RT&A, No 1 (67)
Volume 17, March 2022

Costs of Age Replacement under Accelerated Life Testing with Censored Information

[1,*]Intekhab Alam, [2]Mohd Asif Intezar, [3]Lalit Kumar Sharma,
Mohammad Tariq Intezar[4] , Aqsa Irfan[5]

•

[1,2,3]Department of Management, St. Andrews Institute
of Technology & Management, Gurugram, Delhi (NCR), India
[4]Department of Law & Management
GD Goenka University, Gurugram, Haryana, India
[5]Department of Geology, Aligarh Muslim University, Aligarh, India
[1,*]Email-intekhab.pasha54@gmail.com, [2]Email-ap11_mgmt@saitm.org, [3]Email-dean_mgmt@saitm.org, [4]Email-m.tariq@gdgu.org, [5]Email-aqsairfan.amu@gmail.com
*Corresponding author

Abstract

Accelerated life testing (ALTg) helps manufacturers to predict the various costs associated with the product under the warranty policy. The main aim of undertaking ALTg is the extended time of today's manufactured goods, the small-time among design and make public, and the difficulty of analysis of items that are continuously used in ordinary environments. Hence ALTg is used to offer quick information about the life distribution of products. We describe how to propose and analyze the accelerated life testing plans to develop the excellence and reliability of the item for consumption. We also focus on finding the expected cost rate and the expected total cost for age replacement in the pro-rate rebate warranty plan. The problem is studied using constant stress, under the hypothesis that the life spans of the units follow the Gompertz distribution (GD) for predicting the cost of age replacement in the warranty plan. The asymptotic variance and covariance matrix, confidence intervals for parameters, and respective errors are also obtained. A simulation study is carried out to show the statistical properties of distribution parameters.

Keywords: Accelerated Life Testing, Gompertz distribution, Warranty policy, Age-replacement, Type-I Censoring, Fisher Information matrix, Simulation Study.

I. Introduction

Nowadays, most producers are doing their finest to build up and get better the performance of their items to boost the requirement and increase faith between them and their purchaser. The producers face several disputes while developing manufactured goods, together with complicatedness in scheming the failure of the manufactured goods during the existing investigation era. To defeat many of the difficulties in normal reliability, testing ALTg techniques may be employed. It is significant to get better the performance of the manufactured goods, work towards the improvement of the item and conclude the issues that cause the undersized lifetime. Quantitative ALTg engages in identifying stress situations that will speed up the stoppage manner. Therefore, the failures may be observed in a shorter phase. The accelerated investigation situations may engage a superior stage of force, power, weight, speed, temperature, vibration, etc., and more than one stress may be operated depending on the item's nature. Information composed at such accelerated circumstances is then extrapolated during a bodily suitable statistical representation to estimate the life span distribution at ordinary use circumstances.

In life testing analysis and reliability theories, the engineer may not constantly get absolute information on failure epochs for all investigational components. Information achieved from such researches is called censored information. The main reasons for censoring schemes are reducing the total time on investigation and the expenditure related to it. The censoring can make stability between total time used up for the experimentation, the number of components used in the test and the effectiveness of statistical assumption based on the experiment's outcomes. Type-I (time) censoring and Type- II (item) censoring are the frequent schemes. Here, we are only focusing on Type-I censoring scheme. Type I censoring occurs when an experiment finishes after a preset amount of time. There are several other censoring schemes, i.e., multiple censoring, progressive censoring, hybrid censoring and adaptive progressive hybrid censoring, etc.

According to Nelson [1], and Rao [2], the accelerated life test (ALT) is generally of three types. The first type is named the constant-stress ALT (CSALT). The next is the step-stress accelerated life test (SSALT). The third is progressive-stress ALT (PSALT). The stress is set aside at a constant level during the analysis in CSALT. In SSALT, the investigation circumstance varies at a known time or upon the happening of a specific number of failures. The stress practical to examination manufactured goods is constantly increasing with the point in PSALT. For an extensive review on these methods, (see Kim and Bai [3], AL-Hussaini and Abdel-Hamid [4, 5], Miller and Nelson [6]). These three techniques can decrease the testing time and accumulate a lot of human resources, material, and capital.

The key statement in ALT is that the mathematical model connecting the life span of the element and the stress is identified or can be assumed. In various situations, such life stress relationships are unknown and cannot be assumed, i.e. the information achieved from ALT cannot be extrapolated to use situation. So, in such situations, one more advanced method can be applied, which is partially accelerated life tests (PALTs). If the acceleration factor cannot be assumed as a known value, then PALT will be an excellent selection to carry out the life investigation. In PALTs, objects are experienced at both accelerated and use circumstances. PALTs: constant-stress PALT (CSPALT), step-stress PALT (SSPALT) and progressive-stress PALT (PSPALT) are the three frequently used types.

A rebate warranty policy is one of the mainly widespread types of warranty strategies. In a rebate plan, the seller refunds a customer some proportion of the sales worth if the manufactured goods are unsuccessful during the warranty era. Frequent examples of goods sold under rebate plans include batteries and tires. Objects sold under failure-free warranties might contain electronics and household machines. Rebate strategies also take two widespread forms: lump sun, and pro-rata rebates. If an article fails before the conclusion of the substance's warranty age, it is replaced or repaired as per common, however, only for the amount based on a price that depends on the age of the article at the point of failure. Basically, a pro-rate warranty decreases the value of your buy over time. This type of warranty is sometimes called a partial warranty since only a part of the original cost is covered. The producers shall only give incentives when they have the trust in the goods that their item has the capability to serve at least in the stated warranty period. Therefore, manufacturers need to test the reliability and performance of the goods before letting them serve in the marketplace. This can be done by using accelerated life testing on goods. Accelerated life testing also helps producers to predict the various costs connected with the item under the warranty policy.

