A Novel Transformation: Based on Inverse Trigonometric Lindley Distribution

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Abstract

As we see that the present era is directly depending upon various kinds of machines. In other words, we can say that we are fully surrounded by machines. Machines are assembled with many components and each component has its own importance. For proper functioning of a machine, these components should be up to date. Therefore, for smooth functioning, we have to make replacement of the component before its failure. In this present paper, we propose a new transformation which is purely based on inverse trigonometry with lindley distribution for the first time and so, named "Inverse Trigonometric Lindley Distribution". It find its various properties like survival function, hazard rate function, moments, conditional moments, order statistics, entropy measurement etc. Maximum likelihood estimator have also considered for estimation of parameter. To know the paternal behavior of the model, different real datasets have been considered. To understand the behavior of estimators at the long run, simulation study is being performed in detail.

Keywords: Lindley distribution, Renyi entropy, Moments, Maximum likelihood estimator, Simulation Study

1. Introduction

Survival and reliability analysis play an important role in the field of statistics. These are devising several noteworthy real life applications in many areas of applied and medical sciences, such as engineering, public health, actuarial science, biomedical studies, demography, industrial reliability, etc. In this era, each activity is dealt by the machines which are gathered by the help of different components. It is required to keep each and every component up to date, so that machines can run or function efficiently. In addition, before failure of that component, there is the need of replacement. But, the key matter is to estimate the appropriate time of replacement and decide the best policy to adopt with regard to replacement. Therefore, the main objective is to study about the behavior of the life of components individually as well as through suitable lifetime models. In relevant literatures, there is a list of different lifetime distributions along with their theoretical discussion to identify correct guess at which we replace the components so that system could work without failure. There are vital role of Survival and Hazard rate function in lifetime data analysis. Using these functions, we can identify the nature of a chosen model.
If we look back in literature, many authors used several transformations. For example, Power transformation was used by Gupta et al. (1998), Quadratic Rank Transmutation Map (QRTM) developed by Shaw and Buckley (2005), trigonometry based transformations like SS-transformation proposed by Kumar et al. (2015) and Chesneau et al. (2018), Mahmood, Z., and Chesneau, C. (2019), logarthim based transformation proposed by Maurya et al.(2016), DUS-transformation proposed by Kumar et al. (2015) and its generalization have done by Maurya et al. (2017),

generalization through QRTM have done by Yadav et al. (2019) etc. Their numerous statistical results have been obtained and deliberated its various shapes of hazard rate patterns in means of increasing, decreasing, upper side down etc. If we see literature, we get there are only few transformation that are based on inverse trigonometry.

In this paper, we propose a novel transformation based on "Inverse Trigonometric Lindley Distribution", which is related to inverse trigonometric function and lindley distribution. We also discuss its various characteristic and properties as well as identify its various hazard rate patterns. For simplicity, applicability and suitability in real life scenario different datasets have been considered. Simulation study is also being carried out to know the behavior of the estimators at the long-run.

2. A New Transformation Using Trigonometric Function

We introduce a new transformation using trigonometric function and It is denoted by G(x):

$$G(x) = K * \tan^{-1} F(x) \tag{1}$$

where, K is $\frac{1}{\tan^{-1}(1)}$.

Theorem 1. The function G(x) possesses the properties of a cdf.

Proof. Let f(x) be a pdf associated to the cdf F(x) is continuous with F(x)∈[0,1],

$$\lim_{x \to +\infty} F(x) = 1 \qquad \lim_{x \to -\infty} F(x) = 0$$

and f(x)=F′(x) almost everywhere with f(x) ≥ 0. Let us now investigate sufficent conditions for G(x) to be a cdf.

- $G(-\infty) = 0$ and $G(+\infty) = 1$.

$$G(x) = K * \tan^{-1} F(x)$$
$$G(-\infty) = K * \tan^{-1} F(-\infty)$$
$$G(-\infty) = 0$$

and

$$G(+\infty) = K * \tan^{-1} F(x)$$
$$G(+\infty) = K * \tan^{-1} F(+\infty)$$
$$G(+\infty) = 1$$

- G is non decreasing function.

