EXPLICIT TIME DEPENDENT SOLUTION OF A TWO-STATE RETRIAL QUEUEING MODEL WITH HETEROGENOUS SERVERS

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Abstract

In this paper, two dimensional state retrial queueing system with two non - identical parallel servers is considered. Incoming calls (primary calls) arrive at the server according to a Poisson process. Repeating calls also follows the same fashion. Service times of two servers follow exponential distribution with different rates. An incoming call that finds the servers busy, joins an orbit and retries after some random amount of time. Time dependent probabilities of exact number of arrivals and exact number of departures at when the servers are free or when one server is busy or when both servers are busy are derived for the system. Finally busy period distribution obtained to illustrate the system dynamics.

Keywords: Retrial, Queueing, Arrivals, Departures, Heterogeneous Servers

I. Introduction

Recently retrial queues are paid much attention because they have applications in performance analysis of various systems such as call centers, computer networks and telecommunication systems. Retrial queues are characterized by the fact that arriving customers when could not able to receive service may enter a virtual queue (orbit) and retry for service again after some random amount of time. The analyse of retrial queueing models are much more difficult than without retrials and explicit results are obtained only in a few special cases.

Retrial queueing models are often used for the performance and reliability modeling of computer systems and communication networks. The reason is that the return of customers plays a special role in many of these systems or in other practical applications. Some applications of retrial queues can be found in Li and Yang [1], Janssens [2], Tran-Gia and Mandjes [3], Onur et al. [4] and the detailed overviews of retrial queues are given in Falin and Templeton [5], Artalejo [6], Falin and Artalejo [7], Artalejo [8], Falin [9].

Queues with non-identical parallel servers (heterogeneous servers) can be widely used for modeling real systems with heterogeneous environment. Heterogeneous servers are allocated in banks, hospitals, telecommunication and business centers. Customers arrive according to a Poisson process at a rate λ. The servers have a tendency to serve the same type of job but with different service rates μ_1& μ_2.

The classical transient results for the M/M/1 queueing model provide slight perception about the behavior of a queueing system through a fixed time t, but Pegden and Rosenshine [10] have given the probability of exact number of arrivals in the system and exact number of departures from the system by a given time for the classical queueing model M/M/1/∞. This measures supplies better insight into the behavior of a queueing system than the probability of the exact number of units in the system at a given time.

In this paper, we obtained the time dependent probabilities of the exact number of arrivals in the system and exact number of departures from the system for a retrial queueing system with two servers having unequal service rates. Many authors have studied systems with two non-identical parallel servers. Satty [11] studied a continuous time first come first served queueing system with two parallel servers each with different service rate. He obtained the steady state probabilities for the number of units in the system/ queue. Gumbel [12] studied the steady state probabilities for the number of units in the system by considering a more general queueing problem having a finite number of servers, each with different service rate. Morse [13] also considered two servers with different service rates and obtained the steady state solutions.

The paper is organized as follows: In Section 2, the full description of the model is discussed. In Section 3, we defined the two-dimensional state model and derived its difference-differential equations. The time dependent solution for the model is also obtained in this section. Then the main performance measures of the system and a special case are derived in Section 4. In Section 5, several numerical examples are discussed. The busy period distribution for the system is obtained in Section 6 and finally the paper ends with a conclusion.

II. Model Description

I. Assumption and Notation

The queueing system investigated in this paper is described by the following assumptions:

1) The arrival of primary calls follow a Poisson distribution with parameter λ.
2) The repeated calls to each server follow a Poisson distribution with parameter θ.
3) Service times are exponentially distributed with parameters $\mu_1$and μ_2 for the first and second channel respectively.
4) When the channels are empty, an arriving unit/ repeating unit joins the first channel with probability a_1 and the second channel with probability a_2.
5) The stochastic process involved viz. arrival of units, departures of unit and retrials are statistically independent.

Laplace transformation $\bar{f}(s)$ of $f(t)$ given by
$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$, Re (s) > 0
The Laplace inverse of
$\frac{Q(p)}{P(p)}$ is $\sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{t^{m_k-l} e^{a_k t}}{(m_k-l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}} \frac{Q(p)}{P(p)} (p-a_k)^{m_k} \forall$ p=a_k, $a_i \neq a_k$ for i ≠ k

where,

$P(p)= (p-a_1)^{m_1}(p-a_2)^{m_2}\ldots\ldots\ldots(p-a_n)^{m_n}$

Q(p) is a polynomial of degree $<m_1+m_2+m_3+\ldots\ldots\ldots m_n-1$.

The Laplace inverse of $\bar{N}^{a,b,c}_{n_1,n_2,n_3}(s) = \frac{1}{(s+a)^{n_1}(s+b)^{n_2}(s+c)^{n_3}}$ is

$$N^{a,b,c}_{n_1,n_2,n_3}(t) = \sum_{l=1}^{n_3}\sum_{m=1}^{l}\frac{e^{-at}\,t^{n_3-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_2+g_2)\right)}{(n_3-l)!(m-1)!\,(b-a)^{n_2+m-1}(c-a)^{n_1+l-m}}$$
$$+\sum_{l=1}^{n_2}\sum_{m=1}^{l}\frac{e^{-bt}\,t^{n_2-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_2-l)!\,(m-1)!\,(a-b)^{n_3+m-1}(c-b)^{n_1+l-m}}$$
$$+\sum_{l=1}^{n_1}\sum_{m=1}^{l}\frac{e^{-ct}\,t^{n_1-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(n_2+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_1-l)!\,(m-1)!\,(a-c)^{n_3+m-1}(b-c)^{n_2+l-m}}$$

If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$, then

$L^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)G(t-u)du = F * G$, F * G is called the convolution of F and G.

III. The Two-Dimensional State Model

I. Definitions

$P_{i,j,0}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t when both servers are free.

$P_{i,j,1,k}(t)$ = Probability that there are exactly i arrivals in the system, j departures from the system by time t when one server is free and that unit is in the kth channel. **k = 1,2.**

$P_{i,j,2}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t when both servers are busy.

$P_{i,j}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t.

$P_{i,j}(t) = P_{i,j,0}(t) + P_{i,j,1,1}(t) + P_{i,j,1,2}(t) + P_{i,j,2}(t)\ \forall\ i, j \qquad i \geq j$

$P_{i,j,1}(t) = P_{i,j,1,1}(t) + P_{i,j,1,2}(t)$

also

$P_{i,j,0}(t) = 0,\ i < j;\ P_{i,j,1,k}(t) = 0\ \&\ P_{i,j,2}(t) = 0,\ i \leq j.$ **k=1,2**

Initially

$P_{0,0,0}(0) = 1;\ P_{i,j,0}(0) = 0,\ P_{i,j,1,k}(0) = 0\ \&\ P_{i,j,2}(0) = 0,\ i,j\neq 0.$ **k=1,2.**

II. The difference – differential equations governing the system are

$$\frac{d}{dt}P_{i,j,0}(t) = -(\lambda+(i-j)\theta)P_{i,j,0}(t) + \mu_1 P_{i,j-1,1,1}(t) + \mu_2 P_{i,j-1,1,2}(t) \qquad i \geq j \geq 0 \qquad (1)$$

$$\frac{d}{dt}P_{1,0,1,1}(t) = -(\lambda+\mu_1)P_{1,0,1,1}(t) + \lambda a_1 P_{0,0,0}(t) \qquad (2)$$

$$\frac{d}{dt}P_{1,0,1,2}(t) = -(\lambda+\mu_2)P_{1,0,1,2}(t) + \lambda a_2 P_{0,0,0}(t) \qquad (3)$$

$$\frac{d}{dt}P_{i,j,1,1}(t) = -(\lambda+\mu_1+(i-j-1)\theta)P_{i,j,1,1}(t) + \lambda a_1 P_{i-1,j,0}(t) + (i-j)\theta a_1 P_{i,j,0}(t) + \mu_2 P_{i,j-1,2}(t) \qquad i > j > 0 \qquad (4)$$