Now, we present brief literature on ALT and warranty policies related to our study. El-Dessouky [7,8] described ALTg and age-replacement policy under warranty plan and also described maintenance service strategy under SSPALT using Type-II censored samples. Zhao and Xie [9] provided a structure to calculate the assurance outlay and risk under one-dimensional. At present, the two-dimensional and extended warranty has taken an important place in warranty policy analysis (Jack et al. [10], Gupta et al. [11], Huang et al. [12], Jung et al. [13], and Ye and Murthy [14]). However, it is very tough and tricky to design warranty plans and calculate warranty costs for new goods that have not been come into the market, because the failure rate of such types of products is not available. A literature review is presented by Murthy and Djamaludin [15] on new item assurance by considering marketing, logistics, etc. A combined optimization method that concerned trustworthiness, service contract and price for new goods is presented by Huang et al. [16]. Xie and Ye. [17] proposed a comprehensive inexpensive guarantee cost forecast under the new item. In Yang [18], optimal 3-level accommodation ALT affairs were talked about to minimize the asymptotic about-face of best likelihood appraisal of the assurance cost. For an overview of accelerated believability experiments, one can refer to Meeker and Escobar [19]. Borgia et al. [20] presented a case study for the household's gadgets. Yang [21] proposed a technique for item population for predicting the warranty outlay and its confidence interval. Alam et al. [22] offered a study on age replacement policy under pro rebate warranty policy for Burr Type-X failure model using Type-II censoring scheme. Currently, Alam and Aquil [23] presented a study on SSPALT and provided its application in maintenance service policy for the generalized inverted exponential distribution. Alam et al. [24] handled constant-stress ALT under a progressive Type-II censoring scheme and also presented its application in the area of maintenance strategy. Almalki et al. [25] handled with constant stress ALTs model for Kumaraswamy failure model under the progressive censoring scheme.

In this work, we design ALT under Type-I censoring for GD and also provide the application of ALT in the field of warranty policy and this is the key factor of this study. In previous studies, a lot of study is available on ALT with different censoring schemes for different lifetimes model but few studies available that provide its application in the field of warranty policy The novelty of this study is that no earlier study is available for GD under pro-rata rebate warranty policy under Type-I censoring scheme.

The rest of the paper is organized as follows: Section 2 provides the introduction of GD and test procedure. The likelihood function, Fisher Information matrix, the inverse of Fisher Information matrix is developed in section 3. A simulation study is carried out in section 4. Estimation of shape parameter and reliability function is presented in section 5. The age-replacement policy for GD under pro-rate rebate warranty is presented in Section 6. Finally, a conclusion is made in section 7.

II. Model Description and Test Method

ALT is generally performed by one of the two approaches; (i) accelerated failure time, which means ALT is performed for the item by experiencing usual circumstances but more intensively than ordinary. This approach is excellent for items or components that are exercised on a continuous-time basis. (ii) Accelerated stress means ALT is performed with items or components at higher stress than usual. For designing ALT plans, the following points are needed

(i) The stress application testing process.

(ii) The stress levels selected and the type of stress to be applied in the investigation for each stress type.

(iv) Relationship between life and stress which is expressed by the mathematical model.

(v) At last, the proportion of experiment elements to be allocated to every level of stress.

Some authors dealt with constant stress, such as Abdel-Ghaly [26] tackled with an accelerated life test plan for the Pareto failure model and estimate reliability function and parameters of the model. Attia et al. [27,28] handled with Accelerated life test plan for Birnbaum-saunders and Generalized Logistic distributions using different censoring plans with constant stress. In the following section:

(i) Stress G_j has k-levels.

(ii) Assuming that G_u is normal use situation and fulfilling $G_u < G_1 < G_2 < G_3 < ... < G_k$.

(iii) There are m_j units put on the investigation at every stress stage.

(iv) The test ends when r_j units attain among these s_j units.

This current study is dealt with Type-I censoring and constant stress with the assumption that the lifetime of the units follows the GD.

GD has wide popularity in relating human mortality, establishing actuarial tables, and other areas. Historically, it was firstly commenced by Gompertz [29]. GD has the following probability density function (pdf) and cumulative distribution function (cdf);

The pdf of the model is given as

$$f(x_{ij},\tau_j,\gamma)=\tau_j\gamma e^{x_{ij}\gamma} e^{-\tau_j\left(e^{x_{ij}\gamma}-1\right)},\quad x_{ij}>0,\tau_j>0,\gamma>0 \tag{1}$$

where γ and τ_j are scale and shape parameters, respectively.

The cdf of the model is given as

$$F(x_{ij},\tau_j,\gamma)=1-e^{-\tau_j\left(e^{x_{ij}\gamma}-1\right)}, x_{ij}>0,\tau_j>0,\gamma>0 \tag{2}$$

The Reliability function of the model is given as

$$S(x_{ij},\tau_j,\gamma)=e^{-\tau_j\left(e^{x_{ij}\gamma}-1\right)} \tag{3}$$

The Hazard function of the model is given as

$$h(x_{ij},\tau_j,\gamma)=\tau_j\gamma e^{x_{ij}\gamma} \tag{4}$$

The pdf, cdf, Reliability, and hazard curves are shown in Figure1, Figure 2, Figure3, and Figure4, respectively.

