$\psi_j = \sum_{i=j}^{\infty} (-1)^i \binom{n}{i} \binom{\frac{n-i}{\alpha}}{j} p^j B^{j\sigma}$. ■

3.3. Moment generating function

The moment generating function (mgf) of a random variable with well-defined density function, $f(x)$, is defined by$M_X(t) = E(e^{tX})$. For a random variable with pdf defined as in equation(3) then, the mgf is given by

$$\begin{aligned} M_X(t) &= \int_0^{\infty} e^{tx} \frac{Bp^{\frac{1}{\sigma}} \alpha \left(1 + \frac{x}{\lambda}\right)^{\alpha - 1} e^{-Bp^{\frac{1}{\sigma}} \left[\left(1 + \frac{x}{\lambda}\right)^{\alpha} - 1 \right]^{-\frac{1}{\sigma}}}}{\lambda\sigma \left[\left(1 + \frac{x}{\lambda}\right)^{\alpha} - 1 \right]^{\frac{1}{\sigma} + 1}} \\ &= \frac{Bp^{\frac{1}{\sigma}}\alpha}{\lambda\sigma} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!} \binom{\frac{i}{\sigma} + \frac{1}{\sigma} + j}{j} \left(Bp^{\frac{1}{\sigma}}\right)^i \int_0^{\infty} e^{tx} \left(1 + \frac{x}{\lambda}\right)^{-\alpha\left(\frac{i}{\sigma} + \frac{1}{\sigma} + j\right) - 1} dx \\ M_X(t) &= \sum_{j=0}^{\infty} \varphi_j \Gamma\left(-\frac{\alpha\left(j\sigma + i + 1\right)}{\sigma}, t\lambda \right), \end{aligned}$$

where
$\varphi_j = \frac{Bp^{\frac{1}{\sigma}}\alpha}{\sigma} \sum_{j=i}^{\infty} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!} \binom{\frac{i}{\sigma} + \frac{1}{\sigma} + j}{j} \left(Bp^{\frac{1}{\sigma}}\right)^i (-1\lambda)^{\frac{\alpha(j\sigma+i+1)}{\sigma}}$.

3.4. Mode

The mode of a distribution plays an important role in life distribution. It defines the most likely failure time of an object when failure is of consideration. The mode of GMO-Lomax is obtained as the value of x that satisfies $\frac{\partial log(g(x))}{\partial x} = 0$ given in equation (11)

$$\frac{\alpha-1}{\lambda\left(1+\frac{x}{\lambda}\right)}+\frac{B\alpha p^{\frac{1}{\sigma}}}{\sigma\lambda}\left(1+\frac{x}{\lambda}\right)^{\alpha-1}\left[\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right]^{-\left(\frac{1}{\sigma}+1\right)}-\frac{\left(\frac{1}{\sigma}+1\right)\alpha\left(1+\frac{x}{\lambda}\right)^{\alpha-1}}{\lambda\left[\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right]}=0 \tag{11}$$

4. Reliability measures

4.1. Hazard rate function

Generally, the hazard rate function is defined as the conditional probability of failure, given that a component has survived up to time x. [4] note that the hazard rate function is an important quantity which characterizes life phenomena. Denoting the hazard rate function as $R(x)$, the hazard rate function is defined as $\frac{g(x)}{S(x)}$, where $S(x)$ represents the survival function. Suppose a random variable X follows GMO-Lomax distribution, the hazard rate function associated to GMO-Lomax is given by

$$R(x)=\frac{Bp^{\frac{1}{\sigma}}\left(1+\frac{x}{\lambda}\right)^{\alpha-1}}{\lambda\sigma\left[\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right]^{\frac{1}{\sigma}+1}\left[e^{Bp^{\frac{1}{\sigma}}\left[\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right]^{-\frac{1}{\sigma}}}-1\right]}.$$

Figure 2 shows some possible shapes of the GMO-Lomax hazard rate function which include decreasing hazard rate function which captures the high failure rate at the initial phase (infant mortality), the constant hazard rate function representing the period of stability of the component, and the increasing hazard rate function capturing the increase in failure rate as the component begins to wear-out.