$$\frac{d}{dt}P_{i,j,1,2}(t) = -(\lambda+\mu_2+(i-j-1)\theta)P_{i,j,1,2}(t) + \lambda a_2 P_{i-1,j,0}(t) + (i-j)\theta a_2 P_{i,j,0}(t) + \mu_1 P_{i,j-1,2}(t) \qquad i > j > 0 \qquad (5)$$

$$\frac{d}{dt}P_{i,0,2}(t) = -(\lambda+\mu_1+\mu_2)P_{i,0,2}(t) + \lambda\,\delta_{i,2}\{P_{i-1,0,1,1}(t) + P_{i-1,0,1,2}(t)\} + \lambda(1-\delta_{i-2,j})P_{i-1,0,2}(t) \qquad i \geq 2 \qquad (6)$$

$$\frac{d}{dt}P_{i,j,2}(t) = -(\lambda+\mu_1+\mu_2)P_{i,j,2}(t) + (i-j-1)\theta\{P_{i,j,1,1}(t) + P_{i,j,1,2}(t)\} + \lambda\{P_{i-1,j,1,1}(t) + P_{i-1,j,1,2}(t)\} + \lambda(1-\delta_{i-2,j})P_{i-1,j,2}(t) \qquad i>2,\ i > j > 0 \qquad (7)$$

$$\text{where } \delta_{i-2,j} = \begin{cases}1, & when\ i-2=j\\ 0, & otherwise\end{cases}$$

$$\delta_{i,2} = \begin{cases} 1, & when\ i = 2 \\ 0, & otherwise \end{cases}$$

Using Laplace transformation $\bar{f}$ (s) of $f(t)$ given by

$\bar{f}$ (s) = $\int_0^\infty e^{-st} f(t)$ dt, Re (s) > 0

in the equations (1) - (7) along with the initial conditions. We have

$(s + \lambda + (i-j)\theta)\bar{P}_{i,j,0}(s) = \mu_1 \bar{P}_{i,j-1,1,1}(s) + \mu_2 \bar{P}_{i,j-1,1,2}(s) + P_{i,j,0}(0)$ $\quad i \geq j \geq 0$ (8)

$(s + \lambda + \mu_1)\bar{P}_{1,0,1,1}(s) = \lambda a_1 \bar{P}_{0,0,0}(s)$ (9)

$(s + \lambda + \mu_2)\bar{P}_{1,0,1,2}(s) = \lambda a_2 \bar{P}_{0,0,0}(s)$ (10)

$(s + \lambda + \mu_1 + (i-j-1)\theta)\bar{P}_{i,j,1,1}(s) = \lambda a_1 \bar{P}_{i-1,j,0}(s) + (i-j)\theta a_1 \bar{P}_{i,j,0}(s) + \mu_2 \bar{P}_{i,j-1,2}(s)$ $\quad i > j > 0$ (11)

$(s + \lambda + \mu_2 + (i-j-1)\theta)\bar{P}_{i,j,1,2}(s) = \lambda a_2 \bar{P}_{i-1,j,0}(s) + (i-j)\theta a_2 \bar{P}_{i,j,0}(s) + \mu_1 \bar{P}_{i,j-1,2}(s)$ $\quad i > j > 0$ (12)

$(s+\lambda+\mu_1+\mu_2)\bar{P}_{i,0,,2}(s) = \lambda\delta_{i,,2}\{\bar{P}_{i-1,0,1,1}(s)+\bar{P}_{i-1,0,1,2}(s)\} + \lambda(1-\delta_{i-2,j})\bar{P}_{i-1,j,2}(s)$ $\quad i \geq 2$ (13)

$(s + \lambda + \mu_1+\mu_2)\bar{P}_{i,j,2}(s) = (i-j-1)\theta\{\bar{P}_{i,j,1,1}(s) + \bar{P}_{i,j,1,2}(s)\} + \lambda\{\bar{P}_{i-1,j,1,1}(s) + \bar{P}_{i-1,j,1,2}(s)\} + \lambda(1-\delta_{i-2,j})\bar{P}_{i-1,j,2}(s)$ $\quad i>2,\ i > j > 0$ (14)

where $\delta_{i-2,j} = \begin{cases} 1, & when\ i-2 = j \\ 0, & otherwise \end{cases}$

$$\delta_{i,2} = \begin{cases} 1, & when\ i = 2 \\ 0, & otherwise \end{cases}$$

III. Solution of the Problem

Solving equations (8) to (14) recursively, we have

$\bar{P}_{0,0,0}(s) = \frac{1}{s+\lambda}$ (15)

$\bar{P}_{1,1,0}(s) = \left(\frac{\lambda a_1}{(s+\lambda+\mu_1)}\frac{\mu_1}{(s+\lambda)^2} + \frac{\lambda a_2}{(s+\lambda+\mu_2)}\frac{\mu_2}{(s+\lambda)^2}\right)$ (16)

$\bar{P}_{i,2,0}(s) = \frac{\mu_1\mu_2}{(s+\lambda+(i-2)\theta)}\left[\frac{1}{(s+\lambda+\mu_1+(i-2)\theta)} + \frac{1}{(s+\lambda+\mu_2+(i-2)\theta)}\right]\bar{P}_{i,0,2}(s)$ $\quad i \geq 3$ (17)

$\bar{P}_{1,0,1,1}(s) = \frac{1}{s+\lambda}\frac{\lambda a_1}{(s+\lambda+\mu_1)}$ (18)

$\bar{P}_{1,0,1,2}(s) = \frac{1}{s+\lambda}\frac{\lambda a_2}{(s+\lambda+\mu_2)}$ (19)

$\bar{P}_{2,1,1,1}(s) = \frac{\lambda a_1}{(s+\lambda+\mu_1)}\bar{P}_{1,1,0}(s) + \left[\frac{\mu_2}{(s+\lambda+\mu_1)}\left(\frac{\lambda}{(s+\lambda+\mu_1+\mu_2)}\right)\{\bar{P}_{1,0,1,1}(s) + \bar{P}_{1,0,1,2}(s)\}\right]$ (20)

$\bar{P}_{2,1,1,2}(s) = \frac{\lambda a_2}{(s+\lambda+\mu_2)}\bar{P}_{1,1,0}(s) + \left[\frac{\mu_1}{(s+\lambda+\mu_2)}\left(\frac{\lambda}{(s+\lambda+\mu_1+\mu_2)}\right)\{\bar{P}_{1,0,1,1}(s) + \bar{P}_{1,0,1,2}(s)\}\right]$ (21)

$\bar{P}_{i,1,1,1}(s) = \left(\frac{\mu_2}{(s+\lambda+\mu_1+(i-2)\theta)}\frac{\lambda^{i-1}}{(s+\lambda+\mu_1+\mu_2)^{i-1}}\{\bar{P}_{1,0,1,1}(s) + \bar{P}_{1,0,1,2}(s)\}\right)$ $\quad i > 2$ (22)

$\bar{P}_{i,1,1,2}(s) = \left(\frac{\mu_1}{(s+\lambda+\mu_2+(i-2)\theta)}\frac{\lambda^{i-1}}{(s+\lambda+\mu_1+\mu_2)^{i-1}}\{\bar{P}_{1,0,1,1}(s) + \bar{P}_{1,0,1,2}(s)\}\right)$ $\quad i > 2$ (23)

$\bar{P}_{i,0,2}(s) = \frac{\lambda^{i-1}}{(s+\lambda+\mu_1+\mu_2)^{i-1}}\{\bar{P}_{1,0,1,1}(s) + \bar{P}_{1,0,1,2}(s)\}$ $\quad i>1$ (24)

$$\bar{P}_{i,j,2}(s) = \left(\sum_{k=1}^{i-j}\left(\frac{\lambda}{s+\lambda+\mu_1+\mu_2}\right)^{i-j-k}\eta'_k(s)\{\bar{P}_{j+k,j,1,1}(s) + \bar{P}_{j+k,j,1,2}(s)\}\right)$$