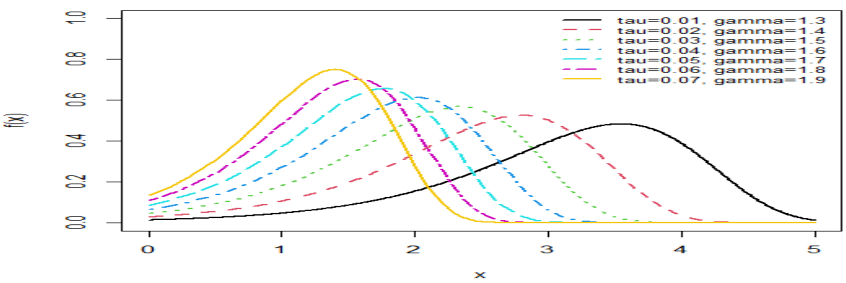

Figure 1: Probability density curve of GD

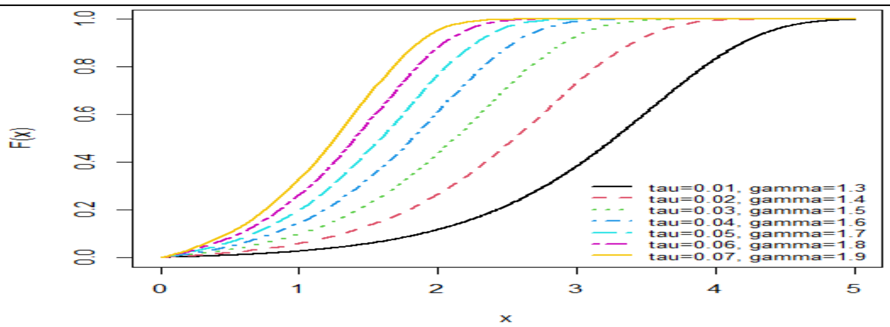

Figure2: Cumulative distribution curve of GD

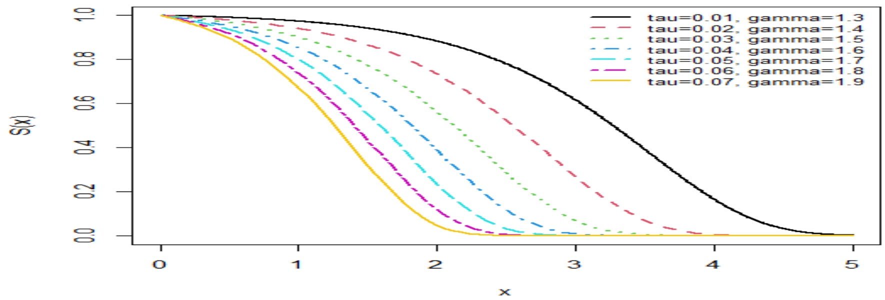

Figure3: Reliability curve of GD

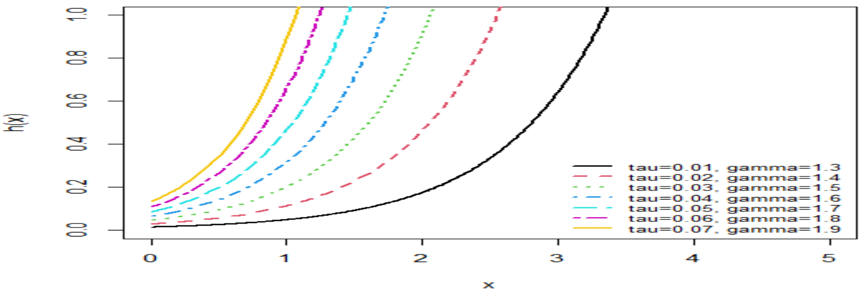

Figure4: Hazard curve of GD

GD possesses a unimodal pdf and has an increasing hazard curve for increasing values of τ and γ . Willekens [30] handled the associations of GD with other failure models. Wu et al. [31] proposed the weighted and unweighted least squares estimations for GD under censored and complete information. Chang and Tsai [32] intended the maximum likelihood estimates (MLEs), and accomplished the establishment for the exact confidence interval. Mohie El-Din [33] presented a study under generalized progressively hybrid censoring for GD.

This study is based on constant stress and Type-I censoring scheme. We have considered the Stress $G_j, j = 1,2,...,k$ which affects the shape parameter of the used distribution, μ_j through the following equation (5) called the power rule model.

$$\tau_j = RG_j^{-f}\ ; \quad R > 0\,, f > 0\,, \; j = 1,2,...,k \tag{5}$$

where R and f are the proportionality constant; and power of the applied stress respectively.

III. Estimation Process

In this section, the maximum likelihood (ML) estimation method is used. The reliability practitioner used this method because it is very robust and provides estimates of the parameters with excellent properties. At the stress level G_j, the authors constructed the likelihood function of an observation x (time to failure) and at each stress level G_j, s_j units were put on the test.

Therefore, $N=\sum_{j=1}^{k} s_j$ is the total number of components in the test. When a Type-I (time) censoring scheme is adopted at each stress level, the experiment ends once the censoring time "x_0"is reached. It is assumed that $r_j(\le s_j)$ units are observed at the jth stress level before the test is terminated and $(s_j - r_j)$ units still carry on till the end of the analysis. In this situation, the likelihood function of the testing for GD model is taking the following form

$$L(x_{ij}, R, \gamma, f)=\prod_{j=1}^{k}\frac{s_j}{(s_j - r_j)!}\left[\prod_{i=1}^{r_j} f(x_{ij}, R, \gamma, f)\right][1-F(x_0)\ ]^{s_j - r_j} \tag{6}$$

where x_0 is the time of cessation of the experiment and $\ln L(x_{ij}, R, \gamma, f)=\ln L$.

