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Sharma-Mittal Entropy Properties on Generalized (k) Record Values

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Abstract

In this paper, we derive Sharma-Mittal entropy of generalized (k) record values and analyse some of its important properties. We establish some bounds for the Sharma-Mittal entropy of generalized (k) record values. We generate a characterization result based on the properties of Sharma-Mittal entropy of generalized (k) record values for the exponential distribution. We further establish some distribution-free properties of Sharma-Mittal divergence information between the distribution of a generalized (k) record value and the parent distribution. We extend the concept of Sharma-Mittal entropy to the concomitants of generalized (k) record values arising from a Farlie-Gumbel-Morgenstern (FGM) bivariate distribution. Also, we consider residual Sharma-Mittal Entropy and used it to describe some properties of generalized (k) record values.

Keywords: Generalized (k) record values, Sharma-Mittal entropy, Maximum entropy principle, Characterization, Concomitants of generalized (k) record values, Residual Sharma-Mittal entropy.

1. Introduction

In equilibrium thermodynamics, physicists originally developed the notion of entropy, which was later extended through the development of statistical mechanics. Shannon [30] introduced a generalization of Boltzmann-Gibbs entropy, and later it was known as Shannon entropy or Shannon information measure. Shannon entropy represents an absolute limit on any communication's best possible lossless compression. More generally, the concept of entropy is a measure of uncertainty associated with a random variable. For a continuous random variable X with probability density function (pdf) f, the Shannon entropy is defined by

$$H(X) = -\int_0^\infty f(x) \log f(x) \, \mathrm{d}x. \tag{1}$$

In the continuous case, $H(X)$ is also referred to as the differential entropy. It is known that $H(X)$ measures the uniformity of f. When $H(X_1) > H(X_2)$, for any two random variables with pdf f_1 and f_2 respectively, then we conclude that it is more difficult to predict outcomes of X_1, as compared with predicting outcomes of X_2 [see, 37]. One main drawback of $H(X)$ is that for some probability distributions, it may be negative and then it is no longer an uncertainty measure. This drawback is removed in the generalized entropies like Rényi entropy [29], Tsallis entropy [36] and so on.

Subsequently Sharma-Mittal entropy [31] was introduced as a two parameter measure $H_{\alpha,\beta}(X)$

of a random variable X with pdf f as

$$H_{\alpha,\beta}(X) = \frac{1}{1-\beta}\left\{\left(\int_{-\infty}^{\infty}\{f(x)\}^{\alpha}\,\mathrm{d}x\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}, \tag{2}$$

with α, $\beta > 0$, $\alpha \neq 1 \neq \beta$ and $\alpha \neq \beta$. It is clear to be note that if we take limit $\beta \to 1$ in (2) then Sharma-Mittal entropy becomes Rényi entropy [29] which is given by

$$H_{\alpha,1}(X) = \frac{1}{1-\alpha}\log\int\{f(x)\}^{\alpha}\,\mathrm{d}x. \tag{3}$$

If we take limit as $\beta \to \alpha$, in (2), then the resulting expression is Tsallis entropy [36] and is given by

$$H_{\alpha,\alpha}(X) = \frac{1}{1-\alpha}\left\{\int_{-\infty}^{\infty}\{f(x)\}^{\alpha}\,\mathrm{d}x - 1\right\}. \tag{4}$$

In the limiting case when both parameters approach 1, we recover the ordinary Shannon entropy [30] as given in (1).

One may observe several applications of Sharma-Mittal entropy from the available literature. Frank and Daffertshofer [10] have established the relation between anomalous diffusion process and Sharma-Mittal entropy. Masi [17] explained how this entropy measure unifies Rényi and Tsallis entropies. For more details on the applications of this entropy see, Aktürk et al. [4] and Kosztołowicz and Lewandowska [14]. Nielsen and Nock [21] obtained a closed-form formula for the Sharma-Mittal entropy of any distribution belonging to the exponential family of distributions.

Successive extremes occurring in a sequence of Independent and identically distributed (*iid*) random variables have been called by Chandler [8] as the record values of the sequence. Properties of record statistics arising from a distribution help to understand the intrinsic properties of the parent distribution as well. A limitation that one encounters in dealing with statistical inference problems based on classical record values is about their limited occurrence, as the expected value of inter arrival times of records is infinite [see, 11]. Also the occurrence of an outlier in a sequence of random variables arrests the subsequent realization of record values. However one may observe that generally the *kth* record values as introduced by Dziubdziela and Kopocinski [9] occur more frequently than those of the classical records. The reason for this is that the generation of the sequence of upper (k) records makes $k-1$ of the upper extreme values (outliers) of the sequence incapacitated from their occurrence in the constructed record sequence. Similar property holds with the generated sequence of lower (k) record values as well. Suppose $\{X_n\}$ is a sequence of *iid* random variables. Then for a positive integer $k \geq 1$, the sequence of upper *kth* record times $\{T_{U(n,k)}, n \geq 1\}$ is defined as [see, 20, p. 82]:-

$$T_{U(1,k)} = k,$$

and, for n≥ 1

$$T_{U(n+1,k)} = min\{j : j > T_{U(n,k)}, X_j > X_{T_{U(n,k)}-k+1:T_{U(n,k)}}\},$$

where $X_{i:m}$ denotes the i-th order statistic in a sample of size m. Now if we write

$$X_{U(n,k)} = X_{T_{U(n,k)}-k+1:T_{U(n,k)}}, \text{ for } n = 1, 2, \ldots$$

then $\{X_{U(n,k)}\}$ is known as the sequence of the *kth* upper record values. In a similar manner we can define the sequence $\{X_{L(n,k)}\}$ of *kth* lower record values as well. It is to be noted that *kth* member of sequence of the classical record values is also called as *kth* record value. This contradicts with the *kth* record values as defined in [9]. Pointing out this conflict in the usage of *kth* record values of Dziubdziela and Kopocinski [9], and as it generates the classical record values for $k = 1$, Minimol and Thomas [18, 19], Paul [22], Paul and Thomas [23, 24, 25] and Thomas and