$i \geq j+2,\ j \geq 1$ (25)

where $\eta'_k(s) = \begin{cases} 1 & for\ k = 1 \\ \left(1 + \frac{(k-1)\theta}{s+\lambda+\mu_1+\mu_2}\right) & for\ k = 2\ to\ i-j-1 \\ \frac{(k-1)\theta}{s+\lambda+\mu_1+\mu_2} & for\ k = i-j \end{cases}$

$\bar{P}_{i,i-1,1,1}(s) = \frac{\lambda a_1}{(s+\lambda+\mu_1)}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta a_1}{(s+\lambda+\mu_1)}\bar{P}_{i,i-1,0}(s) + \frac{\mu_2}{(s+\lambda+\mu_1)}\bar{P}_{i,i-2,2}(s)$ $\quad i > 2$ (26)

$$\bar{P}_{i,i-1,1,2}(s) = \frac{\lambda a_2}{(s+\lambda+\mu_2)}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta a_2}{(s+\lambda+\mu_2)}\bar{P}_{i,i-1,0}(s) + \frac{\mu_1}{(s+\lambda+\mu_2)}\bar{P}_{i,i-2,2}(s) \qquad i>2 \tag{27}$$

$$\bar{P}_{i,j,1,1}(s) = \frac{\lambda a_1}{(s+\lambda+\mu_1+(i-j-1)\theta)}\bar{P}_{i-1,j,0}(s) + \frac{(i-j)\theta a_1}{(s+\lambda+\mu_1+(i-j-1)\theta)}\bar{P}_{i,j,0}(s) + \frac{\mu_2}{(s+\lambda+\mu_1+(i-j-1)\theta)}\left(\sum_{k=0}^{i-j}\left(\frac{\lambda}{s+\lambda+\mu_1+\mu_2}\right)^{i-j-k}\eta'_k(s)\{\bar{P}_{j+k,j-1,1,1}(s) + \bar{P}_{j+k,j-1,1,2}(s)\}\right)$$
$$i \geq j+2,\ j \geq 2 \tag{28}$$

$$\text{where } \eta'_k(s) = \begin{cases} 1 & \text{for } k=0 \\ \left(1+\frac{k\theta}{s+\lambda+\mu_1+\mu_2}\right) & \text{for } k=1 \text{ to } i-j-1 \\ \frac{k\theta}{s+\lambda+\mu_1+\mu_2} & \text{for } k=i-j \end{cases}$$

$$\bar{P}_{i,j,1,2}(s) = \frac{\lambda a_2}{(s+\lambda+\mu_2+(i-j-1)\theta)}\bar{P}_{i-1,j,0}(s) + \frac{(i-j)\theta a_2}{(s+\lambda+\mu_2+(i-j-1)\theta)}\bar{P}_{i,j,0}(s) + \frac{\mu_1}{(s+\lambda+\mu_2+(i-j-1)\theta)}\left(\sum_{k=0}^{i-j}\left(\frac{\lambda}{s+\lambda+\mu_1+\mu_2}\right)^{i-j-k}\eta'_k(s)\{\bar{P}_{j+k,j-1,1,1}(s) + \bar{P}_{j+k,j-1,1,2}(s)\}\right)$$
$$i \geq j+2,\ j \geq 2 \tag{29}$$

$$\text{where } \eta'_k(s) = \begin{cases} 1 & \text{for } k=0 \\ \left(1+\frac{k\theta}{s+\lambda+\mu_1+\mu_2}\right) & \text{for } k=1 \text{ to } i-j-1 \\ \frac{k\theta}{s+\lambda+\mu_1+\mu_2} & \text{for } k=i-j \end{cases}$$

$$\bar{P}_{i,i,0}(s) = \left(\frac{\lambda}{(s+\lambda)}\right)\left[\frac{\mu a_1}{(s+\lambda+\mu_1)} + \frac{\mu a_2}{(s+\lambda+\mu_2)}\right]\bar{P}_{i-1,i-1,0}(s) + \left(\frac{\theta}{(s+\lambda)}\right)\left[\frac{\mu a_1}{(s+\lambda+\mu_1)} + \frac{\mu a_2}{(s+\lambda+\mu_2)}\right]\bar{P}_{i,i-1,0}(s) + \left(\frac{\mu_1+\mu_2}{(s+\lambda)}\right)\left[\frac{1}{(s+\lambda+\mu_1)} + \frac{1}{(s+\lambda+\mu_2)}\right]\bar{P}_{i,i-2,2}(s) \qquad i>1 \tag{30}$$

$$\bar{P}_{i,j,0}(s) = \frac{\mu_1}{(s+\lambda+(i-j)\theta)}\left\{\frac{\lambda a_1}{(s+\lambda+\mu_1+(i-j)\theta)}\bar{P}_{i-1,j-1,0}(s) + \frac{(i-j+1)\theta a_1}{(s+\lambda+\mu_1+(i-j)\theta)}\bar{P}_{i,j-1,0}(s) + \frac{\mu_2}{(s+\lambda+\mu_1+(i-j)\theta)}\left(\sum_{k=0}^{i-j+1}\left(\frac{\lambda}{s+\lambda+\mu_1+\mu_2}\right)^{\{(i-j)+1\}-k}\eta'_k(s)\{\bar{P}_{(j-1)+k,j-2,1,1}(s) + \bar{P}_{(j-1)+k,j-2,1,2}(s)\}\right)\right\}$$
$$+\frac{\mu_2}{(s+\lambda+(i-j)\theta)}\left\{\frac{\lambda a_2}{(s+\lambda+\mu_2+(i-j)\theta)}\bar{P}_{i-1,j-1,0}(s) + \frac{(i-j+1)\theta a_2}{(s+\lambda+\mu_2+(i-j)\theta)}\bar{P}_{i,j-1,0}(s) + \frac{\mu_1}{(s+\lambda+\mu_2+(i-j)\theta)}\left(\sum_{k=0}^{i-j+1}\left(\frac{\lambda}{s+\lambda+\mu_1+\mu_2}\right)^{\{(i-j)+1\}-k}\eta'_k(s)\{\bar{P}_{(j-1)+k,j-2,1,1}(s) + \bar{P}_{(j-1)+k,j-2,1,2}(s)\}\right)\right\}$$
$$i > j \geq 3 \tag{31}$$

$$\text{where } \eta'_k(s) = \begin{cases} 1 & \text{for } k=0 \\ \left(1+\frac{k\theta}{s+\lambda+\mu_1+\mu_2}\right) & \text{for } k=1 \text{ to } i-j \\ \frac{k\theta}{s+\lambda+\mu_1+\mu_2} & \text{for } k=i-j+1 \end{cases}$$

Taking the Inverse Laplace transform of equations (15) to (31), we have

$$P_{0,0,0}(t) = e^{-\lambda t} \tag{32}$$

$$P_{1,1,0}(t) = \lambda\,\mu_1 a_1(te^{-\lambda t})e^{-(\lambda+\mu_1)t} + \lambda\,\mu_2\, a_2(te^{-\lambda t})e^{-(\lambda+\mu_2)t} \tag{33}$$

$$P_{i,2,0}(t) = \mu_1\mu_2 e^{-(\lambda+(i-2)\theta)t}\left(\frac{1}{(\mu_1+(i-2)\theta)} - \frac{e^{-(\mu_1+(i-2)\theta)t}}{(\mu_1+(i-2)\theta)}\right) * P_{i,0,2}(t) + \mu_1\mu_2 e^{-(\lambda+(i-2)\theta)t}\left(\frac{1}{(\mu_2+(i-2)\theta)} -\right.$$