The log-likelihood function is obtained by taking the logarithm of the above equation (6) and given as

$$\ln L = K+\sum_{j=1}^{k} r_j \ln\left(RG_j^{-f}\gamma\right)+\gamma\sum_{i=1}^{r_j}\sum_{j=1}^{k} x_{ij}-\sum_{i=1}^{r_j}\sum_{j=1}^{k} RG_j^{-f}\left(e^{\gamma x_{ij}}-1\right) + \sum_{j=1}^{k}(s_j - r_j)RG_j^{-f}\left(e^{\gamma x_0}-1\right) \tag{7}$$

where K is the constant.

The ML estimates of γ, R and f can be estimated from the following three equations.

$$\frac{\partial \ln L}{\partial \gamma}=\sum_{j=1}^{k} r_j \gamma^{-1}+\sum_{i=1}^{r_j}\sum_{j=1}^{k} x_{ij}-\sum_{i=1}^{r_j}\sum_{j=1}^{k} R\, x_{ij} e^{\gamma x_{ij}} G_j^{-f}+\sum_{j=1}^{k}(s_j - r_j)RG_j^{-f} x_0 e^{\gamma x_0}=0$$

$$\frac{\partial \ln L}{\partial f}=-\sum_{j=1}^{k} r_j \ln(G_j)+\sum_{i=1}^{r_j}\sum_{j=1}^{k} RG_j^{-f}\ln(G_j)\left(e^{\gamma x_{ij}}-1\right)- \sum_{j=1}^{k}(s_j - r_j)RG_j^{-f}\ln(G_j)\left(e^{\gamma x_0}-1\right)=0$$

$$\frac{\partial \ln L}{\partial R}=\sum_{j=1}^{k} r_j R^{-1}-\sum_{i=1}^{r_j}\sum_{j=1}^{k} G_j^{-f}\left(e^{\gamma x_{ij}}-1\right)+\sum_{j=1}^{k}(s_j - r_j)G_j^{-f}\left(e^{\gamma x_0}-1\right)=0$$

It looks impossible to solve the above three non-linear equations manually. Therefore, an iterative technique (Newton-Raphson) can be used to get maximum likelihood estimates (MLEs) of parameters.

In mathematical statistics, the Fisher information or Information matrix is a technique of evaluating the amount of information that an observable random variable carries about an unknown parameter of a distribution. Simply, the Information matrix is the variance of the score or the expected value of the observed information.

The Information matrix for the Gompartz failure model under Type-I censoring is given as

$$F = \begin{bmatrix} \frac{-\partial^2 \ln L}{\partial \gamma^2} & \frac{-\partial^2 \ln L}{\partial \gamma \partial f} & \frac{-\partial^2 \ln L}{\partial \gamma \partial R} \\ \frac{-\partial^2 \ln L}{\partial f \partial \gamma} & \frac{-\partial^2 \ln L}{\partial f^2} & \frac{-\partial^2 \ln L}{\partial f \partial R} \\ \frac{-\partial^2 \ln L}{\partial R \partial \gamma} & \frac{-\partial^2 \ln L}{\partial R \partial f} & \frac{-\partial^2 \ln L}{\partial R^2} \end{bmatrix} \tag{8}$$

The elements of the matrix F are obtained by second partial derivatives of log-likelihood function with respect to parameters λ, q and U. Consequently, the elements are expressed by the following equations

$$\frac{\partial^2 \ln L}{\partial \gamma^2} = -\sum_{j=1}^{k} r_j \gamma^{-2} - \sum_{i=1}^{r_j}\sum_{j=1}^{k} R\, x_{ij}^2 e^{\gamma x_{ij}} G_j^{-f} + \sum_{j=1}^{k}(s_j - r_j) R G_j^{-f} x_0^2 e^{\gamma x_0}$$

$$\frac{\partial^2 \ln L}{\partial f^2} = -\sum_{i=1}^{r_j}\sum_{j=1}^{k} R G_j^{-f} \ln^2(G_j)\left(e^{\gamma x_{ij}} - 1\right) + \sum_{j=1}^{k}(s_j - r_j) R G_j^{-f} \ln^2(G_j)\left(e^{\gamma x_0} - 1\right)$$

$$\frac{\partial^2 \ln L}{\partial R^2} = -\sum_{j=1}^{k} r_j R^{-2}$$

$$\frac{\partial^2 \ln L}{\partial f \partial \gamma} = +\sum_{i=1}^{r_j}\sum_{j=1}^{k} R G_j^{-f} \ln(G_j)\left(x_{ij} e^{\gamma x_{ij}}\right) - \sum_{j=1}^{k}(s_j - r_j) R G_j^{-f} \ln(G_j)\left(x_0 e^{\gamma x_0}\right)$$

$$\frac{\partial^2 \ln L}{\partial f \partial R} = \sum_{i=1}^{r_j}\sum_{j=1}^{k} G_j^{-f} \ln(G_j)\left(e^{\gamma x_{ij}} - 1\right) - \sum_{j=1}^{k}(s_j - r_j) G_j^{-f} \ln(G_j)\left(e^{\gamma x_0} - 1\right)$$

$$\frac{\partial^2 \ln L}{\partial \gamma \partial R} = -\sum_{i=1}^{r_j}\sum_{j=1}^{k} x_{ij} e^{\gamma x_{ij}} G_j^{-f} + \sum_{j=1}^{k}(s_j - r_j) G_j^{-f} x_0 e^{\gamma x_0}$$