Paul [34, 35] have called the kth record values as defined in Dziubdziela and Kopocinski [9] as the generalized (k) record values. Agreeing with the contention of above authors, we also call the kth record values of [9] as generalized(k)record values all through this paper.

Suppose $\{X_i, i \geq 1\}$ is a sequence of random variables with absolutely continuous cdf $F(x)$ and pdf $f(x)$. Let $\{X_{U(n,k)}\}$ be the sequence of GURV's generated from the sequence $\{X_i\}$. Then the pdf $f_{X_{U(n,k)}}(x)$ of $X_{U(n,k)}$ is given by [see, 6]

$$f_{X_{U(n,k)}}(x) = \frac{k^n}{\Gamma(n)} \left[-\ln\{1 - F(x)\}\right]^{n-1} \left[1 - F(x)\right]^{k-1} f(x), -\infty < x < \infty,\ n = 1, 2, \ldots. \quad (5)$$

for $n \geq 2$. In a similar manner we can define generalized lower (k) record values (GLRV's) as well. If we write $X_{L(n,k)}$ to denote the nth GLRV, then the pdf $f_{X_{L(n,k)}}(x)$ of $X_{L(n,k)}$ is given by [see, 28]

$$f_{X_{L(n,k)}}(x) = \frac{k^n}{\Gamma(n)} \left[-\ln\{F(x)\}\right]^{n-1} \left[F(x)\right]^{k-1} f(x), -\infty < x < \infty,\ n = 1, 2, \ldots. \quad (6)$$

Generalized (k) record values arise naturally in problems such as industrial stress testing, meteorological analysis, hydrology, sporting, stock markets, athletic events and seismology. Anderson et al. [5] have attributed some connection between record statistics and the strain released in quakes. Majumdar and Ziff [16] have enlisted the detailed involvement of record theory in its multiple applications in spin glasses, adaptive process, evolutionary models of a biological population. See also Sibani and Henrik [33] for some record dynamics arising in some physical systems. For more details on applications of record, values see, Arnold et al. [6], Nevzorov [20] and the references therein.

Of late several articles have been published on various information measures associated with record values. [7] studied some information properties of records based on Shannon entropy. Abbasnejad and Arghami [1] studied the Rényi entropy properties of records and compared the same information with that of the iid observations. Baratpour et al. [7], Ahmadi and Fashandi [2] and Paul and Thomas [23, 24, 26, 27] have obtained some characterization results based on Shannon, Rényi, Tsallis and Mathai-Haubold entropies of record values. Shannon information in k-records was studied by Madadi and Tata [15].

The rest of this paper is organized as follows. In section 2, we express the Sharma-Mittal entropy of nth generalized upper (k) record arising from an arbitrary distribution in terms of Sharma-Mittal entropy of nth generalized upper (k) record arising from a standard exponential distribution. Section 3 provides bounds for Sharma-Mittal entropy of generalized (k) records. Section 4 characterizes exponential distribution by maximizing Sharma-Mittal entropy of generalized (k) record values arising from a specified class of distributions. Section 5 contains expressions for some measures associated with Sharma-Mittal entropy on generalized (k) records and concomitants of generalized (k) records. In subsection 5.1, it is shown that the Sharma-Mittal divergence information between generalized (k) record value and the parent distribution is distribution-free. Section 5.2 contains the representation of Sharma-Mittal entropy of concomitants of generalized (k) record values arising from the FGM family of bivariate distributions. In section 5.3, we provide an expression for the residual Sharma-Mittal entropy of nth generalized upper (k) record arising from an arbitrary distribution in terms of the corresponding expressions for the nth generalized upper (k) record arising from a standard uniform distribution.

2. Sharma-Mittal Entropy of Generalized (k) Record Values

In this section, we describe some properties of Sharma-Mittal entropy of generalized(k)record values. In the following theorem, we express Sharma-Mittal entropy of nth generalized upper (k) record arising from an arbitrary distribution in terms of Sharma-Mittal entropy of nth generalized upper (k) record arising from standard exponential distribution. In the theorem and in the remaining part of this paper we use the notation $G(a, b)$ to denote the well known gamma distribution with pdf

$$g_{a,b}(x) = \frac{a^b}{\Gamma(b)} e^{-ax} x^{b-1}, \ a > 0, \ b > 0, \ x > 0.$$

Theorem 1. Let $\{X_i, i \geq 1\}$ be a sequence of iid continuous random variables from a distribution with cdf $F(x)$, pdf $f(x)$ and quantile function $F^{-1}(.)$. Let $\{X_{U(n,k)}\}$ be the associated sequence of generalized upper (k) record values. Then the Sharma-Mittal entropy of $X_{U(n,k)}$ can be expressed as