$\frac{e^{-(\mu_2+(i-2)\theta)t}}{(\mu_2+(i-2)\theta)}\Big) * P_{i,0,2}(t)$ $\quad i \geq 3 \quad$ (34)

$$P_{1,0,1,1}(t) = \lambda a_1 e^{-\lambda t} \times \left(\frac{1}{\mu_1} - \frac{e^{-\mu_1 t}}{\mu_1}\right) \quad (35)$$

$$P_{1,0,1,2}(t) = \lambda a_2 e^{-\lambda t} \times \left(\frac{1}{\mu_2} - \frac{e^{-\mu_2 t}}{\mu_2}\right) \quad (36)$$

$$P_{2,1,1,1}(t)=\lambda\, a_1 e^{-(\lambda+\mu_1)t} * P_{1,1,0}(t) + \left(\lambda\mu_2 e^{-(\lambda+\mu_1)t}\left(\frac{1}{\mu_1+\mu_2} - \frac{e^{-(\mu_1+\mu_2)t}}{\mu_1+\mu_2}\right) * \left\{P_{1,0,1,1}(t) + P_{1,0,1,2}(t)\right\}\right) \quad (37)$$

$$P_{2,1,1,2}(t)=\lambda\, a_2 e^{-(\lambda+\mu_2)t} * P_{1,1,0}(t) + \left(\lambda\mu_1 e^{-(\lambda+\mu_2)t}\left(\frac{1}{\mu_1+\mu_2} - \frac{e^{-(\mu_1+\mu_2)t}}{\mu_1+\mu_2}\right) * \left\{P_{1,0,1,1}(t) + P_{1,0,1,2}(t)\right\}\right) \quad (38)$$

$$P_{i,1,1,1}(t)=\left[\left(\mu_2\lambda^{i-1}e^{-(\lambda+\mu_1+(i-2)\theta)t}\left\{\frac{1}{(\mu_1+\mu_2)^{i-1}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-2}\frac{(t)^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-r}}\right\}\right) * \left\{P_{1,0,1,1}(t) + P_{1,0,1,2}(t)\right\}\right] \quad i>2 \quad (39)$$

$$P_{i,1,1,2}(t)=\left[\left(\mu_1\lambda^{i-1}e^{-(\lambda+\mu_2+(i-2)\theta)t}\left\{\frac{1}{(\mu_1+\mu_2)^{i-1}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-2}\frac{(t)^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-r}}\right\}\right) * \left\{P_{1,0,1,1}(t) + P_{1,0,1,2}(t)\right\}\right] \quad i>2 \quad (40)$$

$$P_{i,0,2}(t) = \left(\lambda^{i-1}\frac{t^{i-2}}{(i-2)!}e^{-(\lambda+\mu_1+\mu_2)t}\right) * P_{1,0,1,1}(t) + \left(\lambda^{i-1}\frac{t^{i-2}}{(i-2)!}e^{-(\lambda+\mu_1+\mu_2)t}\right) * P_{1,0,1,2}(t) \quad i>1 \quad (41)$$

$$P_{i,j,2}(t) = \left(\left(\lambda^{i-j-1}\frac{t^{i-j-2}}{(i-j-2)!}e^{-(\lambda+\mu_1+\mu_2)t}\right) * \left\{P_{j+1,j,1,1}(t) + P_{j+1,j,1,2}(t)\right\}\right) + \left(\sum_{k=2}^{i-j-1}\left(\lambda^{i-j-k}\frac{t^{i-j-k-1}}{(i-j-k-1)!}e^{-(\lambda+\mu_1+\mu_2)t}\right) * \left\{P_{j+k,j,1,1}(t) + P_{j+k,j,1,2}(t)\right\}\right) + \left(\sum_{k=2}^{i-j-1}\left(\lambda^{i-j-k}(k-1)\theta\frac{t^{i-j-k}}{(i-j-k)!}e^{-(\lambda+\mu_1+\mu_2)t}\right) * \left\{P_{j+k,j,1,1}(t) + P_{j+k,j,1,2}(t)\right\}\right) + \left(\left((i-j-1)\theta e^{-(\lambda+\mu_1+\mu_2)t}\right) * \left\{P_{i,j,1,1}(t) + P_{i,j,1,2}(t)\right\}\right) \quad i \geq j+2,\ j \geq 1 \quad (42)$$

$$P_{i,i-1,1,1}(t) = \lambda\, a_1 e^{-(\lambda+\mu_1)t} * P_{i-1,i-1,0}(t) + \theta a_1 e^{-(\lambda+\mu_1)t} * P_{i,i-1,0}(t) + \mu_2 e^{-(\lambda+\mu_1)t} * P_{i,i-2,2}(t) \quad i>2 \quad (43)$$

$$P_{i,i-1,1,2}(t) = \lambda\, a_2 e^{-(\lambda+\mu_2)t} * P_{i-1,i-1,0}(t) + \theta a_2 e^{-(\lambda+\mu_2)t} * P_{i,i-1,0}(t) + \mu_1 e^{-(\lambda+\mu_2)t} * P_{i,i-2,2}(t) \quad i>2 \quad (44)$$

$$P_{i,j,1,1}(t) = \lambda\, a_1 e^{-(\lambda+\mu_1+(i-j-1)\theta)t} * P_{i-1,j,0}(t) + (i-j)\theta a_1 e^{-(\lambda+\mu_1+(i-j-1)\theta)t} * P_{i,j,0}(t) + \left[\mu_2\lambda^{i-j}e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-1}\frac{(t)^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-r}}\right\} * \left\{P_{j,j-1,1,1}(t) + P_{j,j-1,1,2}(t)\right\}\right] + \left[\mu_2 e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-k}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-k-1}\frac{(t)^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-k-r}}\right\} * \left\{P_{j+k,j-1,1,1}(t) + P_{j+k,j-1,1,2}(t)\right\}\right] + \left[\mu_2 e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}(k\theta)\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-k+1}} - e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-k}\frac{(t)^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-k+1-r}}\right\} * \left\{P_{j+k,j-1,1,1}(t) + P_{j+k,j-1,1,2}(t)\right\}\right]$$

$$+\left[\mu_2(i-j)\theta e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\times\left(\frac{1}{(\mu_1+\mu_2)}-\frac{e^{-(\mu_1+\mu_2)t}}{(\mu_1+\mu_2)}\right)*\left\{P_{i,j-1,1,1}(t)+P_{i,j-1,1,2}(t)\right\}\right]$$

$i\geq j+2,\ j\geq 2$ (45)

$$P_{i,j,1,2}(t)=\lambda a_2 e^{-(\lambda+\mu_2+(i-j-1)\theta)t}*P_{i-1,j,0}(t)+(i-j)\theta a_2 e^{-(\lambda+\mu_2+(i-j-1)\theta)t}*P_{i,j,0}(t)$$
$$+\left[\mu_1\lambda^{i-j}e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j}}-e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-1}\frac{(t)^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-r}}\right\}*\left\{P_{j,j-1,1,1}(t)+P_{j,j-1,1,2}(t)\right\}\right]+$$
$$\left[\mu_1 e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-k}}-e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-k-1}\frac{(t)^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-k-r}}\right\}*\left\{P_{j+k,j-1,1,1}(t)+P_{j+k,j-1,1,2}(t)\right\}\right]+$$
$$\left[\mu_1 e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}(k\theta)\left\{\frac{1}{(\mu_1+\mu_2)^{i-j-k+1}}-e^{-(\mu_1+\mu_2)t}\sum_{r=0}^{i-j-k}\frac{(t)^r}{r!}\frac{1}{(\mu_1+\mu_2)^{i-j-k+1-r}}\right\}*\left\{P_{j+k,j-1,1,1}(t)+P_{j+k,j-1,1,2}(t)\right\}\right]$$
$$+\left[\mu_1(i-j)\theta e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\times\left(\frac{1}{(\mu_1+\mu_2)}-\frac{e^{-(\mu_1+\mu_2)t}}{(\mu_1+\mu_2)}\right)*\left\{P_{i,j-1,1,1}(t)+P_{i,j-1,1,2}(t)\right\}\right]$$