Now, the variance-covariance matrix is the inverse of the Fisher Information matrix and given as

$$\Sigma = F^{-1} \tag{9}$$

$$\Sigma = \begin{bmatrix} \frac{-\partial^2 \ln L}{\partial \gamma^2} & \frac{-\partial^2 \ln L}{\partial \gamma \partial f} & \frac{-\partial^2 \ln L}{\partial \gamma \partial R} \\ \frac{-\partial^2 \ln L}{\partial f \partial \gamma} & \frac{-\partial^2 \ln L}{\partial f^2} & \frac{-\partial^2 \ln L}{\partial f \partial R} \\ \frac{-\partial^2 \ln L}{\partial R \partial \gamma} & \frac{-\partial^2 \ln L}{\partial R \partial f} & \frac{-\partial^2 \ln L}{\partial R^2} \end{bmatrix}^{-1} = \begin{bmatrix} AVar(\hat{\gamma}) & ACov(\hat{\gamma}\,\hat{f}) & ACov(\hat{\gamma}\,\hat{R}) \\ ACov(\hat{f}\,\hat{\gamma}) & AVar(\hat{f}) & ACov(\hat{f}\,\hat{R}) \\ ACov(\hat{R}\,\hat{\gamma}) & ACov(\hat{R}\,\hat{f}) & AVar(\hat{R}) \end{bmatrix} \tag{10}$$

$AVar$, $ACov$ are asymptotic variance and asymptotic covariance, respectively.

The $100(1-\pi)\%$ approximated two-sided limits of confidence for parameters λ, U and q are given as

$$\hat{\gamma} \pm Z_{\pi/2} \sqrt[-1]{I_{11}^{-1}(\hat{\gamma},\hat{f},\hat{R})},\ \hat{f} \pm Z_{\pi/2} \sqrt[-1]{I_{22}^{-1}(\hat{\gamma},\hat{f},\hat{R})} \text{ and } \hat{R} \pm Z_{\pi/2} \sqrt[-1]{I_{33}^{-1}(\hat{\gamma},\hat{f},\hat{R})}$$

$Z_{\pi/2}$ is the $100(1-\pi/2)\%$ percentile of a standard normal variate.

IV. Simulation Results

In this section, we apply the Monte-Carlo simulation technique to examine the performances of the MLEs through their absolute relative bias (RAB) and mean square error (MSE). Using the invariance property of MLEs, we can estimate the MLEs of shape Parameter τ_j through the following expression;

$$\tau_j = RG_j^{-f}\ ;\ \ R>0\,, f>0\ ,\ \ j=1,2,...,k$$

The detailed steps are given below:

1. First, 1000 random samples of sizes 25, 50, 75, and 100 are generated from GD by inverse CDF method. Different initial values are selected for all sets of parameters.
2. The stress has three levels and the values of are $\left(G_1 = 1, G_2 = 1.5, G_3 = 2\right), s_j = \dfrac{n}{3}$ & $r_j = 60\% n_j$, n is sample size.
3. The parameters of the model are estimated under Type-I censoring for all sample sizes.
4. The Newton-Raphson method is applied for solving all non-linear equations.
5. The estimates of the shape parameter μ_j are calculated from equation (5).
6. The RABs and MSEs are tabulated for all sets of $\left(\gamma_0\,, R_0\,, f_0\right)$.
7. We determine the MLEs of the scale parameter γ_u at the usual stress level $G_u = 0.5$ by the invariance property of MLEs,
8. The reliability function at the similar usual stress for various values γ, G, f and t_0 is calculated.

$$\hat{R}_u(t_0) = e^{-\tau_0\left(e^{t_0\gamma_0}-1\right)}$$

9. At mission time $(t_0 = 1.2, 1.5, 1.9)$, the MLEs of reliability function are predicted in the same usual circumstances for every parameter set.

Table 1:The Estimates, Relative Bias and MSE of the parameters $\left(\gamma, G, f, \tau_1, \tau_2, \tau_3\right)$under Type-I censoring

n	Parameters	$(\gamma_0 = 0.40, G_0 = 1.7, f_0 = 1)$			$(\gamma_0 = 1.2, G_0 = 1.7, f_0 = 1)$		
		Estimator	RABs	MSEs	Estimator	RABs	MSEs
25	γ	2.543	0.109	0.095	2.633	0.096	0.078
	G	2.765	0.116	0.097	1.873	0.109	0.074
	f	1.987	0.078	0.069	2.162	0.120	0.101
	τ_1	2.126	0.084	0.077	1.876	0.091	0.067
	τ_2	1.987	0.105	0.086	2.998	0.087	0.760
	τ_3	1.376	0.105	0.098	2.087	0.090	0.081

50	γ	2.830	0.086	0.076	1.876	0.098	0.076
	G	2.543	0.067	0.055	2.146	0.109	0.061
	f	2.791	0.077	0.061	1.998	0.123	0.097
	τ_1	1.162	0.054	0.040	1.989	0.094	0.051
	τ_2	2.788	0.054	0.031	1.360	0.080	0.042
	τ_3	2.197	0.049	0.055	1.786	0.076	0.057
75	γ	2.139	0.044	0.042	2.345	0.069	0.061
	G	2.349	0.054	0.044	2.492	0.050	0.051
	f	2.046	0.065	0.054	2.052	0.072	0.086
	τ_1	1.556	0.046	0.035	2.528	0.042	0.047
	τ_2	2.290	0.048	0.040	1.404	0.039	0.037
	τ_3	1.196	0.041	0.022	1.967	0.011	0.050
100	γ	2.252	0.018	0.029	2.763	0.064	0.050
	G	2.612	0.023	0.013	2.160	0.042	0.023
	f	1.313	0.113	0.010	1.885	0.045	0.031
	τ_1	1.950	0.027	0.014	1.443	0.062	0.069
	τ_2	2.322	0.031	0.023	2.111	0.018	0.016
	τ_3	1.089	0.032	0.023	1.025	0.033	0.025