4.2. Mean residual life function

Given that a random variable, X, denotes the lifetime of a component. The mean residual life function denoted by $m(t)$ defines the expected value of the remaining lifetime of a component after a fixed point t . Suppose the random variable, X, follows GMO-Lomax distribution, then

$$\begin{aligned} m(t) &= E\left(X-t|X>t\right) \\ &= \frac{1}{1-G(t)}\int_t^{\infty}1-G(u)du, \end{aligned} \tag{12}$$

where $G(.)$ is as defined in equation(2), substituting in equation(12) and simplifying yields

$$m(t)=\sum_{k=0}^{\infty}\psi_k\left(1+\frac{t}{\lambda}\right)^{-\alpha\left(\frac{j}{\sigma}+k\right)}\left\{\sum_{j=0}^{\infty}\psi_j\frac{\left[\lambda\sigma+t\alpha\left(1+i+j\sigma\right)\right]\left(1+\frac{t}{\lambda}\right)^{-\alpha\left(\frac{1}{\sigma}+\frac{1}{\sigma}+j\right)}}{\left(1+i+j\sigma\right)\left[\alpha\left(1+i+j\sigma\right)-\sigma\right]}-t\right\},$$

where

$$\psi_k=\sum_{i,k=j}^{\infty}\frac{(-1)^j}{j!}\left(iBp^{\frac{1}{\sigma}}\right)^j\binom{\frac{j}{\sigma}+k-1}{k}$$

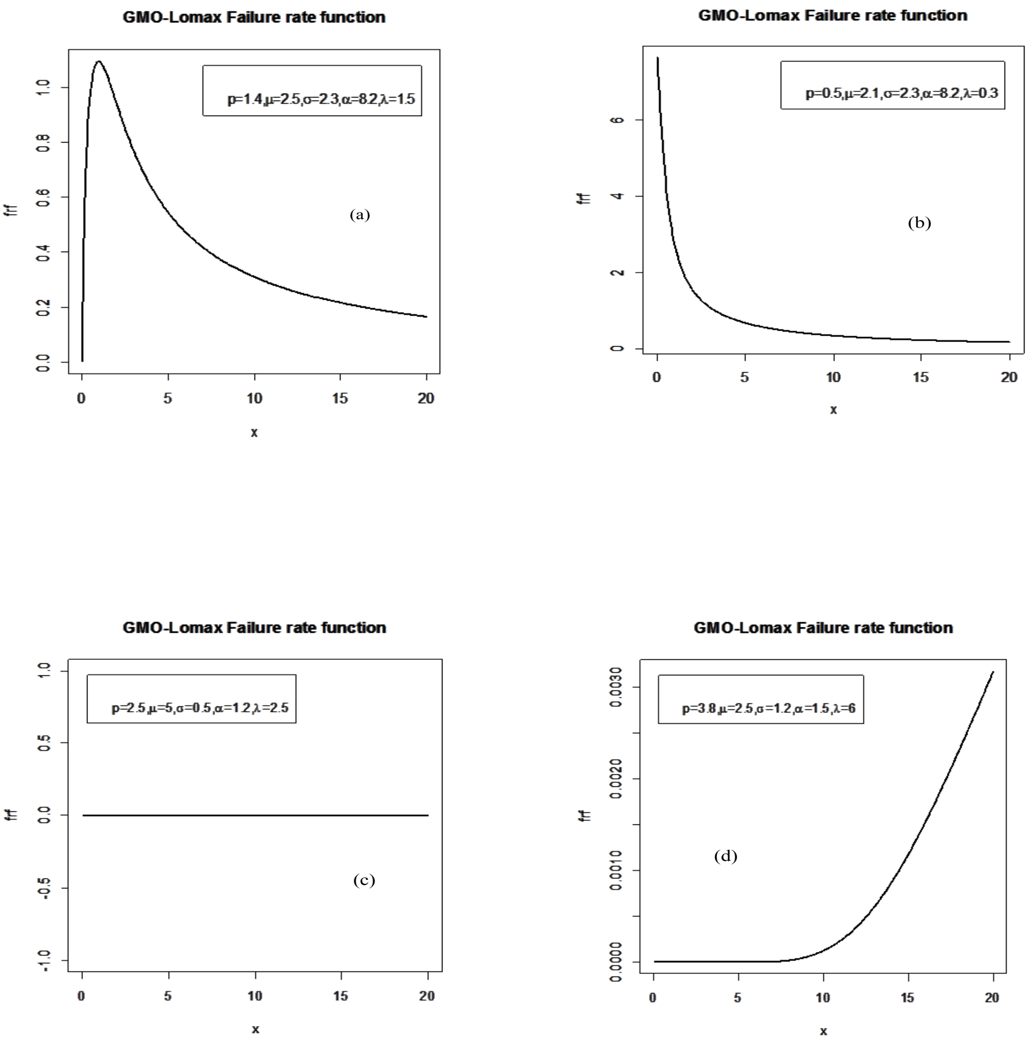


Figure 2: *Some possible shapes of GMO-Lomax failuare rate function: a) right-skewed b) monotone decreasing c) constant d) monotone increasing.*