$i\geq j+2,\ j\geq 2$ (46)

$$P_{i,i,0}(t)=\left(\lambda\mu_1 a_1 e^{-\lambda t}\left(\frac{1}{\mu_1}-\frac{e^{-\mu_1 t}}{\mu_1}\right)+\lambda\mu_2 a_2 e^{-\lambda t}\left(\frac{1}{\mu_2}-\frac{e^{-\mu_2 t}}{\mu_2}\right)\right)*P_{i-1,i-1,0}(t)+$$
$$\left(\mu_1\theta a_1 e^{-\lambda t}\left(\frac{1}{\mu_1}-\frac{e^{-\mu_1 t}}{\mu_1}\right)+\mu_2\theta a_2 e^{-\lambda t}\left(\frac{1}{\mu_2}-\frac{e^{-\mu_2 t}}{\mu_2}\right)\right)*P_{i,i-1,0}(t)+$$
$$\left(\mu_1\mu_2 e^{-\lambda t}\left(\frac{1}{\mu_1}-\frac{e^{-\mu_1 t}}{\mu_1}\right)+\mu_1\mu_2 e^{-\lambda t}\left(\frac{1}{\mu_2}-\frac{e^{-\mu_2 t}}{\mu_2}\right)\right)*P_{i,i-2,2}(t)$$

$i>1$ (47)

$$P_{i,j,0}(t)=\mu_1\lambda a_1 e^{-(\lambda+(i-j)\theta)t}\left(\frac{1}{\mu_1+(i-j)\theta}-\frac{e^{-(\mu_1+(i-j)\theta)t}}{\mu_1+(i-j)\theta}\right)*P_{i-1,j-1,0}(t)+$$
$$\mu_1(i-j+1)\theta a_1 e^{-(\lambda+(i-j)\theta)t}\left(\frac{1}{\mu_1+(i-j)\theta}-\frac{e^{-(\mu_1+(i-j)\theta)t}}{\mu_1+(i-j)\theta}\right)*P_{i,j-1,0}(t)+$$
$$\left[\mu_1\mu_2\lambda^{i-j+1}\left[\sum_{l=1}^{i-j+1}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\,t^{(i-j+1)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{((i-j+1)-l)!(m-1)!\,(\mu_1)^m(\mu_1+\mu_2-(i-j)\theta)^{1+l-m}}-\frac{e^{-(\lambda+\mu_1+(i-j)\theta)t}}{(\mu_1)^{(i-j+1)}(\mu_1-(i-j)\theta)}+\frac{e^{-(\lambda+\mu_1+\mu_2)t}}{(\mu_1+\mu_2-(i-j)\theta)^{(i-j+1)}(\mu_1-(i-j)\theta)}\right]*\left\{P_{j-1,j-2,1,1}(t)+P_{j-1,j-2,1,2}(t)\right\}\right]+$$
$$\left[\mu_1\mu_2\sum_{k=1}^{i-j}\lambda^{(i-j+1)-k}\left[\sum_{l=1}^{(i-j+1)-k}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\,t^{((i-j+1)-k)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{(((i-j+1)-k)-l)!(m-1)!\,(\mu_1)^m(\mu_1+\mu_2-(i-j)\theta)^{1+l-m}}-\frac{e^{-(\lambda+\mu_1+(i-j)\theta)t}}{(\mu_1)^{((i-j+1)-k)}(\mu_1-(i-j)\theta)}+\frac{e^{-(\lambda+\mu_1+\mu_2)t}}{(\mu_1+\mu_2-(i-j)\theta)^{((i-j+1)-k)}(\mu_1-(i-j)\theta)}\right]*\left\{P_{(j-1)+k,j-2,1,1}(t)+P_{(j-1)+k,j-2,1,2}(t)\right\}\right]+$$
$$\mu_1\mu_2\sum_{k=1}^{i-j}(k\theta)\lambda^{(i-j+1)-k}$$
$$\left[\left\{\sum_{l=1}^{((i-j+1)-k)+1}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\,t^{(((i-j+1)-k)+1)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{((((i-j+1)-k)+1)-l)!\,(m-1)!\,(\mu_1)^m(\mu_1+\mu_2-(i-j)\theta)^{1+l-m}}-\frac{e^{-(\lambda+\mu_1+(i-j)\theta)t}}{(\mu_1)^{(((i-j+1)-k)+1)}(\mu_1-(i-j)\theta)}+\frac{e^{-(\lambda+\mu_1+\mu_2)t}}{(\mu_1+\mu_2-(i-j)\theta)^{(((i-j+1)-k)+1)}(\mu_1-(i-j)\theta)}\right\}*\left\{P_{(j-1)+k,j-2,1,1}(t)+P_{(j-1)+k,j-2,1,2}(t)\right\}\right]$$

$$+\Big[\mu_1\mu_2(i-j+1)\theta\Big[\frac{e^{-(\lambda+(i-j)\theta)t}}{(\mu_1)(\mu_1+\mu_2-(i-j)\theta)}-\frac{e^{-(\lambda+\mu_1+(i-j)\theta)t}}{(\mu_1)(\mu_1-(i-j)\theta)}+\frac{e^{-(\lambda+\mu_1+\mu_2)t}}{(\mu_1+\mu_2-(i-j)\theta)(\mu_1-(i-j)\theta)}\Big] * \{P_{i,j-2,1,1}(t)+P_{i,j-2,1,2}(t)\}\Big] +$$

$$\mu_2\lambda a_2 e^{-(\lambda+(i-j)\theta)t}\left(\frac{1}{\mu_2+(i-j)\theta}-\frac{e^{-(\mu_2+(i-j)\theta)t}}{\mu_2+(i-j)\theta}\right) * P_{i-1,j-1,0}(t)+$$

$$\mu_2(i-j+1)\theta a_2 e^{-(\lambda+(i-j)\theta)t}\left(\frac{1}{\mu_2+(i-j)\theta}-\frac{e^{-(\mu_2+(i-j)\theta)t}}{\mu_2+(i-j)\theta}\right) * P_{i,j-1,0}(t)+$$

$$\Big[\mu_1\mu_2\lambda^{i-j+1}\Big[\sum_{l=1}^{i-j+1}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\,t^{(i-j+1)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{((i-j+1)-l)!(m-1)!\,(\mu_2)^m(\mu_1+\mu_2-(i-j)\theta)^{1+l-m}}-\frac{e^{-(\lambda+\mu_2+(i-j)\theta)t}}{(\mu_2)^{(i-j+1)}(\mu_2-(i-j)\theta)}+\frac{e^{-(\lambda+\mu_1+\mu_2)t}}{(\mu_1+\mu_2-(i-j)\theta)^{(i-j+1)}(\mu_2-(i-j)\theta)}\Big] * \{P_{j-1,j-2,1,1}(t)+P_{j-1,j-2,1,2}(t)\}\Big]+$$

$$\Big[\mu_1\mu_2\sum_{k=1}^{i-j}\lambda^{(i-j+1)-k}\Big[\sum_{l=1}^{(i-j+1)-k}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\,t^{((i-j+1)-k)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{(((i-j+1)-k)-l)!(m-1)!\,(\mu_2)^m(\mu_1+\mu_2-(i-j)\theta)^{1+l-m}}-\frac{e^{-(\lambda+\mu_2+(i-j)\theta)t}}{(\mu_2)^{((i-j+1)-k)}(\mu_2-(i-j)\theta)}+\frac{e^{-(\lambda+\mu_1+\mu_2)t}}{(\mu_1+\mu_2-(i-j)\theta)^{((i-j+1)-k)}(\mu_2-(i-j)\theta)}\Big] * \{P_{(j-1)+k,j-2,1,1}(t)+P_{(j-1)+k,j-2,1,2}(t)\}\Big] + \mu_1\mu_2\sum_{k=1}^{i-j}(k\theta)\lambda^{(i-j+1)-k}$$

$$\Bigg[\left\{\sum_{l=1}^{((i-j+1)-k)+1}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\,t^{(((i-j+1)-k)+1)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{((((i-j+1)-k)+1)-l)!\,(m-1)!\,(\mu_2)^m(\mu_1+\mu_2-(i-j)\theta)^{1+l-m}} -\frac{e^{-(\lambda+\mu_1+(i-j)\theta)t}}{(\mu_2)^{(((i-j+1)-k)+1)}(\mu_2-(i-j)\theta)}+\frac{e^{-(\lambda+\mu_1+\mu_2)t}}{(\mu_1+\mu_2-(i-j)\theta)^{(((i-j+1)-k)+1)}(\mu_2-(i-j)\theta)}\right\} * \{P_{(j-1)+k,j-2,1,1}(t)+P_{(j-1)+k,j-2,1,2}(t)\}\Bigg]$$

$$+\Big[\mu_1\mu_2(i-j+1)\theta\Big[\frac{e^{-(\lambda+(i-j)\theta)t}}{(\mu_2)(\mu_1+\mu_2-(i-j)\theta)}-\frac{e^{-(\lambda+\mu_2+(i-j)\theta)t}}{(\mu_2)(\mu_2-(i-j)\theta)}+\frac{e^{-(\lambda+\mu_1+\mu_2)t}}{(\mu_1+\mu_2-(i-j)\theta)(\mu_2-(i-j)\theta)}\Big] * \{P_{i,j-2,1,1}(t)+P_{i,j-2,1,2}(t)\}\Big] \quad i>j\geq 3 \qquad (48)$$