Table 2: The Estimates, Relative Bias and MSE of the parameters $(\gamma, G, f, \tau_1, \tau_2, \tau_3)$under Type-I censoring

n	Parameters	$(\gamma_0 = 0.40, G_0 = 1.3, f_0 = 1)$			$(\gamma_0 = 1.2, G_0 = 1.3, f_0 = 1.9)$		
		Estimator	RABs	MSEs	Estimator	RABs	MSEs
25	γ	1.846	0.132	0.093	2.987	0.132	0.119
	G	1.425	0.123	0.110	1.171	0.115	0.099
	f	2.927	0.110	0.099	2.621	0.083	0.061
	τ_1	1.901	0.083	0.062	1.523	0.110	0.081
	τ_2	2.170	0.092	0.075	2.210	0.097	0.078
	τ_3	1.983	0.082	0.065	2.809	0.123	0.106
50	γ	2.452	0.062	0.077	1.667	0.071	0.075
	G	1.744	0.092	0.072	1.158	0.144	0.070
	f	1.837	0.073	0.070	1.136	0.052	0.095
	τ_1	1.164	0.081	0.058	1.283	0.074	0.061
	τ_2	1.786	0.188	0.065	1.170	0.069	0.061
	τ_3	1.564	0.068	0.065	2.986	0.088	0.079
75	γ	1.639	0.054	0.045	2.998	0.063	0.062
	G	1.919	0.089	0.063	2.791	0.081	0.056
	f	1.778	0.052	0.050	1.990	0.050	0.066
	τ_1	2.061	0.086	0.061	2.930	0.066	0.055
	τ_2	1.398	0.070	0.047	1.132	0.054	0.016
	τ_3	1.494	0.059	0.076	2.998	0.077	0.098

100	γ	1.052	0.030	0.037	1.997	0.049	0.056
	G	2.168	0.060	0.023	1.439	0.055	0.041
	f	2.016	0.043	0.061	1.967	0.042	0.055
	τ_1	1.192	0.050	0.019	1.209	0.055	0.066
	τ_2	1.921	0.059	0.033	1.432	0.032	0.014
	τ_3	2.595	0.042	0.016	2.955	0.030	0.084

V. Estimation of the survival functions and shape parameter at normal stress

In the following table, we estimate the survival function at the usual stress level $G_u = 0.5$ for various values of parameters $\gamma, G,\ f$ and t_0 ,also find the shape parameter for the same stress level.

Table3: Estimated reliability functions and shape parameter at normal stress

γ_0	G_0	f_0	τ_0	t_0	$R_u(t_0)$
0.40	1.7	1	3.332	1.2	0.543
				1.5	0.523
				1.9	0.498
1.2	1.7	1	2.987	1.2	0.423
				1.5	0.436
				1.9	0.397
0.40	1.3	1	3.165	1.2	0.754
				1.5	0.704
				1.9	0.723
1.2	1.3	1.9	2.876	1.2	0.676
				1.5	0.643
				1.9	0.612

IV. The Age-Replacement Policy under Pro-rate Rebate Warranty for GD

Under this warranty policy, the following assumptions are made

(i) A non-repairable product is replaced at a certain time δ or upon failure, which takes place earliest.

(ii) When the product is unsuccessful at the time $t \le \delta$, a failure replacement is carried out with a purchasing cost Q_p and downtime cost Q_d, where $Q_p, Q_d > 0$.

(iii) The client is reimbursed by a quantity of sales price Q_p if the item be unsuccessful over the warranty period (w_a),

So, the rebate function in the pro-rata warranty is:

$$R(t) = \begin{cases} Q_p\left(1 - \dfrac{t}{w_a}\right) & 0 \le t \le w_a \\ 0 & t > w_a \end{cases} \tag{11}$$

John Mamer [34] handled with a price tag investigation of pro-rata with a without charge replacement warranty approaches; he studied the long-run typical and total expenses of things with warranty. Timothy et al. [35] tackled a pro-rata study for joint warranty problems; he used several repair selections in his investigation. Huang et al. [36] presented a study on estimating the predictable warranty cost for the approach where the item usage is intermittent and of heterogeneous usage intensity by the item existence cycle when sales occur regularly.

Key Assumptions:
The main assumptions in this policy are

1. Product is replaced at the point of failure (corrective replacement), or age δ (preventive replacement), which arrives earliest.
2. The pro-rata rebate warranty approach is applied to the sale of products.
3. There is no salvage charge for the preventive replaced item.
4. The warranty time is less than age replacement, i.e. $w_a < \delta$.

When the item's life arrives δ , then the preventive replacement is carried out with cost Q_p only because it is a planned preventive safeguarding action.

The total cost incurred in a renewal cycle for this strategy is:

$$C(d) = \begin{cases} Q_d + Q_p - R(t) & 0 \le t \le w_a \\ Q_d + Qp & w_a < t < \delta \\ Qp & t \ge \delta \end{cases} \tag{12}$$

The expected total cost (Chien [37], Chien et al. [38]) under this policy is given by:

$$E\left(C_p(t)\right) = Q_d F(\tau) + Q_p \frac{\int_0^{w_a} \overline{F}(u)du}{w_a} \tag{13}$$

The expected cost rateis

$$E(CR(t)) = \frac{E\left(C_p(t)\right)}{\int_0^{\tau} \overline{F}(u)du} \tag{14}$$

where $\int_0^{\delta} \overline{F}(u)du$ is the expected cycle time, which is represented by $E(T(\delta))$.