and

$$\psi_j = Bp^{\frac{1}{\sigma}} \sum_{j=i}^{\infty} \frac{(-1)^i}{i!} \left(Bp^{\frac{1}{\sigma}}\right)^i \binom{\frac{j}{\sigma} + k - 1}{k}$$

.

5. Reliability

Suppose the random variables, X and Y, represent, respectively, the strength and stress of a component. The measure of performance of the component (that is the component reliability) having strength, X when subjected to random stress, Y, denoted by R is defined as $R = P(Y < X)$. Let X and Y, respectively, follow GMO-Lomax with some different parameters, then, R, is defined by

$$\begin{aligned} R &= \int_0^\infty g(x; B_1, p_1, \sigma, \alpha, \lambda) P(Y < X) dx \\ &= \int_0^\infty g(x; B_1, p_1, \sigma, \alpha, \lambda) G(x; B_2, p_2, \sigma, \alpha, \lambda) dx \\ &= \int_0^\infty \frac{B_1 p_1^{\frac{1}{\sigma}} \alpha}{\lambda\sigma \left[\left(1 + \frac{x}{\lambda}\right)^\alpha - 1 \right]^{\frac{1}{\sigma}+1}} e^{-\left(B_1 p_1^{\frac{1}{\sigma}} + B_2 p_2^{\frac{1}{\sigma}} \right) \left[\left(1+\frac{x}{\lambda}\right)^\alpha - 1 \right]^{-\frac{1}{\sigma}}} dx \\ &= \sum_{j=0}^\infty C_j B_1 p_1^{\frac{1}{\sigma}}, \end{aligned}$$

where $B_1 = e^{\frac{\mu_1}{\sigma}}, B_2 = e^{\frac{\mu_2}{\sigma}}$ and $C_j = \sum_{j=i}^\infty \frac{(-1)^i}{i!} \binom{\frac{i}{\sigma} + \frac{1}{\sigma} + j}{j} \frac{\left(B_1 p_1^{\frac{1}{\sigma}} + B_2 p_2^{\frac{1}{\sigma}} \right)^i}{(1+i+j\sigma)}$.

5.1. Lorenz curve

The Lorenz curve was established by[14] to graphical represent the distribution of wealth in a population. However, [15] established relationship between the Lorenz curve and the total time on test (TTT). The TTT graphically detects the possible change in the pattern of failures [16]. Hence, if a random variable, X, follows GMO-Lomax such that it denotes the failure times of a component or an individual, then the Lorenz curve is defined as

$$L(\varphi) = \frac{1}{\mu} \int_0^z x f(x) dx \tag{13}$$

Substituting equation (3) in equation (14), we have

$$\begin{aligned} L(\varphi) &= \frac{B p^{\frac{1}{\sigma}} \alpha}{\mu\lambda\sigma} \int_0^z \frac{x \left(1 + \frac{x}{\lambda}\right)^{\alpha-1} e^{-B p^{\frac{1}{\sigma}} \left[\left(1+\frac{x}{\lambda}\right)^\alpha - 1 \right]^{-\frac{1}{\sigma}}}}{\left[\left(1 + \frac{x}{\lambda}\right)^\alpha - 1 \right]^{\frac{1}{\sigma}+1}} dx \\ &= \frac{B p^{\frac{1}{\sigma}} \alpha}{\mu\lambda\sigma} \sum_{i=0}^\infty \frac{(-1)^i}{i!} \left(B p^{\frac{1}{\sigma}} \right)^i \int_0^z x \left(1 + \frac{x}{\lambda}\right)^{\alpha-1} \left[\left(1 + \frac{x}{\lambda}\right)^\alpha - 1 \right]^{-\left(\frac{i}{\sigma} + \frac{1}{\sigma} + 1\right)} dx \\ &= \frac{B p^{\frac{1}{\sigma}} \alpha}{\mu\lambda\sigma} \sum_{i,j=0}^\infty \frac{(-1)^{i+j}}{i!} \left(B p^{\frac{1}{\sigma}} \right)^i \binom{\frac{i}{\sigma} + \frac{1}{\sigma} + 1}{j} \int_0^z x \left(1 + \frac{x}{\lambda}\right)^{-\alpha\left(\frac{i}{\sigma} + \frac{1}{\sigma} + j\right) - 1} dx \\ &= \frac{1}{\mu} \sum_{j=0}^\infty (-1)^j \Psi_j, \end{aligned}$$