Let us prove that $G(x_2) - G(x_1) \geq 0$.

$$G(x_1) = K * \tan^{-1} F(x_1)$$
$$G(x_2) = K * \tan^{-1} F(x_2)$$
$$G(x_2) - G(x_1) = K * [\tan^{-1} F(x_2) - \tan^{-1} F(x_1)]$$
$$G(x_2) - G(x_1) = K * [\tan^{-1} F(x_2) - \tan^{-1} F(x_1)] > 0$$

where, $F(x_2) > F(x_1)$.

The above expression $G(x_2) - G(x_1)$ is positive if $x_2 > x_1$.

- G is Right Continuous.

$\tan^{-1} F(x)$ are right continuous function of x, then G(x) is right continuous function of x.

Thus, we can say that our transformation is a cdf.

The pdf and hazard rate of our transformation is given by

$$g(x) = K * \frac{f(x)}{1 + (F(x))^2} \tag{2}$$

$$h(x) = K * \frac{f(x)}{\{1 - (F(x))^2\}\{1 - K * \tan^{-1} F(x)\}} \tag{3}$$

3. A New Transformation with some related new distributions

• Consider the Uniform Distribution [0,θ], we have $F(x) = \frac{x}{\theta};\ \ 0 \leq x \leq \theta$ then,

$$G(x) = K * \tan^{-1}\left(\frac{x}{\theta}\right)$$

$$g(x) = K * \frac{1}{\theta\left\{1 + (\frac{x}{\theta})^2\right\}}$$

$$h(x) = K * \frac{1}{\theta\left\{1 + (\frac{x}{\theta})^2\right\}\left\{1 - K * \tan^{-1}\left(\frac{x}{\theta}\right)\right\}}$$

• Consider the logistic distribution with parameters $\mu \in R$ and $s > 0$, we have $F(x) = \frac{1}{1+e^{-\frac{x-\mu}{s}}},\ x \in R$ then,

$$G(x) = K * \tan^{-1}\left(\frac{1}{1 + e^{-\frac{x-\mu}{s}}}\right)$$

$$g(x) = K * \frac{e^{-\frac{x-\mu}{s}}}{\left\{s\left(1 + e^{-\frac{x-\mu}{s}}\right)^2\right\}\left\{1 + \left(\frac{1}{1+e^{-\frac{x-\mu}{s}}}\right)^2\right\}}$$

$$h(x) = K * \frac{e^{-\frac{x-\mu}{s}}}{\left\{s\left(1 + e^{-\frac{x-\mu}{s}}\right)^2\right\}\left\{1 + \left(\frac{1}{1+e^{-\frac{x-\mu}{s}}}\right)^2\right\}\left\{1 - K * \tan^{-1}\left(\frac{1}{1+e^{-\frac{x-\mu}{s}}}\right)\right\}}$$

• Consider the Cauchy distribution with parameters $x_0 \in R$ and $a > 0$, we have $F(x) = \frac{1}{\pi}\arctan\left(\frac{x-x_0}{a}\right) + \frac{1}{2},\ x \in R$

$$G(x) = K * \tan^{-1}\left(\frac{1}{\pi}\arctan\left(\frac{x-x_0}{a}\right) + \frac{1}{2}\right)$$

$$g(x) = K * \frac{a}{\pi\left\{a^2 + (x-x_0)^2\right\}\left\{1 + \left(\frac{1}{\pi}\arctan\left(\frac{x-x_0}{a}\right) + \frac{1}{2}\right)^2\right\}}$$

$$h(x) = K * \frac{a}{\pi\left\{a^2 + (x-x_0)^2\right\}\left\{1 + \left(\frac{1}{\pi}\arctan\left(\frac{x-x_0}{a}\right) + \frac{1}{2}\right)^2\right\}\left\{1 - K * \tan^{-1}\left(\frac{1}{\pi}\arctan\left(\frac{x-x_0}{a}\right) + \frac{1}{2}\right)\right\}}$$

• Consider the Normal distribution with parameters $\mu \in R$ and $\sigma > 0$, we have

F(x)=$\int_{-\infty}^{x} - \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt = \Phi(x),\ x \in R,$

$$G(x) = K * \tan^{-1}(\Phi(x))$$

$$g(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}\left\{1 + (\Phi(x))^2\right\}}$$

$$h(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}\left\{1 + (\Phi(x))^2\right\}\left\{1 - K * \tan^{-1}(\Phi(x))\right\}}$$