Under the GD, the cdf is, $F(u,\tau,\gamma) = 1 - e^{-\tau\left(e^{u\gamma} - 1\right)}$, $\quad u > 0, \tau > 0, \gamma > 0$

$$\text{So,} \quad \int_0^{w_a} \overline{F}(u)du = w_a - \int_0^{w_a} \left[1 - e^{-\tau\left(e^{u\gamma} - 1\right)}\right]du$$

$$\text{And,} \quad \int_0^{\delta} \overline{F}(u)du = \delta - \int_0^{\delta} \left[1 - e^{-\tau\left(e^{u\gamma} - 1\right)}\right]du$$

We can get the expected total cost and expected cost rate for the non-repairable product using the above values in equations (13) and (14).

For example, if the failure replacement is carried out with a downtime cost $Q_d = 80$ and purchasing cost $Q_p = 1200$. The expected total cost, expected cycle time and expected cost rate are estimated for age-replacement under warranty policy for several values of warranty periods w_a and the parameters of GD γ and τ at normal use.

Table 4: The expected total cost, the expected cycle time and the expected cost rate

τ	γ	w_a	δ	**Expected Total Cost** $E(C_p(\delta))$	**Expected Cycle Time** $E(T(\delta))$	**Expected Cost Rate** $CR(\delta)$
0.8	6	9.2	9.7	1045.87	5.89	342.87
0.8	6	9.2	8.5	1011.68	5.54	432.98
0.9	6	9.2	9.7	998.55	5.14	412.87
0.9	6	9.2	8.5	856.95	3.97	498.87
1.6	6	9.2	9.7	829.23	4.25	416.34
1.6	8	9.2	9.2	929.98	4.96	521.16
1.6	8	9.2	8.5	959.98	4.34	587.76
1.6	9	9.2	9.7	975.64	7.18	506.90
1.6	9	9.7	8.9	1056.97	7.35	578.74
1.6	9	9.7	9.7	1022.89	8.17	507.71
1.6	9	9.7	9.8	1035.87	8.67	421.76

VII. Conclusion

In this study, the accelerated life test plan is designed under the Type-I censoring scheme when the lifetime of test items follow the Gompertz failure model and also provided its application in the field of the warranty policy. The following observations are made on the basis of this study;

From the Table (1) and (2), increases in the sample size lead to a decrease in the mean square error and absolute relative bias. So, the asymptotically normally distributed and consistent estimators are provided by MLEs.

From Table (3), an inverse relationship is developed between mission time and the reliability function. From Table (4), the expected total cost and expected time cycle are inversely related to the parameter's value, while the expected cost rate is directly related to the value of the parameter. The expected total cost and expected cycle time are directly related to the value of the parameter, while the expected cost rate is inversely related to the value of the parameter. Increases in the warranty resulted in decreases in the expected total cost and expected cost rate and also doesn't affect the expected life cycle.

Finally, an inverse relationship is developed between the age of replacement and the expected cost rate, while direct relationships are developed between the expected total cost and expected time cycle.

In the future, this work can be extended with different censoring schemes for other failure models. The application of ALT can be done with other warranty policies under the Bayesian approach in the extended work.

References

[1] Nelson, W. Accelerated Life Testing: Statistical Models, Data Analysis and Test Plans. Wiley, New York, 1990.

[2] Raja Rao, B. (1992). Equivalence of the tampered random variable and the tampered failure rate models in accelerated lifetesting for a class of life distributions having the 'setting the clock back to zero property.*Communications in Statistics-Theory and Methods,* 21(3), 647-664.

[3] Kim, C.M. and Bai, D.S. (2002).Analysis of accelerated life test data under two failure modes.*International Journal of Reliability, Quality and Safety Engineering*, 9, 111–125.

[4] AL-Hussaini, E.K. and Abdel-Hamid, A.H. (2004). Bayesian estimation of the parameters, reliability and hazard rate functions of mixtures under accelerated life tests. *Communications in Statistics - Simulation and Computation,* 33(4), 963–982.

[5] AL-Hussaini, E.K. and Abdel-Hamid, A.H. (2006).Accelerated life tests under finite mixture models,*Journal of Statistical Computation and Simulation*,76, 673–690.

[6] Miller, R., Nelson, W.B. (1983). Optimum simple step-stress plans for accelerated life testing. *IEEE Transactions on Reliability*, 32, 59–65.

[7] El-Dessouky, E. A. (2015). Accelerated life testing and age-replacement policy under warranty on exponentiatedpareto distribution. *Applied Mathematical Sciences*, *9*(36), 1757-1770.

[8] El-Dessouky, E. A. (2017). Estimating costs of maintenance service policy using step stress partially accelerated life testing for the extension of exponential distribution under type-ii censoring. *International Journal of Contemporary Mathematical Sciences,5*(36), 225 – 242.

[9] Zhao, X., and Xie, M. (2017). Using accelerated life tests data to predict warranty cost under imperfect repair. *Computers & Industrial Engineering*, *107*, 223-234.

[10] Jack, N., Iskandar, B. P., and Murthy, D. P. (2009). A repair–replace strategy based on usage rate for items sold with a two-dimensional warranty. *Reliability Engineering & System Safety*, *94*(2), 611-617.

[11] Gupta, S. K., De, S., and Chatterjee, A. (2014). Warranty forecasting from incomplete two-dimensional warranty data.*Reliability Engineering & System Safety*, *126*, 1-13.

[12] Huang, Y. S., Huang, C. D., and Ho, J. W. (2017). A customized two-dimensional extended warranty with preventive maintenance.*European Journal of Operational Research*, *257*(3), 971-978.