where μ is the first non-central moment and

$$\Psi_j = \sum_{i=j}^{\infty} \frac{(-1)^i \binom{\frac{i}{\sigma}+\frac{1}{\sigma}+1}{j} \left(Bp^{\frac{1}{\sigma}}\right)^{i+1} \left(1+\frac{z}{\lambda}\right)^{-\alpha\left(\frac{i}{\sigma}+\frac{1}{\sigma}+j\right)}}{i!(1+i+j\sigma)[\alpha(1+i+j\sigma)-\sigma]} \times \left[\lambda\sigma\left\{\left(1+\frac{z}{\lambda}\right)^{\alpha\left(\frac{i}{\sigma}+\frac{1}{\sigma}+j\right)}-1\right\} - z\alpha(1+i+j\sigma)\right]$$

6. Order statistics

Suppose $X_1 < X_2 < \cdots < X_n$ are ordered random sample of size n from GMO-Lomax population. The density function of the h^{th} order statistics ($h = 1, 2, \cdots, n$), say, $g_{h:n}(x)$, is obtained as

$$g_{h:n}(x) = \frac{g(x)}{B(h, n-h+1)} \sum_{j=0}^{n-h} (-1)^j \binom{n-h}{j} G(x)^{h+j-1} \tag{14}$$

Substituting equations (2) and (3) in equation(14) and simplifying yields

$$g_{h:n}(x) = \frac{g(x)}{B(h, n-h+1)} \sum_{j=0}^{n-h} (-1)^j \binom{n-h}{j} \sum_{m=0}^{\infty} \varphi_m,$$

where $\varphi_m = \frac{Bp^{\frac{1}{\sigma}}\alpha}{\lambda\sigma} \sum_{k=m}^{\infty} \frac{(-1)^k}{k!} \left[Bp^{\frac{1}{\sigma}}(h+j)\right]^k \left(1+\frac{x}{\lambda}\right)^{-\alpha\left(\frac{k}{\sigma}+\frac{1}{\sigma}+m\right)-1}$.

7. Entropy

Suppose a random variable, X, follows GMO-Lomax, the uncertainty associated with a value of X is measured using entropy. The Rényi entropy introduced by [17] generalizes the Shannon entropy and it is defined by

$$I_R(\gamma) = \frac{1}{1-\gamma} log \left[\int_{\forall} g^{\gamma}(x) dx\right], \tag{15}$$

where $g(x)$ is the pdf of GMO-Lomax, then

$$\begin{aligned} I_R(\gamma) &= \frac{1}{1-\gamma} log \left[\left(\frac{Bp^{\frac{1}{\sigma}}\alpha}{\lambda\sigma}\right)^{\gamma} \int_0^{\infty} \frac{\left(1+\frac{x}{\lambda}\right)^{\gamma(\alpha-1)} e^{-\gamma Bp\left[\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right]^{-\frac{1}{\sigma}}}}{\left[\left(1+\frac{x}{\lambda}\right)^{\alpha}\right]^{\gamma\left(\frac{1}{\sigma}+1\right)}}\right] \\ &= \frac{1}{1-\gamma} log \left(Bp^{\frac{1}{\sigma}}\alpha\right) + log(\lambda\sigma) + \frac{1}{1-\gamma} log \left(\sum_{j=0}^{\infty} \varphi_j\right), \end{aligned}$$

where $\varphi_j = \sum_{j=i}^{\infty} \frac{(-1)^i}{i!} \left(\gamma Bp^{\frac{1}{\sigma}}\right)^i \binom{\frac{i}{\sigma}+\frac{\gamma}{\sigma}+\gamma+j-1}{j}$.