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n)}) &= \frac{1}{1-\beta} \left\{ \left(\frac{k^{n\alpha}\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}[(k-1)\alpha+1]^{(n-1)\alpha+1}} \right.\right. \\ & \left.\left. \times E_{g_{(k-1)\alpha+1,(n-1)\alpha+1}} \left[\left\{ f\left(F^{-1}(1-e^{-U})\right) \right\}^{\alpha-1} \right] \right)^{\frac{1-\beta}{1-\alpha}} - 1 \right\}, \end{aligned} \tag{7}$$

where U is a random variable, with $G((k-1)\alpha+1, (n-1)\alpha+1)$ distribution.

Proof. The Sharma-Mittal entropy of nth generalized upper (k) record value is given by

$$H_{\alpha,\beta}(X_{U(n,k)}) = \frac{1}{1-\beta} \left\{ \left(\int_{-\infty}^{\infty} \left[\frac{k^n\{-\log(1-F(x))\}^{n-1}[1-F(x)]^{k-1}}{(n-1)!} f(x) \right]^{\alpha} \mathrm{d}x \right)^{\frac{1-\beta}{1-\alpha}} - 1 \right\}.$$

On putting $u = -\log[1-F(x)]$, $x = \left[F^{-1}(1-e^{-u})\right]$ and $\mathrm{d}u = \frac{f(x)}{1-F(x)}\mathrm{d}x$ we get

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n,k)}) &= \frac{1}{1-\beta} \left\{ \left(\int_0^{\infty} \frac{k^{n\alpha} e^{-u[(k-1)\alpha+1]} u^{(n-1)\alpha}}{[(n-1)!]^{\alpha}} \left\{ f\left(F^{-1}(1-e^{-U})\right) \right\}^{\alpha-1} \mathrm{d}u \right)^{\frac{1-\beta}{1-\alpha}} - 1 \right\} \\ &= \frac{1}{1-\beta} \left\{ \left(\frac{k^{n\alpha}\Gamma((n-1)\alpha+1)}{[(k-1)\alpha+1]^{(n-1)\alpha}\{\Gamma(n)\}^{\alpha}} \int_0^{\infty} \frac{[(k-1)\alpha+1]^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} \right.\right. \\ & \left.\left. \times\ e^{-u[(k-1)\alpha+1]} u^{(n-1)\alpha} \left\{ f\left(F^{-1}(1-e^{-U})\right) \right\}^{\alpha-1} \mathrm{d}u \right)^{\frac{1-\beta}{1-\alpha}} - 1 \right\} \\ &= \frac{1}{1-\beta} \left\{ \left(\frac{k^{n\alpha}\Gamma((n-1)\alpha+1)}{[(k-1)\alpha+1]^{(n-1)\alpha+1}\{\Gamma(n)\}^{\alpha}} \right.\right. \\ & \left.\left. \times\ E_{g_{(k-1)\alpha+1,(n-1)\alpha+1}} \left[\left\{ f\left(F^{-1}(1-e^{-U})\right) \right\}^{\alpha-1} \right] \right)^{\frac{1-\beta}{1-\alpha}} - 1 \right\}. \end{aligned} \tag{8}$$

■

Now we state the following theorem without proof as the proof is just similar to the proof of theorem 1.

Theorem 2. Let $\{X_i, i \geq 1\}$ be a sequence of iid continuous random variables with common cdf F(x), pdf f(x) and quantile function $F^{-1}(.)$. Let $\{X_{L(n,k)}\}$ be the associated sequence of genralized lower (k) record values. Then the Sharma-Mittal entropy of $X_{L(n,k)}$ can be expressed as

$$H_{\alpha,\beta}(X_{L(n,k)}) = \frac{1}{1-\beta} \left\{ \left(\frac{k^{n\alpha}\Gamma((n-1)\alpha+1)}{[(k-1)\alpha+1]^{(n-1)\alpha+1}\{\Gamma(n)\}^{\alpha}} E_{g_{(k-1)\alpha+1,(n-1)\alpha+1}} \left[\left\{ f\{F^{-1}(e^{-U})\} \right\}^{\alpha-1} \right] \right)^{\frac{1-\beta}{1-\alpha}} - 1 \right\}, \tag{9}$$

where U is a random variable with $G(1, (n-1)\alpha+1)$ distribution.

The following is a corollary to theorem 1.

Corollary 1. Let $\{X_i, i \geq 1\}$ be a sequence of *iid* continuous random variables arising from standard exponential distribution. Let $\{X^*_{U(n,k)}\}$ be the associated sequence of generalized upper (k) record values. Then the Sharma-Mittal entropy of $X_{U(n,k)}$ can be expressed as

$$H_{\alpha,\beta}(X^*_{U(n,k)}) = \frac{1}{1-\beta}\left\{\left(\frac{k^{n\alpha}\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}[k\alpha]^{(n-1)\alpha+1}}\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}. \tag{10}$$

The following theorem follows from theorems 1 and 2 as a consequence of corollary 1.