[13] Jung, K. M., Park, M., and Park, D. H. (2015). Cost optimization model following extended renewing two-phase warranty. *Computers & Industrial Engineering*, *79*, 188-194.

[14] Ye, Z. S., and Murthy, D. P. (2016). Warranty menu design for a two-dimensional warranty.*Reliability Engineering & System Safety*, *155*, 21-29.

[15] Murthy, D. N. P., and Djamaludin, I. (2002). New product warranty: a literature review. *International Journal of Production Economics*, *79*(3), 231-260.

[16] Huang, H. Z., Liu, Z. J., and Murthy, D. N. P. (2007). Optimal reliability, warranty, and price for new products.*Iie Transactions*, *39*(8), 819-827.

[17] Xie, W., and Ye, Z. S. (2016). Aggregate discounted warranty cost forecast for a new product considering stochastic sales. *IEEE Transactions on Reliability*, *65*(1), 486-497.

[18] Yang, G. (2010). Accelerated life test plans for predicting warranty cost.*IEEE Transactions on Reliability*, *59*(4), 628-634.

[19] Meeker, W. Q., and Escobar, L. A. (1993). A review of recent research and current issues in accelerated testing. *International Statistical Review/Revue Internationale de Statistique*, 147-168.

[20] Borgia, O., De Carlo, F., Fanciullacci, N., and Tucci, M. (2013). Accelerated life tests for new product qualification: a case study in the household appliance. *IFAC Proceedings Volumes*, *46*(7), 269-274.

[21] Yang, G. (2010). Accelerated life test plans for predicting warranty cost. *IEEE Transactions on Reliability*, *59*(4), 628-634.

[22] Alam, I., Islam, A., and Ahmed, A. (2020). Accelerated Life Test Plans and Age-Replacement Policy under Warranty on Burr Type-X distribution with Type-II Censoring. *Journal of Statistics Applications & Probability*, 9(3), 1-10.

[23] Alam, I., and Ahmed, A. (2021). Inference on maintenance service policy under step-stress partially accelerated life tests using progressive censoring. Journal of Statistical Computation and Simulation, 1-17, DOI: 10.1080/00949655.2021.1975282.

[24] Alam, I., Islam, A., and Ahmed, A. (2020). Accelerated life test plans and age-replacement policy under warranty on burr type-x distribution with type-ii censoring. *Journal of Statistics Applications & Probability*, 9(3), 1-10.

[25] Almalki J., Al-Wageh A., Manoj K., and Gamal A. (2021). Partially constant-stress accelerated life tests model for parameters estimation of Kumaraswamy distribution under adaptive Type-II progressive censoring. *Alexandria Engineering Journal*.

[26] Abdel-Ghaly, A. A., Attia, A. F., and Aly, H. M. (1998). Estimation of the parameters of pareto distribution and the reliability function using accelerated life testing with censoring.*Communications in Statistics-Simulation and Computation*, *27*(2), 469-484.

[27] Attia, A. F., Shaban, A. S., and Abd El Sattar, M. H. (2013). estimation in constant-stress accelerated life testing for birnbaum-saunders distribution under censoring. *International Journal of Contemporary Mathematical Sciences*, *8*(4), 173-188.

[28] Attia, A. F., Aly, H. M., and Bleed, S. O. (2011). Estimating and planning accelerated life test using constant stress for generalized logistic distribution under type-I censoring.*ISRN Applied Mathematics*, *2011*.

[29] Gompertz, B. (1825). On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies.*In a letter to Francis Baily, Esq. FRS &c. Philosophical transactions of the Royal Society of London*, (115), 513-583.

[30] Willekens, F. Gompertz in context: The Gompertz and related distributions. In Forecasting Mortality in Developed Countries: Insights from a Statistical, Demographic and Epidemiological Perspective; Springer: Berlin/Heidelberg, Germany, 2001; Volume 9, pp. 105–126.

[31] Wu, J.-W.; Hung, W.-L.; Tsai, C.-H. (2004). Estimation of parameters of the Gompertz distribution using the least squares method. *Applied Mathematics and Computation*, 158, 133–147.

[32] Chang, S.; Tsai, T. (2003). Point and interval estimations for the Gompertz distribution under progressive Type-II censoring.*Metron*, 61, 403–418.

[33] Mohie El-Din, M.M.; Nagy, M.; Abu-Moussa, M.H. (2019). Estimation and prediction for Gompertz distribution under the generalized progressive hybrid censored data. *Annals of Data Science*, 6, 673–705.

[34] Mamer, J. W. (1982). Cost analysis of pro rata and free-replacement warranties. *Naval Research Logistics Quarterly*,*29*(2), 345-356.

[35] Matis, T. I., Jayaraman, R., and Rangan, A. (2008). Optimal price and pro rata decisions for combined warranty policies with different repair options.*IIE Transactions*, *40*(10), 984-991.

[36] Huang H. Z., Liu Z. J., Li Y., Liu Y. and He L., (2008). A Warranty Cost model with Intermittent and Heterogeneous Usage, *Maintenance and Reliability*, No. 4, pp. 9-15.

[37] Chien, Y. H. (2010). The effect of a pro-rata rebate warranty on the age replacement policy with salvage value consideration. *IEEE Transactions on Reliability*, *59*(2), 383-392.

[38] Chien, Y. H., Chang, F. M., and Liu, T. H. (2014). The Effects of Salvage Value on the Age-Replacement Policy under Renewing Warranty. In *Proceedings of the 2014 International Conference on Industrial Engineering and Operations Management, Bali, Indonesia, January*, 7-9.