8. Parameter estimation

Let $X_1, X_2, ..., X_n$ be a radom sample of size n from GMO-Lomax population. The unknown parameters of GMO-Lomax are estimated using the maximum likelihood method. The log-likelihood function is obtained as

$$\begin{aligned}\ell(\Theta) &= \frac{n\mu}{\sigma} + \frac{n}{\sigma} log(p) + nlog(\alpha) + (\alpha-1)\sum_{i=1}^{n} log\left(1+\frac{x_i}{\lambda}\right) - e^{\frac{\mu}{\sigma}} p^{\frac{1}{\sigma}} \sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\lambda}\right)^{\alpha}-1\right]^{-\frac{1}{\sigma}} \\ &= -nlog(\lambda) - nlog(\sigma) - \left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\lambda}\right)^{\alpha}-1\right]. \qquad (16)\end{aligned}$$

The corresponding score functions of equation(16) are given below

$$\begin{aligned}\frac{\partial \ell(\Theta)}{\partial \sigma} &= \frac{n\mu}{\sigma^2} - \frac{n}{\sigma^2} log(p) + \frac{e^{\frac{\mu}{\sigma}} p^{\frac{1}{\sigma}}}{\sigma^2}\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\lambda}\right)^{\alpha}-1\right]^{-\frac{1}{\sigma}}[\mu + log(p)] - \frac{n}{\sigma} \\ &-\frac{1}{\sigma^2} e^{\frac{\mu}{\sigma}} p^{\frac{1}{\sigma}} \sum_{i=1}^{n} log\left[\left(1+\frac{x_i}{\lambda}\right)^{\alpha}-1\right]\left[\left(1+\frac{x_i}{\lambda}\right)^{\alpha}-1\right]^{-\frac{1}{\sigma}} + \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\lambda}\right)^{\alpha}-1\right].\end{aligned}$$

$$\frac{\partial \ell(\Theta)}{\partial \mu} = \frac{1}{\sigma}\left[n - e^{\frac{\mu}{\sigma}} p^{\frac{1}{\sigma}} \sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\lambda}\right)^{\alpha}-1\right]^{-\frac{1}{\sigma}}\right].$$

$$\frac{\partial \ell(\Theta)}{\partial p} = \frac{n}{\sigma p}\left[n - e^{\frac{\mu}{\sigma}} p^{\frac{1}{\sigma}} \sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\lambda}\right)^{\alpha}-1\right]^{-\frac{1}{\sigma}}\right].$$

$$\begin{aligned}\frac{\partial \ell(\Theta)}{\partial \lambda} &= \frac{(1-\alpha)}{\lambda^2}\sum_{i=1}^{n}\frac{x_i}{(1+\frac{x_i}{\lambda})} + \frac{\alpha}{\sigma\lambda^2} e^{\frac{\mu}{\sigma}} p^{\frac{1}{\sigma}} \sum_{i=1}^{n} x_i\left(1+\frac{x_i}{\lambda}\right)^{\alpha-1}\left[\left(1+\frac{x_i}{\lambda}\right)^{\alpha}-1\right]^{-\left(\frac{1}{\sigma}+1\right)} \\ &-\left(\frac{1}{\sigma}+1\right)\frac{\alpha}{\lambda^2}\sum_{i=1}^{n} x_i\left(1+\frac{x_i}{\lambda}\right)^{\alpha-1}.\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell(\Theta)}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} log\left(1+\frac{x_i}{\lambda}\right) - \left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n} log\left(1+\frac{x_i}{\lambda}\right)\left(1+\frac{x_i}{\lambda}\right)^{\alpha} \\ &+\frac{e^{\frac{\mu}{\sigma}} p^{\frac{1}{\sigma}}}{\sigma} log(\alpha)\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\lambda}\right)^{\alpha}\right]^{-\left(\frac{1}{\sigma}+1\right)} log\left(1+\frac{x_i}{\lambda}\right)\left(1+\frac{x_i}{\lambda}\right)^{\alpha}.\end{aligned}$$

The maximum likelihood estimators for the nknown parameters of GMO-Lomax are obtained by equating the score functions to zero respectively and solving simultaneously for the parameters. However, the score functions are non-linear to x and there are no closed form solutions for the estimators. The estimates for the parameters can be obtained using iterative numeric optimization methods.