IV. Measures of Effectiveness

I. The Laplace transform of the probability $P_{i.}(t)$ that exactly i units arrive by time t is :

$$\bar{P}_{i.}(s)=\sum_{j=0}^{i}\bar{P}_{i,j}(s)=\frac{\lambda^i}{(s+\lambda)^{i+1}};\ i>0 \qquad (49)$$

And its Inverse Laplace transform is

$$P_{i.}(t)=\frac{e^{-\lambda t}(\lambda t)^i}{i!} \qquad (50)$$

The basic assumption on primary arrivals is that it forms a Poisson process and above analysis of abstract solution also verifies the same.

II. The probability that exactly j customers have been served by time t. $P_{.j}(t)$ in terms of $P_{i,j}(t)$ is given by:

$$P_{.j}(t)=\sum_{i=j}^{\infty}P_{i,j}(t)$$

III. From the abstract solution of our model, we verified that the sum of all possible probabilities is one i.e. taking summation over i and j on equations (15)-(31) and adding, we get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\{\bar{P}_{i,j,0}(s)+\bar{P}_{i,j,1,1}(s)+\bar{P}_{i,j,1,2}(s)+\bar{P}_{i,j,2}(s)\}=\frac{1}{s}$$

Taking inverse Laplace transformation, we get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\{P_{i,j,0}(t)+P_{i,j,1,1}(t)+P_{i,j,1,2}(t)+P_{i,j,2}(t)\}=1,$$

which is a verification of our results.

IV. Converting two-state model into single state model:

Define $Q_{n,m}(t)$ as the probability that there are n customers in the system at time t and the servers are free or busy according as m=0,1,2.

The probability of exactly n customers in the system at time t in terms of $P_{i,j,0}(t)$ and $P_{i,j,m}(t)$:

When the server is free, it is defined by probability $Q_{n,0}(t)$

$$Q_{n,0}(t)=\sum_{j=0}^{\infty}P_{j+n,j,0}(t)$$

In this case, the number of customers in the orbit is calculated with the help of following formula:
n = (number of arrivals – number of departures)

When only one server (m=1) is busy, it is defined by probability $Q_{n,m,k}(t)$

$$Q_{n,m,k}(t)=\sum_{j=0}^{\infty}P_{j+n+m,j,m,k}(t) \qquad \textbf{(k = 1, 2)}$$

In this case, the number of customers in the orbit is calculated with the help of following formula:
n = (number of arrivals – number of departures – m)

When both servers (m=2) are busy, it is defined by probability $Q_{n,m}(t)$

$$Q_{n,m}(t)=\sum_{j=0}^{\infty}P_{j+n+m,j,m}(t)$$

In this case, the number of customers in the orbit is calculated with the help of following formula:
n = (number of arrivals – number of departures – m)

Using above definitions and letting $\mu_1=\mu_2=1$ from the equations (1) to (7) the set of equations in statistical equilibrium are:

$(\lambda+n\theta)\,Q_{n,0}=Q_{n,1}$ $\quad n\geq 0 \quad$ (51)

$(\lambda+n\theta+1)\,Q_{n,1,1}=\lambda a_1\,Q_{n,0}+(n+1)\,\theta\,a_1Q_{n+1,0}+Q_{n,2}$ $\quad n\geq 0 \quad$ (52)

$(\lambda+n\theta+1)\,Q_{n,1,2}=\lambda a_2\,Q_{n,0}+(n+1)\,\theta\,a_2Q_{n+1,0}+Q_{n,2}$ $\quad n\geq 0 \quad$ (53)

$(\lambda+2)\,Q_{n,2}=\lambda\,Q_{n,1}+(n+1)\theta\,Q_{n+1,1}+\lambda\,Q_{n-1,2}(1-\delta_{n,0})n\geq 0$ $\quad$ (54)

where $\delta_{n,0}=\begin{cases}1, & when\ n=0\\ 0, & when\ n\geq 1\end{cases}$

Using $Q_{n,1,1}+Q_{n,1,2}=Q_{n,1}$ and letting $a_1=a_2=\frac{1}{2}$ in equations (51) to (54) then the set of equations are:

$(\lambda+n\theta)\,Q_{n,0}=Q_{n,1}$ $\quad n\geq 0 \quad$ (55)

$(\lambda+n\theta+1)\,Q_{n,1}=\lambda Q_{n,0}+(n+1)\,\theta\,Q_{n+1,0}+2Q_{n,2}$ $\quad n\geq 0 \quad$ (56)

$(\lambda+2)\,Q_{n,2}=\lambda\,Q_{n,1}+(n+1)\,\theta\,Q_{n+1,1}+\lambda\,Q_{n-1,2}(1-\delta_{n,0})$ $\quad n\geq 0 \quad$ (57)

where $\delta_{n,0}=\begin{cases}1, & when\ n=0\\ 0, & when\ n\geq 1\end{cases}$

which coincide with the results (2.1) - (2.3) of Falin and Templeton [5].

V. Special Case:

When there are two servers then various probabilities can be obtained from equations (32) to (48) by letting $\mu_1=\mu_2=\mu$, $a_1=a_2=\frac{1}{2}$and using the relation $P_{i,j,1,1}(t)+P_{i,j,1,2}(t)=P_{i,j,1}(t)$, we get

$P_{1,0,1}(t)=\lambda e^{-\lambda t}\times\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right)$ $\quad$ (58)

$P_{1,1,0}(t)=\lambda\mu(te^{-\lambda t})e^{-(\lambda+\mu)t}$ $\quad$ (59)

$$P_{i,0,2}(t) = \left(\lambda^{i-1}\frac{t^{i-2}}{(i-2)!}e^{-(\lambda+2\mu)t}\right)^{*} P_{1,0,1}(t) \qquad i>1 \qquad (60)$$

$$P_{i,i,0}(t) = \lambda\mu e^{-\lambda t}\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right) * P_{i-1,i-1,0}(t) + \mu\theta e^{-\lambda t}\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right) * P_{i,i-1,0}(t) + 2\mu^2 e^{-\lambda t}\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right) * P_{i,i-2,2}(t) \qquad i>1 \qquad (61)$$

$$P_{2,1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{1,1,0}(t) + 2\lambda\mu e^{-(\lambda+\mu)t}\left(\frac{1}{2\mu}-\frac{e^{-2\mu t}}{2\mu}\right) * P_{1,0,1}(t) \qquad (62)$$

$$P_{i,1,1}(t) = \left[2\mu\lambda^{i-1}e^{-(\lambda+\mu+(i-2)\theta)t}\left\{\frac{1}{(2\mu)^{i-1}} - e^{-2\mu t}\sum_{r=0}^{i-2}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-r}}\right\}\right] * P_{1,0,1}(t) \qquad i>2 \qquad (63)$$

$$P_{i,i-1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu)t} * P_{i,i-1,0}(t) + 2\mu e^{-(\lambda+\mu)t} * P_{i,i-2,2}(t) \qquad i>2 \qquad (64)$$

$$P_{i,2,0}(t) = 2\mu^2 e^{-(\lambda+(i-2)\theta)t}\left(\frac{1}{(\mu+(i-2)\theta)} - \frac{e^{-(\mu+(i-2)\theta)t}}{(\mu+(i-2)\theta)}\right) * P_{i,0,2}(t) \qquad i\ge 3 \qquad (65)$$