• Consider the Exponential Distribution with parameter $\theta > 0$, we have $F(x) = 1 - e^{-\theta x};\ \theta > 0, x > 0$ then,

$$G(x) = K * \tan^{-1}(1 - e^{-\theta x})$$

$$g(x) = K * \frac{\theta e^{-\theta x}}{1 + (1 - e^{-\theta x})^2}$$

$$h(x) = K * \frac{\theta e^{-\theta x}}{\left\{1 + (1 - e^{-\theta x})^2\right\}\left\{1 - K * \tan^{-1}(1 - e^{-\theta x})\right\}}$$

• Consider the Lindley Distribution with parameter $\theta > 0$, we have $F(x) = 1 - \left\{\left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right\};\ \theta > 0, x > 0$ then,

$$G(x) = K * \tan^{-1}\left\{1 - \left(\left(1 + \frac{\theta x}{\theta + 1}\right)e^{-\theta x}\right)\right\} \quad (4)$$

$$g(x) = k * \left(\frac{\theta^2}{\theta + 1}\right)\frac{(1 + x)e^{-\theta x}}{1 + \left\{1 - \left(\left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right\}^2} \quad (5)$$

$$h(x) = \left(\frac{\theta^2}{\theta + 1}\right)\frac{(1 + x)e^{-\theta x}}{1 + \left\{1 - \left(\left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right\}^2\left\{1 - K * \tan^{-1}\left\{1 - \left(\left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right\}\right\}} \quad (6)$$

In order to illustrate the potential of applicability of $IT_{lin}(\theta)$, The shapes of the cdf and pdf are shown in Figures 1 and 2 respectively for different value of θ.

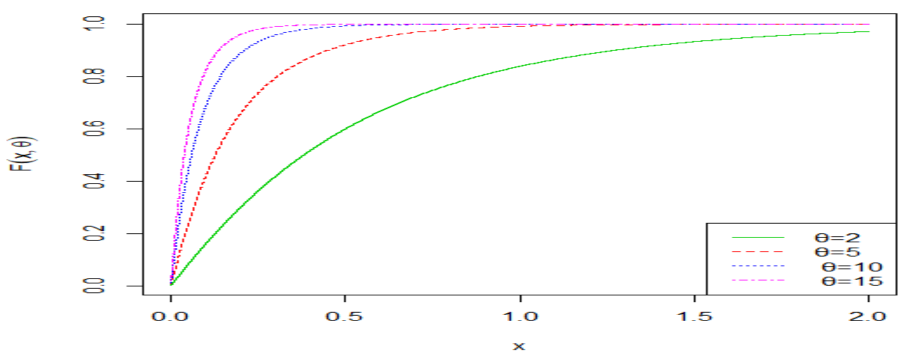

Figure 1: *Plots of Cumulative distribution function for different values of θ*

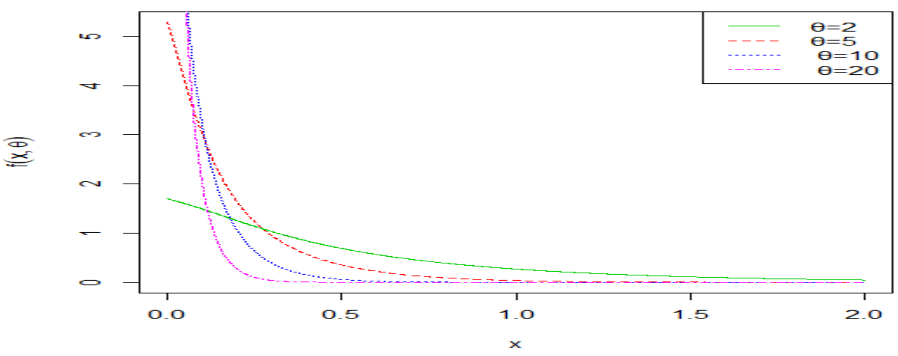

Figure 2: *Plots of Probability density function for different values of θ*

4. Survival Analysis

In this section we present the survival function, the hazard function and the cumulative hazard rate function for the $IT_{lin}(\theta)$-distribution.

Survival Function

It is define as the probability that an individual survives larger than, t: S(t)= P(an individual survives larger than t)
=P(T>t), where T denote the survival time.
The cdf F(t) of T, is given as S(t)=1-P(an individulas fails before t) = 1-F(t).
Hence, the Survival Function of $IT_{lin}(\theta)$-distribution is obtained as follows:-

$$S(t) = 1 - \left[K * \tan^{-1} \left\{ 1 - \left(\left(1 + \frac{\theta t}{\theta + 1} \right) e^{-\theta t} \right) \right\} \right] \tag{7}$$

The behavior of the survival rate function, for different values of θ is shown are Fig. 3.