9. Simulation

The maximum likelihood estimates of GMO-Lomax parameters were examined for asymptotic consistence using simulation study. Random samples of sizes 50, 75, 125 and 200 were generated using equation(5) with initial parameter values $\Omega = ($ p = 2.3, μ=2, σ=1.8, α=0.5, λ=1.2). For each sample size and $N = 1000$, the parameter estimates $\hat{\Omega}_i = (\hat{p}_i, \hat{\mu}_i, \hat{\sigma}_i, \hat{\alpha}_i, \hat{\lambda}_i)$ were evaluated for $i = 1, 2, \cdots, N$. The Mean value $\overline{\hat{\Omega}}$, Bias, Mean Square Error (MSE) were all computed. The values in Table 1 indicate that as the sample size increases, the MSE decreases and the Mean value converges to the initial parameter values as required under first asymptotic theorem.

Table 1: *Summary of the simulation study.*

Initial paramater value	Sample size (n)	Mean value	Bias	MSE
	50	2.2598	-0.0402	0.0022
		1.9324	-0.0676	0.0085
		1.8466	0.0466	0.0033
		0.5232	0.0232	0.0007
		3.5942	2.3943	5.7900
p=2.3	75	2.3081	0.0081	0.0006
$\mu = 2$		1.9758	-0.0242	0.0038
$\sigma = 1.8$		1.8537	0.0537	0.0037
$\alpha = 0.5$		0.5239	0.0239	0.0007
$\lambda = 1.2$		3.3466	2.1466	4.6753
	125	2.3016	0.0016	0.0003
		1.9344	-0.0655	0.0065
		1.7939	-0.0061	0.0006
		0.5028	0.0028	0.0001
		2.9596	1.7596	3.1669
	200	2.2918	-0.0082	0.0003
		1.8786	-0.1214	0.0161
		1.7641	-0.0359	0.0018
		0.4903	-0.0097	0.0001
		2.7963	1.5962	2.6032

10. Applications

In this section, we illustrate the applicability of the GMO-Lomax using two real-life datasets. Comparison with other existing distributions including McDonald Lomax (McLomax) Beta-Lomax, Lomax of Lomax, Marshall-Olkin Lomax(MOL), Logistic Lomax(logisticL), and exponentiated Lomax(Exp Lomax)) are done using goodness-of-fit statistics including Cramer-von Misses (W), Anderson Darling (A), Kolmogorov Smirnov (K-S) test, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC). Generally, the smaller the values of these statistics, the better the distribution fits the data set. The total test on time (TTT) to illustrate the empirical failure rate behavior of the two data sets was done.

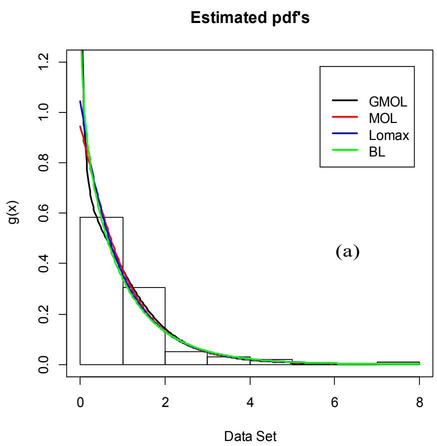
First data set used which was reported by [18] is on the Kevlar 49/epoxy strands failure when the pressure is at 90% stress level while the second data set reported by [19] is on the lifetimes of 50 industrial device put on life test at time zero. The estimated cramer-von Misses (W*), and Anderson Darling (A*) together with the computed K-S, AIC, BIC, and negative log-likelihood of the two datasets are shown in Tables 3 and 5. The parameter estimates of the competing distributions with the standard errors in parentheses for the first and second data set are respectively shown in Table 2 and 4. Tables 3 and 5 show that the goodness-of-fit statistics values associated with GMO-Lomax are the least among the competing distribution, implying that GMO-Lomax distribution provided adequate fit for the two data sets respectively. The plots of the estimated pdfs with the histograms of the datasets and cdfs with the empirical cdf of the two data sets are shown in Figures 3 and 4. Figure 3 showed a close fit of the dataset's histogram, however, the goodness-of-fit statistics values in Table 3 indicate the numerical difference of how well the various competing distriutions actually fit the dataset. Figure 4 clearly show that the GMO-Lomax provided a better fit on the histogram of the second dataset among other competing distributions. Furthermore, the empirical TTT of the failure rates for the two datasets are shown in Figure 5. The Figure 5 shows that the datasets constitute constant and monotone-increasing failure rate.