$$P_{i,j,2}(t) = \left(\lambda^{i-j-1}\frac{t^{i-j-2}}{(i-j-2)!}e^{-(\lambda+2\mu)t}\right)^{*} P_{j+1,j,1}(t) + \sum_{k=2}^{i-j-1}\left(\lambda^{i-j-k}\frac{t^{i-j-k-1}}{(i-j-k-1)!}e^{-(\lambda+2\mu)t}\right)^{*} P_{j+k,j,1}(t) + \sum_{k=2}^{i-j-1}\left(\lambda^{i-j-k}(k-1)\theta\frac{t^{i-j-k}}{(i-j-k)!}e^{-(\lambda+2\mu)t}\right)^{*} P_{j+k,j,1}(t) + \left((i-j-1)\theta e^{-(\lambda+2\mu)t}\right)^{*} P_{i,j,1}(t) \qquad i\ge j+2,\ j\ge 1 \qquad (66)$$

$$P_{i,j,1}(t) = \lambda e^{-(\lambda+\mu+(i-j-1)\theta)t} * P_{i-1,j,0}(t) + (i-j)\theta e^{-(\lambda+\mu+(i-j-1)\theta)t} * P_{i,j,0}(t) + 2\mu\lambda^{i-j}e^{-(\lambda+\mu+(i-j-1)\theta)t}\left\{\frac{1}{(2\mu)^{i-j}} - e^{-2\mu t}\sum_{r=1}^{i-j-1}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-r}}\right\} * P_{j,j-1,1}(t) + 2\mu e^{-(\lambda+\mu+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\left\{\frac{1}{(2\mu)^{i-j-k}} - e^{-2\mu t}\sum_{r=0}^{i-j-k-1}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-k-r}}\right\} * P_{j+k,j-1,1}(t) + 2\mu e^{-(\lambda+\mu+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}(k\theta)\left\{\frac{1}{(2\mu)^{i-j-k+1}} - e^{-2\mu t}\sum_{r=0}^{i-j-k}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-k+1-r}}\right\} * P_{j+k,j-1,1}(t) + 2\mu(i-j)\theta e^{-(\lambda+\mu+(i-j-1)\theta)t} \times\left(\frac{1}{2\mu}-\frac{e^{-2\mu t}}{2\mu}\right) * P_{i,j-1,1}(t) \qquad i\ge j+2,\ j\ge 2 \qquad (67)$$

$$P_{i,j,0}(t) = \lambda\mu e^{-(\lambda+(i-j)\theta)t}\left(\frac{1}{\mu+(i-j)\theta}-\frac{e^{-(\mu+(i-j)\theta)t}}{\mu+(i-j)\theta}\right) * P_{i-1,j-1,0}(t) + \mu(i-j+1)\theta e^{-(\lambda+(i-j)\theta)t}\left(\frac{1}{\mu+(i-j)\theta}-\frac{e^{-(\mu+(i-j)\theta)t}}{\mu+(i-j)\theta}\right) * P_{i,j-1,0}(t) + 2\mu^2\lambda^{i-j+1}$$
$$\left[\sum_{l=1}^{i-j+1}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\, t^{(i-j+1)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{((i-j+1)-l)!(m-1)!\,(\mu)^m(2\mu-(i-j)\theta)^{1+l-m}} - \frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)^{(i-j+1)}(\mu-(i-j)\theta)} + \frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)^{(i-j+1)}(\mu-(i-j)\theta)}\right] * P_{j-1,j-2,1}(t) + 2\mu^2\sum_{k=1}^{i-j}\lambda^{(i-j+1)-k}$$
$$\left[\sum_{l=1}^{(i-j+1)-k}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\, t^{((i-j+1)-k)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{(((i-j+1)-k)-l)!(m-1)!\,(\mu)^m(2\mu-(i-j)\theta)^{1+l-m}} - \frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)^{(i-j+1)-k}(\mu-(i-j)\theta)} + \frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)^{(i-j+1)-k}(\mu-(i-j)\theta)}\right] * P_{(j-1)+k,j-2,1}(t) + 2\mu^2\sum_{k=1}^{i-j}\lambda^{(i-j+1)-k}(k\theta)$$
$$\left[\sum_{l=1}^{((i-j+1)-k)+1}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\, t^{(((i-j+1)-k)+1)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{((((i-j+1)-k)+1)-l)!(m-1)!\,(\mu)^m(2\mu-(i-j)\theta)^{1+l-m}} - \frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)^{((i-j+1)-k)+1}(\mu-(i-j)\theta)} + \frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)^{((i-j+1)-k)+1}(\mu-(i-j)\theta)}\right] * P_{(j-1)+k,j-2,1}(t) + 2\mu^2(i-j+1)\theta$$
$$\left[\frac{e^{-(\lambda+(i-j)\theta)t}}{(\mu)(2\mu-(i-j)\theta)} - \frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)(\mu-(i-j)\theta)} + \frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)(\mu-(i-j)\theta)}\right] * P_{i,j-2,1}(t) \qquad i>j\ge 3 \qquad (68)$$

The above equations coincide with that of Singla & Kalra [14].

V. Numerical Solution

Using MATLAB programming the numerical results are generated for the case when $\rho\left(=\frac{\lambda}{\mu_1+\mu_2}\right)=0.3$, $\eta\left(=\frac{\theta}{\mu_1+\mu_2}\right)=0.6$, $r_1\left(=\frac{\mu_1}{\mu_1+\mu_2}\right)=0.3$, $a_1=0.4, a_2=0.6$. From the numerical results, it is found that the sum of all the probabilities at any instant approaches to one. In table 1, we show some of the significant probabilities at different instants of time whose sum is found close to one.

Table 1: Some significant probabilities at different instants of time.

At time t=1

$P_{0,0,0}$	$P_{1,1,0}$	$P_{1,0,1,1}$	$P_{2,1,1,1}$	$P_{1,0,1,2}$	$P_{2,1,1,2}$	$P_{2,0,2}$
0.7408	0.0495	0.0768	0.0069	0.0959	0.0046	0.0204

$P_{3,0,2}$	$P_{3,1,2}$	Sum
0.0018	0.0008	0.9975

At time t=5

$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,2,0}$	$P_{3,3,0}$	$P_{1,0,1,1}$	$P_{2,1,1,1}$	$P_{3,2,1,1}$	$P_{1,0,1,2}$	$P_{2,1,1,2}$
0.2231	0.2097	0.0947	0.0260	0.0693	0.0759	0.0330	0.0556	0.0463

$P_{3,2,1,2}$	$P_{2,0,2}$	$P_{3,0,2}$	$P_{3,1,2}$	$P_{4,1,2}$	$P_{4,2,2}$	$P_{5,3,2}$	Sum
0.0196	0.0341	0.0087	0.0315	0.0079	0.0107	0.0024	0.9147

At time t=10

$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,2,0}$	$P_{3,3,0}$	$P_{5,2,0}$	$P_{5,5,0}$	$P_{1,0,1,1}$
0.0498	0.1176	0.1378	0.1052	0.0579	0.0242	0.0189

$P_{2,1,1,1}$	$P_{3,2,1,1}$	$P_{4,2,1,1}$
0.0480	0.0572	0.0047

$P_{4,3,1,1}$	$P_{1,0,1,2}$	$P_{2,1,1,2}$	$P_{3,2,1,2}$	$P_{4,3,1,2}$	$P_{5,4,1,2}$	$P_{6,5,1,2}$	$P_{2,0,2}$	$P_{3,0,2}$	$P_{3,1,2}$
0.0433	0.0128	0.0288	0.0331	0.0249	0.0134	0.0055	0.0094	0.0028	0.0216

$P_{4,1,2}$	$P_{4,2,2}$	$P_{5,3,2}$	$P_{6,3,2}$	$P_{6,4,2}$	$P_{7,4,2}$	$P_{7,5,2}$	Sum
0.0069	0.0241	0.0174	0.0063	0.0089	0.0074	0.001	0.9062