Theorem 3. Let $\{X_i, i \geq 1\}$ be a sequence of iid continuous random variables having a common cdf $F(x)$, pdf $f(x)$ and quantile function $F^{-1}(.)$. Let $\{X_{U(n,k)}\}$ and $\{X_{L(n,k)}\}$ be the associated sequences of generalized upper and lower(k)record values respectively. Then the Sharma-Mittal entropy of $X_{U(n,k)}$ and $X_{L(n,k)}$ can be expressed as

$$H_{\alpha,\beta}(X_{U(n,k)}) = \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1} \times E_{g_{1,(n-1)\alpha+1}}\left[\left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1}\right]\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \tag{11}$$

$$H_{\alpha,\beta}(X_{L(n,k)}) = \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1} \times E_{g_{1,(n-1)\alpha+1}}\left[\left\{f\left(F^{-1}(e^{-U})\right)\right\}^{\alpha-1}\right]\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}, \tag{12}$$

where $X^*_{U(n,k)}$ denotes the *nth* generalized upper (k) record value arising from the standard exponential distribution and U is a random variable, with $G((k-1)\alpha+1, (n-1)\alpha+1)$ distribution.

3. Bounds for Sharma-Mittal Entropy of Generalized (k) Record Values

Baratpour et al. [7] and [1] have obtained bounds for Shannon entropy of records and Rényi entropy of records respectively. In this section, we use the relation (7) for deriving some bounds on Sharma-Mittal entropy of generalized upper (k) record values.

Theorem 4. If X has pdf $f(x)$ and the Sharma-Mittal entropy $H_{\alpha,\beta}(X_{U(n,k)})$ of $X_{U(n,k)}$ arising from $f(x)$ is such that $H_{\alpha,\beta}(X_{U(n,k)}) < \infty$ then we have

(a) for all $\alpha > 1$ and $0 < \beta < 1$, $H_{\alpha,\beta}(X_{U(n,k)}) \leq \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)$
$\times \left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1} B_n S_{\alpha,\beta}(f)\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}$, and

(b) for $0 < \alpha < 1$ and $\beta > 1$, $H_{\alpha,\beta}(X_{U(n,k)}) \geq \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)$
$\times \left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1} B_n S_{\alpha,\beta}(f)\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}$, where,

(i) $X^*_{U(n,k)}$ denotes the *nth* generalized upper (k) record value arising from the standard exponential distribution

(ii) $B_n = \frac{e^{-((n-1)\alpha)}\{(n-1)\alpha\}^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}$ and

(iii) $S_\alpha(f) = \int_{-\infty}^{\infty} \lambda_F(x)\,\{f(x)\}^{\alpha-1}\,dx$, where $\lambda_F(x)$ is the hazard function of X.

Proof. The Sharma-Mittal entropy of *nth* generalized upper (k) record value is given by

$$H_{\alpha,\beta}(X_{U(n,k)}) = \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1} \times E_{g_{(k-1)\alpha+1,(n-1)\alpha+1}}\left[\left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1}\right]\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta},$$

where $g_{(k-1)\alpha+1,(n-1)\alpha+1}$ is the pdf corresponding to the $G((k-1)\alpha+1,\,(n-1)\alpha+1)$ distribution. Since the mode of the distribution with pdf $g_{(k-1)\alpha+1,(n-1)\alpha+1}$ is $m_n = \frac{(n-1)\alpha}{(k-1)\alpha+1}$ we have

$$g_{(k-1)\alpha+1,(n-1)\alpha+1}(m_n) = \frac{e^{-(n-1)\alpha}[(n-1)\alpha]^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} = B_n.$$

Hence we have $g_{(k-1)\alpha+1,(n-1)\alpha+1}(u) \leq B_n$. Now for $\alpha > 1$ and $0 < \beta < 1$ the entropy is

$$\begin{aligned}
H_{\alpha,\beta}(X_{U(n,k)}) &= \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1}\right. \\
&\quad \left.\times \int_0^\infty g_{(k-1)\alpha+1,(n-1)\alpha+1}(u)\left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1} \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \\
&\leq \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1} B_n\right. \\
&\quad \left.\times \int_0^\infty \left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1} \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \\
&= \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1} B_n\right. \\
&\quad \left.\times \int_{-\infty}^\infty \lambda_F(y)\left\{f(y)\right\}^{\alpha-1} \mathrm{d}y\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \\
&= \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1} B_n S_\alpha(f)\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}.
\end{aligned}$$

For $0 < \alpha < 1$ and $\beta > 1$ the proof is similar. ■

4. Characterization Property by the Sharma-Mittal Entropy of Generalized (k) Record Values

Sometimes we may observe the uncertainty prevailing in the system under study as so large that we are curious to know the type of distribution which governs the system. That is, in such a system, we look for a distribution that is capable of possessing maximum entropy as suggested in Jaynes [12]. This section derives exponential distribution as the distribution that maximizes the Sharma-Mittal entropy of record values under some information constraints. Let C be a class of all distributions with cdf $F(x)$ over the support set $\mathbb{R}^+$ with $F(0) = 0$ such that

(i) $\lambda_F(x,\theta) = a(\theta)b(x)$

(ii) $b(x) \leq M$, where M is a positive real constant with $b(x) = B^{'}(x)$ such that $b(x)$ and $a(\theta)$ are non-negative functions of x and θ receptively.

Now we prove the following theorem.