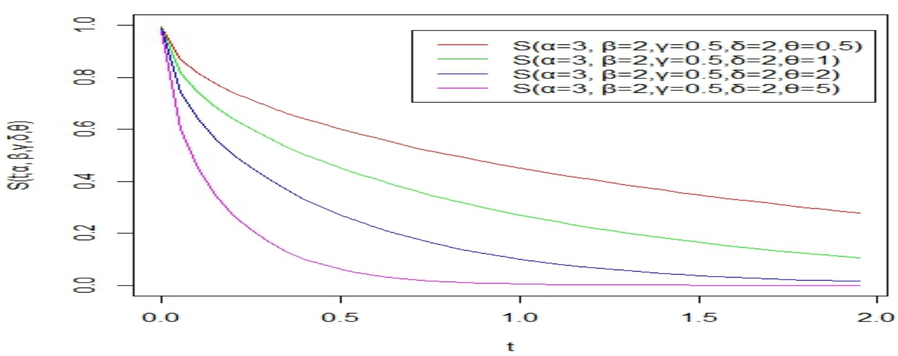

Figure 3: *Plots of Survival Rate Function for different values of θ*

Hazard Rate Function

It is defined as the probability of failure during a very small time interval, assuming that the individual has survived to the begining of the interval, or as the limit of the probability that an individual fails in a very short interval, $t+\Delta t$, given that the individual has survived to time t:

$$h(t) = \lim_{\Delta x \to 0} \frac{\Pr(an\ individual\ fails\ an\ interval\ (t+\Delta t)\ given\ the\ individual\ has\ survived\ to\ t)}{\Delta t}$$

It is defined as the terms of CDF F(t) and PDF f(t) is given as,

$$h(t) = \frac{f(t)}{1-F(t)}$$

The hazard rate function, h(t) of the $IT_{lin}(\theta)$-distribution are given by

$$h(t) = \left(\frac{\theta^2}{\theta+1}\right) \frac{(1+t)e^{-\theta t}}{1+\left\{1-\left(\left(1+\frac{\theta t}{\theta+1}\right)e^{-\theta t}\right)\right\}^2 \left\{1-K*\tan^{-1}\left\{1-\left(\left(1+\frac{\theta t}{\theta+1}\right)e^{-\theta t}\right)\right\}\right\}} \tag{8}$$

Figure 4 illustrates these shapes for selected paramter values.

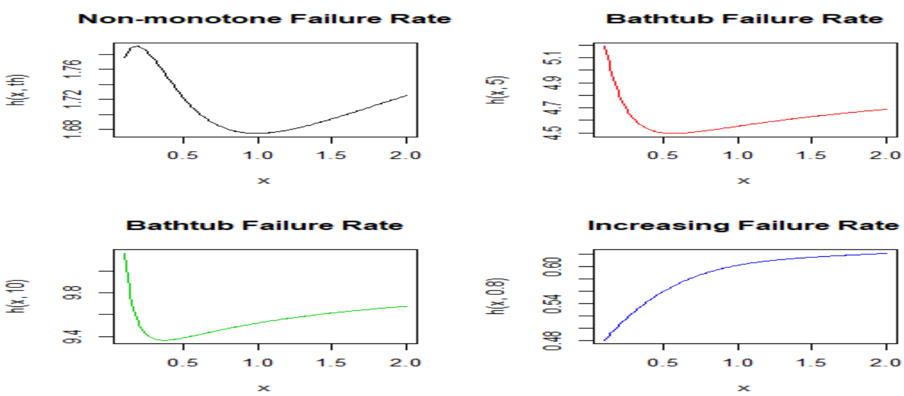

Figure 4: *Plots of Hazard Rate Function for different values of* θ

5. Moments Properties

The expression of the moment, are defined as

$$E(X^r) = \int_0^\infty x^r g(x)dx$$

$$= k*\left(\frac{\theta^2}{\theta+1}\right)\int_0^\infty x^r \frac{(1+x)e^{-\theta x}}{1+\left\{1-\left(\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right\}^2}dx$$

$$= k*\left(\frac{\theta^2}{\theta+1}\right)\int_0^\infty x^r(1+x)e^{-\theta x}\left[1+\left\{1-\left(\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right\}^2\right]^{-1} dx$$