At time t=20

$P_{1,1,0}$	$P_{2,2,0}$	$P_{3,3,0}$	$P_{4,4,0}$	$P_{5,5,0}$	$P_{6,6,0}$	$P_{7,6,0}$
0.0132	0.0354	0.0627	0.0886	0.0876	0.0754	0.0071

$P_{7,7,0}$	$P_{3,2,1,1}$	$P_{4,3,1,1}$
0.2357	0.0148	0.0265

$P_{5,4,1,1}$	$P_{6,5,1,1}$	$P_{7,6,1,1}$	$P_{3,2,1,2}$	$P_{4,3,1,2}$	$P_{5,4,1,2}$	$P_{6,5,1,2}$	$P_{1,0,2}$	$P_{5,3,2}$
0.0353	0.0372	0.0741	0.0088	0.0155	0.0204	0.0213	0.0310	0.0119

$P_{6,4,2}$	$P_{7,4,2}$	$P_{7,5,2}$	Sum
0.0155	0.0080	0.0237	0.9497

At time t=40

$P_{5,5,0}$	$P_{6,6,0}$	$P_{7,7,0}$	$P_{7,6,1,1}$	$P_{1,0,2}$	$P_{7,5,2}$	Sum
0.0096	0.0180	0.9183	0.0242	0.0075	0.0027	0.9803

V. Busy Period Probabilities

In this section, we discuss some interesting numerical results about busy period distribution of the server and busy period distribution of the system.
The probability when the one or both servers are busy is given

P (Servers one or both busy) $=\sum_{i>j\geq 0}\left(P_{i,j,1,1}(t)+P_{i,j,1,2}(t)+P_{i,j,2}(t)\right)$

The probability when the system is busy is given by

$$\text{P (System is busy)} = \sum_{i>j\geq 0}\left(P_{i,j,0}(t)+P_{i,j,1,1}(t)+P_{i,j,1,2}(t)+P_{i,j,2}(t)\right)$$

The numerical results are generated using MATLAB programming for the desired probabilities. The probability when system is busy and the probability when one or both servers are busy for different values of $\rho\left(=\frac{\lambda}{\mu_1+\mu_2}\right)$at $\eta\left(=\frac{\theta}{\mu_1+\mu_2}\right)=0.6$, $r_1\left(=\frac{\mu_1}{\mu_1+\mu_2}\right)=0.3$, $a_1=0.4, a_2=0.6$ are listed in Table 2.

Table 2: Probability of system busy and one or both servers busy (r_1 = 0.3, a_1 = 0.4, a_2 = 0.6).

	Probability (System busy) η=0.6			Probability (Servers busy) η=0.6		
t	ρ=0.3	ρ=0.6	ρ=0.9	ρ=0.3	ρ=0.6	ρ=0.9
0	0	0	0	0	0	0
1	0.2082	0.3734	0.5045	0.2082	0.3732	0.5039
2	0.314	0.532	0.6826	0.3135	0.5294	0.6768
3	0.3754	0.6164	0.7685	0.3737	0.6088	0.754
4	0.4145	0.6684	0.8184	0.4111	0.6547	0.7949
5	0.441	0.7035	0.8504	0.4357	0.6838	0.8176
6	0.4598	0.7283	0.8699	0.4525	0.7028	0.8266
7	0.4733	0.7452	0.8759	0.4644	0.714	0.8207

In figure 1, probability (system busy) and probability (one or both servers busy) are studied by plotting these against time for the case (ρ=0.6, η=0.6,r_1 = 0.3, a_1 = 0.4, a_2 = 0.6). From this figure it is apparent that the probability when the system is busy always remains more than the probability when the (server / servers)are busy.

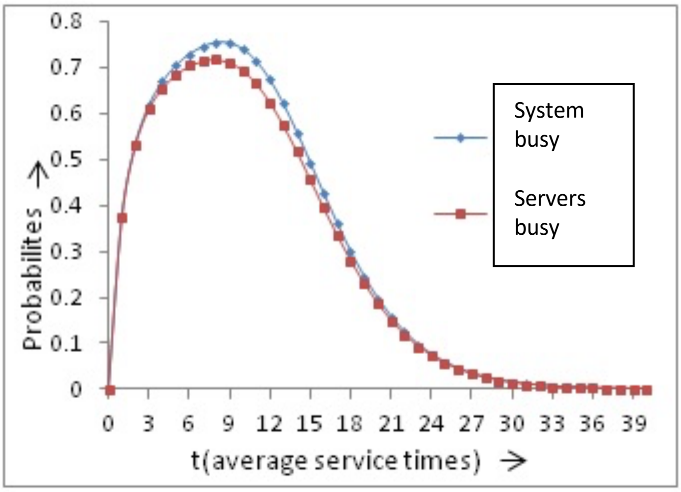

Figure 1 : Probability (system busy) and Probability (server/servers busy) against time for ρ=0.6, η=0.6

The probability (system busy) and the probability (one or both servers are busy) are plotted in figures 2 and 3 for different values of ρ for the case (η=0.6, $r_1 = 0.3$, $a_1 = 0.4$,$a_2 = 0.6$). From these figures it is clearly visible for higher values of value of ρ both the probabilities achieved greater highest values for some t, but this trend reverses for higher values of t.

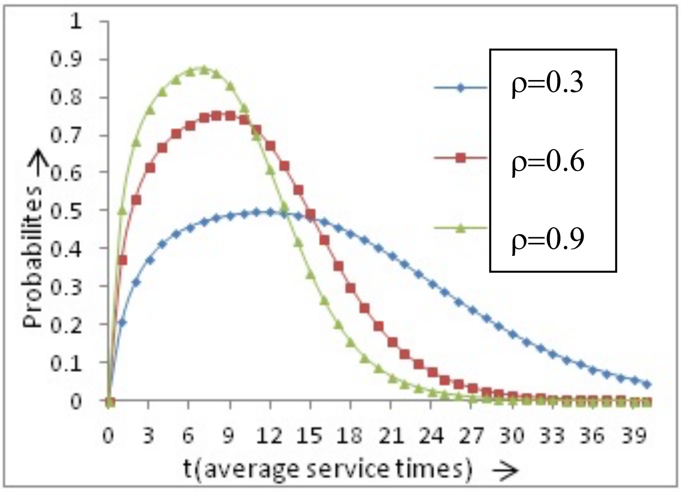

Figure 2: Effect of ρ on probability (system busy) against time

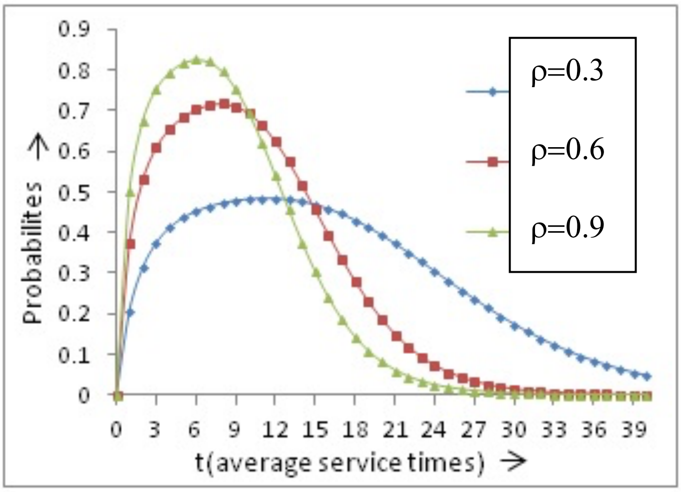

Figure 3: Effect of ρ on probability (servers busy) against time

VI. Conclusion

This paper considers a two state retrial queueing system having two non- identical parallel servers, which can be used in practical modeling of computer and communication systems. The transient state solution of the model is obtained and some measures of performance are derived. Due to the two-dimensional nature of the model under study, factors are clearly understood and well quantified. Further, the model can be converted into a model with the total number of customers in the system. Numerical results and busy period distribution demonstrate the influence of changing arrival rate on behavior of the system.

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