Theorem 5. Under the conditions described above Sharma-Mittal entropy $H_{\alpha,\beta}(X_{U(n,k)})$ arising from the distribution $F(x)$ is maximum in C, if and only if $F(x;\theta) = 1 - e^{-Ma(\theta)x}$.

Proof. Let $X_{U(n,k)}$ be the nth genralized upper (k) record value arising from the cdf $F(x;\theta) \in C$. Then by (7) we have

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n,k)}) &= \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1} \right. \\ &\quad \left. \times E_{g_{(k-1)\alpha+1,(n-1)\alpha+1}}\left[\left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1}\right]\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \\ &= \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1} \frac{[(k-1)\alpha+1]^{(n-1)\alpha+1}}{\Gamma((n-1)\alpha+1)} \right. \\ &\quad \left. \times \int_0^\infty e^{-u[(k-1)\alpha+1]} u^{(n-1)\alpha} \left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1} \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \\ &= \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1} \frac{[(k-1)\alpha+1]^{(n-1)\alpha+1}}{\Gamma((n-1)\alpha+1)} \right. \\ &\quad \left. \times \int_0^\infty e^{-u[(k-1)\alpha+1]} u^{(n-1)\alpha} \left\{a(\theta) b\left[B^{-1}\left\{\frac{u}{a(\theta)}\right\}\right] e^{-a(\theta)B\left[B^{-1}\left\{\frac{u}{a(\theta)}\right\}\right]}\right\}^{\alpha-1} \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} \\ &\quad - \frac{1}{1-\beta} \end{aligned}$$

$$\begin{aligned} &= \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left(\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{(n-1)\alpha+1} \frac{[(k-1)\alpha+1]^{(n-1)\alpha+1}}{\Gamma((n-1)\alpha+1)} \right. \\ &\quad \left. \times \int_0^\infty e^{-uk\alpha} u^{(n-1)\alpha} [a(\theta)]^{\alpha-1} b^{\alpha-1}\left[B^{-1}\left\{\frac{u}{a(\theta)}\right\}\right] \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}. \end{aligned} \tag{13}$$

Noting that $b(x) \leq M$ we have

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n,k)}) &\leq \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left(\frac{[a(\theta)M]^{\alpha-1}[k\alpha]^{(n-1)\alpha+1}}{\Gamma((n-1)\alpha+1)} \int_0^\infty e^{-uk\alpha} u^{(n-1)\alpha} \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} \\ &\quad - \frac{1}{1-\beta} \\ &\leq \left(H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{1-\beta}\right)\left\{[a(\theta)]^{\alpha-1} M^{\alpha-1}\right\}^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}. \end{aligned} \tag{14}$$

Then clearly

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n,k)}) &\leq \frac{1}{1-\beta}\left(\frac{k^{n\alpha}\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}[k\alpha]^{(n-1)\alpha+1}}\left\{[a(\theta)]M\right\}^{\alpha-1}\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \\ &\leq \frac{1}{1-\beta}\left\{\left(\frac{k^{n\alpha}\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}[k\alpha]^{(n-1)\alpha+1}}\left\{[a(\theta)]M\right\}^{\alpha-1}\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}. \end{aligned} \tag{15}$$

This proves the necessary part of the theorem.

On the other hand, suppose the nth generalized upper (k) record value arising from $F(x;\theta) = 1 - e^{-Ma(\theta)x}$ has maximum Sharma-Mittal entropy in class C. Then we have

$$H_{\alpha,\,\beta}(X_{U(n,k)}) = \frac{1}{1-\beta}\left\{\left(\frac{k^{n\alpha}\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}[k\alpha]^{(n-1)\alpha+1}}\left\{[a(\theta)]\,M\right\}^{\alpha-1}\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}. \quad (16)$$

It is clear to be note that the maximum entropy of nth generalized upper (k) record value ($X_{U(n,k)}$) arising from any arbitrary distribution under conditions (i) and (ii) will holds the inequality (15). As (16) is the expression on the right side of (15), it then follows that exponential distribution attains the maximum Sharma-Mittal entropy in the class C.

■

5. Some Properties of Sharma-Mittal Entropy on Generalized (k) Record Values

This section provides exact expressions for the Sharma-Mittal divergence measure on generalized (k) record values. Further in this section, we derive expressions for Sharma-Mittal entropy of concomitants of generalized upper and lower (k) record values arising from the Farlie-Gumbel-Morgenstern family. In the last part of this section, we derive an expression for residual Sharma-Mittal entropy of generalized upper (k) record values arising from an arbitrary distribution.

5.1. Sharma-Mittal Divergence Measure on Generalized(k)Record Values

Sharma and Mittal in 1977 introduced a two parameter divergent measure viz. Shrma-Mittal divergence measure denoted by $D_{\alpha,\,\beta}(f : g)$, between two distributions $f(x)$ and $g(x)$ and is defined by

$$D_{\alpha,\,\beta}(f : g) = \frac{1}{\beta-1}\left\{\left(\int\limits_{-\infty}^{\infty}\left(\frac{f(x)}{g(x)}\right)^{\alpha-1} f(x)\mathrm{d}x\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}, \ \forall\, \alpha > 0,\ \alpha \neq 1 \neq \beta. \quad (17)$$

[3] shown that, most of the widely used divergence measures such as Rényi, Tsallis, Bhattacharya and Kullback-Liabler divergences are special cases of Sharma-Mittal divergence measure.