Using expansion $\left[1+\{F(x)\}^2\right]^{-1} = \sum_{k=0}^{\infty}(-1)^k \{F(x)\}^{2k}$ in equation (21) we get,

$$= \sum_{k=0}^{\infty}(-1)^k K*\left(\frac{\theta^2}{\theta+1}\right)\int_0^\infty x^r(1+x)e^{-\theta x}\left\{1-\left(\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right\}^{2k} dx$$

Using expansion of $\{1-F(x)\}^{2k} = \sum_{l=0}^{2k}\binom{2k}{l}(-1)^l\{F(x)\}^l$ in equation (22) we get,

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{2k}(-1)^{k+l}\binom{2k}{l}K*\left(\frac{\theta^2}{\theta+1}\right)\frac{1}{(\theta+1)^l}\int_0^\infty x^r(1+x)e^{-\theta x(1+l)}\{1+\theta+\theta x\}^l dx$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{2k}\sum_{m=0}^{l}(-1)^{k+l}\binom{2k}{l}\binom{l}{m}K*\left(\frac{\theta^2}{\theta+1}\right)\frac{\theta^m}{(\theta+1)^l}\int_0^\infty x^r e^{-\theta x(1+l)}\{1+x\}^{m+1}dx$$

Using expansion of $(1+x)^{m+1} = \sum_{n=0}^{m+1}\binom{m+1}{n}x^n$ and gamma function we get expression of r^{th} moments as,

$$E(X^r) = K*\frac{\theta^2}{\theta+1}\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\sum_{m=0}^{l}\sum_{n=0}^{m+1}(-1)^{k+l}\binom{2k}{l}\binom{l}{m}\binom{m+1}{n}\frac{\theta^m}{(1+\theta)^l}\frac{(n+r)!}{(\theta+\theta l)^{n+r+1}} \tag{9}$$

where , K is $\frac{1}{\tan^{-1}(1)}$.

In particular, the first four moments of X are as follows:

$$E(X) = K*\frac{\theta^2}{\theta+1}\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\sum_{m=0}^{l}\sum_{n=0}^{m+1}(-1)^{k+l}\binom{2k}{l}\binom{l}{m}\binom{m+1}{n}\frac{\theta^m}{(1+\theta)^l}\frac{(n+1)!}{(\theta+\theta l)^{n+2}} \tag{10}$$

$$E(X^2) = K*\frac{\theta^2}{\theta+1}\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\sum_{m=0}^{l}\sum_{n=0}^{m+1}(-1)^{k+l}\binom{2k}{l}\binom{l}{m}\binom{m+1}{n}\frac{\theta^m}{(1+\theta)^l}\frac{(n+2)!}{(\theta+\theta l)^{n+3}}$$

$$E(X^3) = K*\frac{\theta^2}{\theta+1}\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\sum_{m=0}^{l}\sum_{n=0}^{m+1}(-1)^{k+l}\binom{2k}{l}\binom{l}{m}\binom{m+1}{n}\frac{\theta^m}{(1+\theta)^l}\frac{(n+3)!}{(\theta+\theta l)^{n+4}}$$

$$E(X^4) = K*\frac{\theta^2}{\theta+1}\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\sum_{m=0}^{l}\sum_{n=0}^{m+1}(-1)^{k+l}\binom{2k}{l}\binom{l}{m}\binom{m+1}{n}\frac{\theta^m}{(1+\theta)^l}\frac{(n+4)!}{(\theta+\theta l)^{n+5}}$$

The variance, skewness and kurtosis of T can be obtained using the following relationships: $Var(X) = E(X^2) - (E(X))^2$, $Skewness(X) = \frac{E(X-E(X))^3}{(Var(X))^{3/2}}$ and $Kurtosis(X) = \frac{E(X-E(X))^4}{(Var(X))^2}$.