Table 2: *Results of parameter estimates for the first dataset(standard errors).*

Distribution					
GMO-Lomax(p,$\mu,\sigma,\alpha,\lambda$)	5.0148	0.5808	3.5797	33.6301	6.5039
	(371.7516)	(74.1372)	(0.6937)	(27.8848)	(6.5015)
McLomax(a,b,α, λ, c)	0.8243	6.0317	1.6613	4.1831	3.1728
	(0.1279)	(17.3009)	(4.5598)	(7.2706)	(2.8795)
Beta-Lomax(a,b,α, λ)	0.8897	4.2914	7.6109	36.09837	
	(0.1177)	(108.5245)	(189.4789)	(94.9773)	
Lomax(α,λ)	15.4125	14.7618			
	(20.9761)	(21.3217)			
MOL(p,α,λ)	1.3640	8.9718	6.9621		
	(0.8281)	(10.9643)	(11.1955)		
LogisticsL(β,α,λ)	1.2869	38.9985	24.4089		
	(0.1089)	(31.6599)	(20.3205)		
Exp-Lomax(θ,α,λ)	0.8846	31.0501	33.3998		
	(0.1201)	(71.2834)	(80.0430)		

Table 3: *Results of the goodness-of-fit-statistics for the first dataset.*

Distribution	W^*	A^*	K-S	AIC	BIC	$-\ell$
GMO-Lomax	0.0985	0.5926	0.0653	208.9945	209.6261	99.4973
McLomax	0.1440	0.8452	0.0967	213.9501	227.0257	101.9751
Beta-Lomax	0.1934	1.0843	0.0925	213.6633	224.1238	102.8817
Lomax	0.2107	1.1665	0.0864	210.4693	215.6995	103.2346
MOL	1.515	8.2009	0.6336	212.2111	220.0565	103.1056
LogisticsL	0.5828	3.1709	0.1065	233.0110	240.8564	113.5055
Exp-Lomax	0.1914	1.0749	0.0926	211.6259	219.4713	102.8129



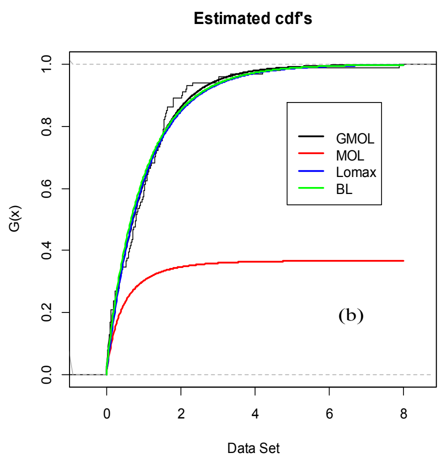


Figure 3: *Estimated plots for the first dataset: a) competing pdfs b) empirical cdf with competing cdf.*

Table 4: *Results of parameter estimates for the second dataset (standard errors).*

Distribution					
GMO-Lomax(p,$\mu,\sigma,\alpha,\lambda$)	7.3403	1.8975	5.5264	60.0204	341,1701
	(83.8036)	(11.5748)	(1.5003)	(38.3307)	(211.1367)
McLomax(a,b,α, λ, c)	0.8345	63.7855	1.1889	105.2354	8.1853
	(0.1398)	(54.4193)	(0.6347)	(51.0148)	(4.0484)
Beta-Lomax(a,b,α,λ)	0.5273	0.0915	37.2292	162.6509	
	(0.1464)	(0.0277)	(11.0337)	(26.1937)	
Lomax(α,λ)	5.1659	205.1413			
	(2.5299)	(110.9948)			
MOL(p,α,λ)	3.9229	4.3019	83.7042		
	(2.3716)	(1.8419)	(52.2756)		
LogisticsL(β,α,λ)	8.7631	0.1069	0.0022		
	(1.1127)	(0.0038)	(0.0005)		
Exp-Lomax(θ,α,λ)	0.8464	3.9194	176.1126		
	(0.1547)	(1.6727)	(88.7161)		