In this section we study the Sharma-Mittal divergence between the probability distribution of nth generalized upper (k) record value and the parent distribution from which it arises.

Theorem 6. The Sharma-Mittal divergence between the nth generalized upper (k) record and the parent distribution is given by the following representation

$$D_{\alpha,\,\beta}(f_{U(n,k)}, f) = \frac{1}{\beta-1}\left\{\left(\frac{\Gamma((n-1)\alpha+1)}{(\Gamma(n))^{\alpha}}\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}. \quad (18)$$

Proof. The Sharma-Mittal information between the nth generalized upper (k) record and the parent distribution is given by

$$D_{\alpha,\,\beta}(f_{U(n,k)}, f) = \frac{1}{\beta-1}\left\{\left(\int_{-\infty}^{\infty}\frac{\left[k^{n}\{-\log[1-F(x)]\}^{n-1}[1-F(x)]^{k-1}\right]^{\alpha}}{((n-1)!)^{\alpha}} f(x)\mathrm{d}x\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}.$$

On putting $u = -\log[1-F(x)]$, we get $x = \left[F^{-1}(1-e^{-u})\right]$, $\mathrm{d}u = \frac{f(x)}{1-F(x)}dx$ and hence we have

$$D_{\alpha,\ \beta}(f_{U(n,k)}, f) = \frac{1}{\beta-1}\left\{\left(\int_0^\infty \frac{k^{n\alpha}e^{-u[(k-1)\alpha+1]}u^{(n-1)\alpha}}{((n-1)!)^\alpha}\mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\} \tag{19}$$

$$= \frac{1}{\beta-1}\left\{\left(\frac{k^{n\alpha}\Gamma((n-1)\alpha+1)}{(\Gamma(n))^\alpha[(k-1)\alpha+1]^{(n-1)\alpha+1}}\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}.$$

Hence the theorem. ■

Note 1. The Sharma-Mittal divergence between the *nth* upper record and the parent distribution can also be represented as

$$D_{\alpha,\ \beta}(f_{U(n)}, f) = \left\{H_{\alpha,\beta}(X^*_{U(n,k)}) + \frac{1}{\beta-1}\right\}\left[\frac{k\alpha}{(k-1)\alpha+1}\right]^{\frac{((n-1)\alpha+1)(1-\beta)}{1-\alpha}} - \frac{1}{\beta-1} \tag{20}$$

where, $X^*_{U(n,k)}$ denotes the *nth* generalized upper (k) record value arising from the standard exponential distribution.

Remark 1. The Sharma-Mittal information between the *nth* generalized upper (k) record value $X_{U(n,k)}$ and the parent distribution as given by 18 and 20 establishes that this information is a distribution free information measure.

5.2. Sharma-Mittal Entropy of Concomitants of Generalized (k) Records from Farlie-Gumbel-Morgenstern (FGM) family of Distributions

Let X and Y be two random variables with cdf's given by F_X(x) and $F_Y(y)$ respectively with corresponding pdf's $f_X(x)$ and $f_Y(y)$ and jointly distributed with cdf $F(x,y)$ given by, [see, 13].

$$F(x,y) = F_X(x)F_Y(y)\{1+\gamma(1-F_X(x))(1-F_Y(y))\},\ -1 \le \gamma \le 1, \tag{21}$$

where γ is known as association parameter. Then the family of distributions having the above form of cdf's is called Farlie-Gumbel-Morgenstern (FGM) family of distributions. It is obvious that (21) includes the case of independence as well when $\gamma = 0$. The joint pdf corresponding to the cdf defined in (21) is given by,

$$f(x,y) = f_X(x)f_Y(y)\{1+\gamma(1-2F_X(x))(1-2F_Y(y))\},\ -1 \le \gamma \le 1. \tag{22}$$

Let (X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n) be two-dimensional random vectors with the common bivariate distribution function $F(x,y)$ as given in (21). If we construct the sequence of GURV's $\{X_{U(n,k)}\}$ from the marginal sequence $\{X_i\}$, then the Y value occuring in an ordered pair with X observations equal to $X_{U(n,k)}$ is called the concomitant of the *nth* generalized upper(k)record value. We write $Y_{U[n,k]}$ to denote concomitant of *nth* GURV $X_{U(n,k)}$. Similarly the concomitant of *nth* GLRV, $X_{L(n,k)}$ as well can be defined and we denote it by $Y_{L[n,k]}$. Then the pdf of $Y_{U[n,k]}$ is denoted by $f_{Y_{U[n,k]}}$ and is given by

$$f_{Y_{U[n,k]}}(y) = \int f_{Y|X}(y|x)\ f_{X_{U(n,k)}(x)}\,dx = f_Y(y)\{1-\gamma_n(1-2F_Y(y))\}, \tag{23}$$

where $\gamma_n = \left(1-2\left\{\frac{k}{k+1}\right\}^n\right)\gamma$. Using (2) and (23) we can represent the Sharma-Mittal entropy of concomitant of *nth* generalized upper (k) record value as follows:

$$H_{\alpha,\beta}(Y_{U[n,k]}) = \frac{1}{1-\beta}\left\{\left(\int_{-\infty}^\infty (f_Y(y)\{1-\gamma_n(1-2F_Y(y))\})^\alpha\mathrm{d}y\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}$$

$$= \frac{1}{1-\beta}\left\{\left(\int_{-\infty}^\infty \{f_Y(y)\}^\alpha(\{1-\gamma_n(1-2F_Y(y))\})^\alpha\mathrm{d}y\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}.$$