6. Order Statistics

Let f(x) and F(x) be the pdf and cdf respectively, then for r=1,2,...n be the pdf $f_r(x)$ of r^{th} order statistics $X_{r:n}$ is

$$\begin{aligned} f_r(x) &= \frac{n!}{(r-1)!(n-r)!}F^{r-1}(x)[1-F(x)]^{n-r}f(x) \\ &= \frac{n!}{(r-1)!(n-r)!}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}F^{r+i+1}(x)f(x) \end{aligned} \tag{11}$$

Now using pdf and cdf in equation we have,

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!}K*\frac{\theta^2}{\theta+1}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\frac{(1+x)e^{-\theta x}}{\left[1+\left\{1-\left(\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right\}^2\right]} \left[K*\tan^{-1}\left\{1-\left(\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right\}\right]^{r+i+1} \tag{12}$$

And corresponding to r^{th} order statistics of cdf $F_r(x)$ is ,

$$\begin{aligned} F_r(x) &= \sum_{i=r}^{n}\binom{n}{i}F^i(x)[1-F(x)]^{n-i} \\ &= \sum_{i=r}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}(-1)^jF^{i+j}(x) \end{aligned} \tag{13}$$

Using equation we get,

$$\sum_{i=r}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}(-1)^j\left[K*\tan^{-1}\left\{1-\left(\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right\}\right]^{i+j} \tag{14}$$

7. Renyi Entropy

An entropy of a random variable X is a measure of variation of the uncertainty. Renyi entropy is defined by

$$R_\gamma=\frac{1}{1-\gamma}\ln\left(\int f^\gamma(x)dx\right) \tag{15}$$

Where $\gamma>0$ and $\gamma\neq 1$. Substitute (5) in above expression, we have

$$R_\gamma=\frac{1}{1-\gamma}log\left(k*\left(\frac{\theta^2}{\theta+1}\right)\frac{(1+x)e^{-\theta x}}{1+\left\{1-\left(\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right)\right\}^2}\right)^\gamma dx \tag{16}$$

Solve the equation(16) anatically and we get final result are:

$$R_\gamma=\frac{1}{1-\gamma}+ln\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\sum_{p=0}^{l}\sum_{m=0}^{\infty}(-1)^{k+l}\binom{2k}{l}\binom{\gamma}{m}\binom{l}{p}\frac{(\gamma+k-1)^{[k]}}{k!}\frac{\Gamma m+p+1}{[\theta(\gamma+l)]^{m+p+1}} \tag{17}$$

8. Estimation

In this section, we briefly discuss the maximum likelihood estimators (MLE's) of the $IT_{lin}(\theta)$ distribution.
Let $\underline{x}=(x_1,...,x_n)$ be a random sample of size n from $IT_{lin}(\theta)$, then the log likelihood function $l(\theta|\underline{x})$ can be written as

$$\begin{aligned}l(\theta|\underline{x})=&-n*ln(\tan^{-1}(1))+n*ln\left(\frac{\theta^2}{1+\theta}\right)-\theta\sum x+\sum(ln(1+x))\\&+\sum\left[ln\left\{1+\left(1-\left(1+\frac{\theta x}{1+\theta}\right)\right)e^{-\theta x}\right\}^2\right]\end{aligned} \tag{18}$$

Therefore, to obtain MLE's of estimated θ, we can maximise the equation directly w.r.t θ or we can solve the following Non- Linear equation by using Newton-Raphson method. Since this equation is not closed form and can not be solved analytically. So, we have to use some numerical technique such as Newton-Raphson method for the solution.

9. Simulation Algorithm and Study

9.1. Inverse cdf method

One of the most simpelst and common method to generating random variates is based on the inverse cdf. For arbitrary cdf, define $F^{-1}(u)=min\{x;F(x)\geq u\}$, see Sharma et al.(2016). In case of $IT_{lin}(\theta)$ distribution, inverse cdf cannot be obatined easily, so we proposed the use of Newton's method for the solution of the Cdf of $IT_{lin}(\theta)$ distribution. The algorithm used for this purpose is as follows:
Step 1. Set n, θ and initial value x^0.
Step 2. Generate U Uniform(0,1).
Step 3. Update x^0 by the using Newton's formula.
$x^*=x^0-H(x^0,\theta)$

where $H(x^0,\theta) = \frac{F(x^0,\theta)-U}{f(x^0,\theta)}$; f(.) and F(.) are given equation respectively.
Step 4. If $|x^0 - x^*| \leq \epsilon$, then store $x = x^*$ as a sample from $IT_{lin}(\theta)$.
Step 5. If $|x^0 - x^*| > \epsilon$, then, set $x^0 = x^*$ and goto step 3.
Step 6. Repeat steps 2-5, n times for $x_1, x_2, ..., x_n$, respectively.