Table 5: *Results of the goodness-of-fit-statistics for the second dataset.*

Distribution	W^*	A^*	K-S	AIC	BIC	$-\ell$
GMO-Lomax	0.3725	2.3066	0.1641	479.9236	489.4837	234.9618
McLomax	0.3898	2.4432	0.2277	481.2248	490.7849	235.6124
Beta-Lomax	0.4871	2.9544	0.2124	492.1212	499.7693	242.0606
Lomax	0.8010	4.5753	0.8014	490.7842	494.6083	243.3921
MOL	1.5131	7.7835	0.8757	491.1396	496.8757	242.5698
LogisticsL	0.8579	4.7819	0.2566	521.2151	526.9511	257.6075
Exp-Lomax	0.5455	3.2668	0.1999	492.8816	498.0960	243.1799

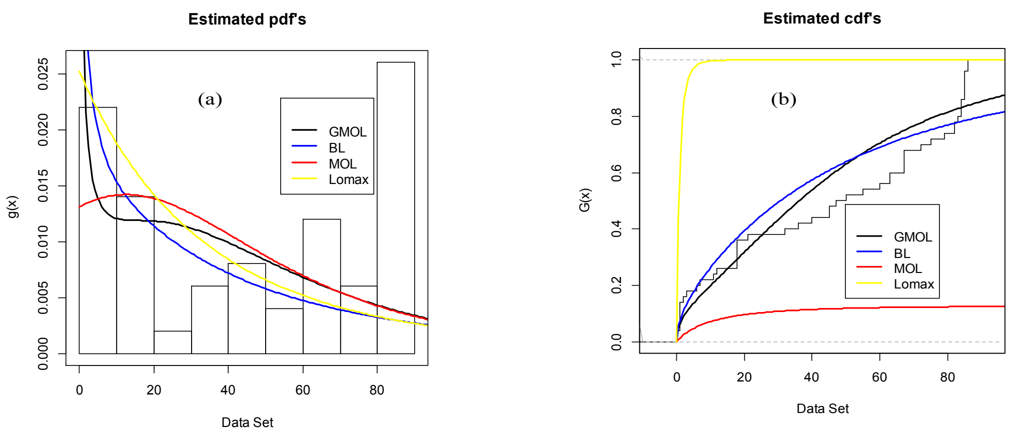


Figure 4: *Estimated plots for the second dataset: a) competing pdfs b) empirical cdf with competing cdf.*

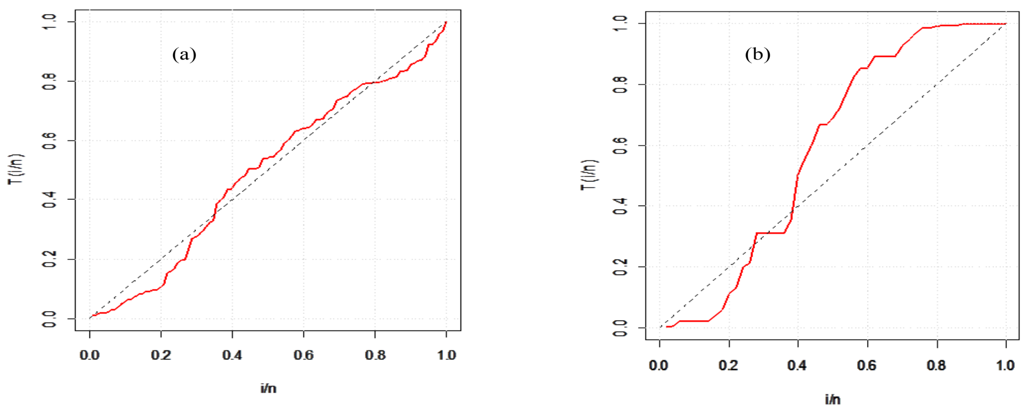


Figure 5: *Plots of Total time on test: a) First dataset b) second dataset.*

11. Conclusion

We have introduced a new five parameter distribution for modeling reliability problems. The statistical properties and some reliability measures of the new distribution are derived. The unknown parameters of the distribution are estimated using the maximum likelihood approach. Furthermore, the maximum likelihood estimates of the new distribution were examined for asymptotic consistence and were found to conform to the first order asymptotic theorem. Two real-life data sets were used to illustrate the applicability of the new distribution and comparison with other existing distributions indicates that the new distribution provided better fit for the two data sets. The constant and monotone-increasing failure shapes shown in the TTT plots are indications of the suitability of GMO-Lomax distribution which has constant and monotone-increasing failure rate shapes amongst other possible shapes in modelling the two datasets.

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