On putting $F_Y(y) = u$, $y = F_y^{-1}(u)$ and $f_y(y)\mathrm{d}y = \mathrm{d}u$, we get

$$\begin{aligned} H_{\alpha,\beta}(Y_{U[n,k]}) &= \frac{1}{1-\beta}\left\{\left(\int_0^1 \left\{f_Y(F_y^{-1}(u))\right\}^{\alpha-1}\{1-\gamma_n(1-2u)\}^{\alpha}\,\mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\} \\ &= \frac{1}{1-\beta}\left\{\left(E_U\left[\left\{f_Y(F_y^{-1}(U))\right\}^{\alpha-1}\{1-\gamma_n(1-2U)\}^{\alpha}\right]\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}, \end{aligned}$$

where U is a uniformly distributed random variable over $(0,1)$. Similarly the Sharma-Mittal entropy of concomitant of *nth* generalized lower(k)record can be represented by

$$H_{\alpha,\beta}(Y_{L[n,k]}) = \frac{1}{1-\beta}\left\{\left(E_U\left[\left\{f_Y(F_y^{-1}(1-U))\right\}^{\alpha-1}\{1+\gamma_n(1-2U)\}^{\alpha}\right]\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}.$$

5.3. The Residual Sharma-Mittal Entropy of Generalized (k) Record Values

Suppose X represents the lifetime of a unit with pdf $f(.)$, then $H_{\alpha,\,\beta}(X)$ as defined in (2) is useful for measuring the associated uncertainty. Suppose a component is known to have survived up to an age t. In that case, information about the remaining lifetime is an important characteristic required for data analysis arising from areas such as reliability, survival studies, economics, business etc. However, for the analysis of uncertainty about the remaining life time of the unit, we will consider residual Sharma-Mittal entropy and is defined by

$$H_{\alpha,\,\beta}(X;t) = \frac{1}{1-\beta}\left\{\left(\int_t^{\infty}\left\{\frac{f(x)}{\bar{F}(t)}\right\}^{\alpha}\mathrm{d}x\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}, \tag{24}$$

where $H_{\alpha,\,\beta}(X;t)$ measures the expected uncertainty contained in the conditional density of $X-t$ given $X > t$ and $\bar{F}(t) = 1 - F(t)$. In this section we derive a closed form representation for the residual Sharma-Mittal entropy of record values in terms of residual Sharma-Mittal entropy of uniform distribution over $[0,1]$. The survival function of the *nth* generalized(k)upper record, denoted by $\bar{F}_{X_{U(n,k)}}(x)$, is given by

$$\bar{F}_{X_{U(n,k)}}(x) = \sum_{j=1}^{n}\frac{[-k\log\bar{F}(x)]^j}{j!}\bar{F}(x)^k = \frac{\Gamma(n+1;-k\log\bar{F}(x))}{\Gamma(n+1)}, \tag{25}$$

where $\Gamma(a;x)$ denotes the incomplete Gamma function and is defined by

$$\Gamma(a;x) = \int_x^{\infty} e^{-u}u^{a-1}du, \;\; a,\, x > 0.$$

Lemma 1. Let $Z_{U(n,k)}$ denote the *nth* generalized upper (k) record value from a sequence of observations from $U(0,1)$. Then

$$H_{\alpha,\beta}(Z_{U(n,k)};t) = \frac{1}{1-\beta}\left\{\left(\frac{k^{n\alpha}\Gamma((n-1)\alpha+1;-[(k-1)\alpha+1]\log(1-t))}{[(k-1)\alpha+1]^{(n-1)\alpha+1}\{\Gamma(n;-k\log(1-t))\}^{\alpha}}\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}. \tag{26}$$

Proof. By considering (5), (24) and (25), the residual Sharma-Mittal entropy of $Z_{U(n,k)}$ is given by

$$H_{\alpha,\beta}(Z_{U(n,k)};t) = \frac{1}{1-\beta}\left\{\left(\int_t^{\infty}\frac{k^{\alpha}[-k\log(1-x)]^{(n-1)\alpha}[1-x]^{(k-1)\alpha}}{\{\Gamma(n;-k\log(1-t))\}^{\alpha}}\mathrm{d}x\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}.$$

On putting $-k\log(1-x) = u$, $x = 1 - e^{-\frac{u}{k}}$ and $k\mathrm{d}x = e^{-\frac{u}{k}}\mathrm{d}u$.

$$H_{\alpha,\beta}(Z_{U(n,k)};t) = \frac{1}{1-\beta}\left\{\left(k^{\alpha-1}\int_{-k\log(1-t)}^{\infty}\frac{u^{(n-1)\alpha}e^{-\frac{u}{k}[(k-1)\alpha+1]}}{\{\Gamma(n;-k\log(1-t))\}^{\alpha}}\mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}.$$

Now we consider the transformation $\frac{u}{k}[(k-1)(2-\alpha)+1] = v$, $u = \frac{vk}{(k-1)(2-\alpha)+1}$ and $\mathrm{d}u = \frac{k}{(k-1)(2-\alpha)+1}\mathrm{d}v$.