9.2. Simulation Study

In this section, we present the results of the long run guarantee of the proposed lifetime distribution. To verify the behaviour of the proposed lifetime model in terms of mean square error (MSE), bias confidence interval and width of the confidence interval of the maximum likelihood estimator (MLE) of θ. Here we generate 15000 different random sample of size n (n= 10,15,20,25,30,40,90,160,250 and 400) for the consider true value of parameter θ (θ= 0.5,1 and 2).

Table 1: *ML estimates, MSE, Bias, Confidence Interval and width of CI for the true value of parameter is 0.5.*

n	mle	mse	Bais	LCL UCL	width
10	0.5600	0.0239	0.0600	(0.4756, 0.6444)	0.1688
15	0.5058	0.0116	0.0058	(0.4553, 0.5563)	0.1009
20	0.5052	0.0071	0.0052	(0.4674, 0.5430)	0.0755
25	0.5088	0.0056	0.0088	(0.4783, 0.5392)	0.0608
30	0.5093	0.0049	0.0093	(0.4839, 0.5247)	0.0507
40	0.5088	0.0037	0.0088	(0.4897, 0.5077)	0.0380
90	0.5011	0.0032	0.0011	(0.4928, 0.5094)	0.0166
160	0.5020	0.0009	0.0020	(0.4973, 0.5067)	0.0094
250	0.5017	0.0006	0.0017	(0.4987, 0.5047)	0.0060
400	0.5013	0.0004	0.0013	(0.4993, 0.5031)	0.0037

Table 2: *ML estimates, MSE, Bias, Confidence Interval and width of CI for the true value of parameter is 1.*

n	mle	mse	bais	LCL UCL	width
10	1.0758	0.0897	0.0758	(0.9059, 1.2456)	0.3397
15	1.0516	0.0530	0.0516	(0.9414, 1.1617)	0.2203
20	1.0377	0.0367	0.0377	(0.9563, 1.1190)	0.1627
25	1.0312	0.0287	0.0312	(0.9666, 1.0958)	0.1292
30	1.0302	0.0237	0.0302	(0.9765, 1.0840)	0.1075
40	1.0233	0.0172	0.0233	(0.9833, 1.0633)	0.0800
90	1.0106	0.0069	0.0106	(0.9931, 1.0282)	0.0350
160	1.0063	0.0038	0.0063	(0.9965, 1.0161)	0.0196
250	1.0041	0.0025	0.0041	(0.9978, 1.0103)	0.0125
400	1.0022	0.0015	0.0022	(0.9983, 1.0061)	0.0078

Table 3: *ML estimates, MSE, Bias, Confidence Interval and width of CI for the true value of parameter is 2.*

n	mle	mse	Bais	Confidence Inteval	width
10	2.1632	0.3061	0.1632	(1.7950, 2.5312)	0.7362
15	2.1555	0.1921	0.1555	(1.9110, 2.3999)	0.4889
20	2.1491	0.1472	0.1491	(1.9663, 2.3318)	0.3655
25	2.1295	0.1102	0.1295	(1.9847, 2.2743)	0.2895
30	2.1025	0.0810	0.1025	(1.9835, 2.2214)	0.2378
40	2.1215	0.0715	0.1215	(2.0314, 2.2116)	0.1802
90	2.0862	0.0274	0.0862	(2.0469, 2.1256)	0.0786
160	2.1008	0.0235	0.1008	(2.0785, 2.1231)	0.0446
250	2.0976	0.0180	0.0976	(2.0833, 2.1118)	0.0285
400	2.0959	0.0144	0.0959	(2.0870, 2.1048)	0.0178

The results are consider in all Tables (Table 1,2 and 3), which show the average of the 15000 MLE's together with their MSE, Bias, 95% of confidence interval and width of the confidence interval of true value of the parameter of the proposed lifetime distribution. For all considered true value of the parameter θ;the MSE and bias are decreases as sample size increased. These results suggest that the MLE have performed consistently 95% confidence interval and width of the paramter θ also decreases as sample size increases.