$$\begin{aligned} H_{\alpha,\beta}(Z_{U(n,k)};t) &= \frac{1}{1-\beta}\left\{\left(\int_{-[(k-1)(2-\alpha)+1]\log(1-t)}^{\infty}\frac{k^{n\alpha}}{[(k-1)\alpha+1]^{(n-1)\alpha+1}}\right.\right. \\ &\quad \left.\left.\times\frac{u^{(n-1)\alpha}e^{-v}}{\{\Gamma(n;-k\log(1-t))\}^{\alpha}}\mathrm{d}v\right)^{\frac{1-\beta}{1-\alpha}}-1\right\} \\ &= \frac{1}{1-\beta}\left\{\left(\frac{k^{n\alpha}\Gamma((n-1)\alpha+1;-[(k-1)\alpha+1]\log(1-t))}{[(k-1)\alpha+1]^{(n-1)\alpha+1}\{\Gamma(n;-k\log(1-t))\}^{\alpha}}\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}. \end{aligned} \tag{27}$$

Hence the lemma. ■

Theorem 7. The residual Sharma-Mittal entropy of $X_{U(n,k)}$ arising from an arbitrary distribution can be written in terms of the residual Sharma-Mittal entropy of $Z_{U(n,k)}$ as follows

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n,k)};t) &= \left\{H_{\alpha,\beta}(Z_{U(n,k)};F(t))+\frac{1}{1-\beta}\right\}\left(E_V\left[\left\{f\left(F^{-1}(1-e^{-\frac{v}{(k-1)(2-\alpha)+1}})\right)\right\}^{\alpha-1}\right]\right)^{\frac{1-\beta}{1-\alpha}} \\ &\quad -\frac{1}{1-\beta}. \end{aligned} \tag{28}$$

where $V \sim \Gamma_{-[(k-1)\alpha+1]\log(1-F(t))}((n-1)\alpha+1;1)$.

Proof. The residual Sharma-Mittal entropy of $X_{U(n)}$ is given by

$$H_{\alpha,\beta}(X_{U(n,k)};t) = \frac{1}{1-\beta}\left\{\left(\int_t^{\infty}\frac{k^{\alpha}[-k\log(1-F(x))]^{(n-1)\alpha}[1-F(x)]^{(k-1)\alpha}}{\{\Gamma(n;-k\log(1-F(t)))\}^{\alpha}}\mathrm{d}x\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}.$$

On putting $u = -k\log[1-F(x)]$, $x = \left[F^{-1}(1-e^{-\frac{u}{k}})\right]$ and $k\mathrm{d}x = e^{-\frac{u}{k}}\mathrm{d}u$ we get

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n,k)};t) &= \frac{1}{1-\beta}\left\{\left(\int_{-k\log(1-F(t))}^{\infty}\frac{k^{\alpha-1}u^{(n-1)\alpha}e^{-\frac{u}{k}[(k-1)\alpha+1]}}{\{\Gamma(n;-k\log(1-F(t)))\}^{\alpha}}\right.\right. \\ &\quad \left.\left.\left\{f\left(F^{-1}(1-e^{-\frac{u}{k}})\right)\right\}^{\alpha-1}\mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}}-1\right\} \end{aligned}$$

Now we consider the transformation $\frac{u}{k}[(k-1)(2-\alpha)+1] = v$, $u = \frac{vk}{(k-1)(2-\alpha)+1}$ and $\mathrm{d}u = \frac{k}{(k-1)(2-\alpha)+1}\mathrm{d}v$.

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n,k)};t) &= \frac{1}{1-\beta}\left\{\left(\int_{-[(k-1)\alpha+1]\log(1-F(t))}^{\infty}\frac{k^{n\alpha}}{[(k-1)\alpha+1]^{(n-1)\alpha+1}}\right.\right.\\ &\quad\left.\left.\times\frac{e^{-v}v^{(n-1)\alpha}}{\{\Gamma(n;-k\log(1-F(t)))\}^{\alpha}}\left\{f\left(F^{-1}(1-e^{-\frac{v}{(k-1)(2-\alpha)+1}})\right)\right\}^{\alpha-1}\mathrm{d}v\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}\\ &= \frac{1}{1-\beta}\left\{\left(\frac{k^{n\alpha}\Gamma((n-1)\alpha+1;-[(k-1)\alpha+1]\log(1-F(t))}{[(k-1)\alpha+1]^{(n-1)\alpha+1}\{\Gamma(n;-k\log(1-F(t)))\}^{\alpha}}\right.\right.\\ &\quad\left.\left.\times E_V\left\{f\left(F^{-1}(1-e^{-\frac{v}{(k-1)(2-\alpha)+1}})\right)\right\}^{\alpha-1}\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}\\ &= \left\{H_{\alpha,\beta}(Z_{U(n,k)};F(t))+\frac{1}{1-\beta}\right\}\left(E_V\left[\left\{f\left(F^{-1}(1-e^{-\frac{v}{(k-1)(2-\alpha)+1}})\right)\right\}^{\alpha-1}\right]\right)^{\frac{1-\beta}{1-\alpha}}\\ &\quad-\frac{1}{1-\beta}. \end{aligned} \tag{29}$$

Hence the theorem. ■

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