10. Real Data Modeling

In the present section, we have considered two data sets, which are initially proposed by Efron, B (1998). The data represents the patients of two groups suffering from head and neck cancer disease. The data set of first group represents the survival times of 51 head and neck cancer patients treated with radiotherapy whereas the other group of data set represents the survival times of 45 head and neck cancer patients treated with combined radiotherapy and chemotherapy. The data sets are as follows:

$\boldsymbol{Data(A)}$: 6.53, 7, 10.42, 14.48, 16.10, 22.70, 34, 41.55, 42, 45.28, 49.40, 53.62, 63, 64, 83, 84, 91, 108, 112, 129, 133,133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 225, 241, 248, 273,277, 297, 405, 417, 420, 440, 523, 583, 594, 1101, 1146, 1417.

$\boldsymbol{Data(B)}$: 12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81,43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776.

To check the validity of the considered model for the above data sets A and B using Kologorov-Smirnov Statistics (K-S), Akaike information criterion (AIC), and Bayes information criterion (BIC) and log-likelihood criterion. We compare the applicability of this model for the above data sets over Inverse Rayleigh distribution (IRD), Generalized inverted exponential distribution (GIED), Inverse exponential distribution (IED) and Lindley distribution (LD) have discussed and it is found that the considered model is quite flexible than the other four, see Table.

Table 4: *The values of different statistical measures for the Data Set (A)*

Distributions	AIC	BIC	-LL	K-S test value
GIED	773.18	777.30	384.59	0.2453
IRD	840.13	842.06	419.06	0.6039
ITLD	721.25	723.31	359.62	0.2433
IED	773.37	775.43	385.68	0.2875
LD	765.74	767.80	381.87	0.2453

Table 5: *The values of different statistical measures for the Data Set (B)*

Distributions	AIC	BIC	-LL	K-S test value
GIED	572.43	576.04	284.21	0.1901
IRD	962.71	964.49	480.35	0.3783
ITLD	559.45	561.25	278.72	0.1223
IED	571.06	572.86	284.53	0.1823
LD	593.23	595.04	295.61	0.2942

10.1. TTT Plot

The total-time-on-test (TTT) plot is a graphical procedure to get some idea about the shape of the hazard function. We have used the empirical version of the scaled TTT plot, [Aarset (1987)]. We have plotted the empirical version of the scaled TTT transform of the both data set (data set A and B) in figure 5 and 6. Since the empirical version of the scaled TTT transform is concave and convex both, it indicates the hazard function is increasing and decreasing both.

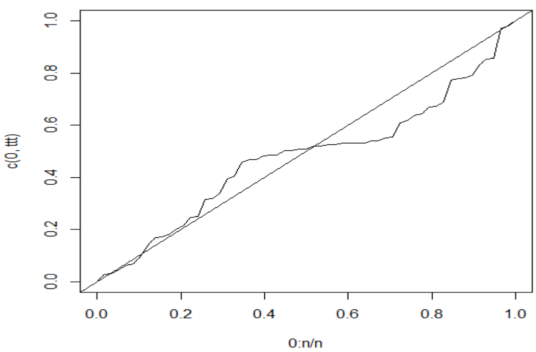

Figure 5: *The empirical scaled TTT transform of the data set A*

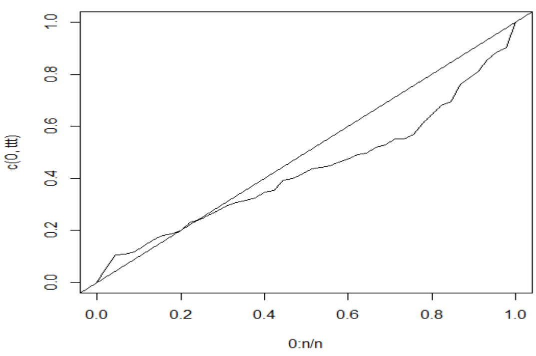

Figure 6: *The empirical scaled TTT transform of the data set B*

11. Conclusions

In this work, we have proposed a new transformation using trigonometric function along with Lindley distribution. So, the name of distribution is "Inverse Trigonometric Lindley Distribution" as it uses lindley distribution as the baseline distribution. Some of its important various statistical/ mathematical properties including shape, survival function, hazard rate, moments and associated measures, order statistics are discussed and renyi entropy of the proposed distribution have been derived. The method of maximum likelihood estimation has also been discussed for estimating the parameter. For depth understanding, we have deliberated real life scenarios as two real data sets of survival of head and neck cancer patients are considered to illustrate the applicability of the discussed model. It is found that our invented model provides better fit to the given data set, which has been well verified by graphical illustrations.

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