ZECH DISTRIBUTION: DERIVATION, PROPERTIES AND APPLICATIONS TO REAL LIFE DATA

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Abstract

The roles of heavy – tailed distribution in modelling real life events, especially in financial and actuarial sciences, cannot be over – emphasized. In this paper, a new heavy right – tailed, three – parameter continuous distribution with increasing hazard rate called Zech distribution is developed. The proposed model is very suitable for modelling heavy right- tailed data. Zech distribution is the reciprocal of the random variable which follows Gompertz- Inverse – Exponential (GoIE) distribution and it does not involve addition of extra parameter, thereby removing the cumbersomeness in the estimation process posed by other methods involving additional extra parameters, especially where more than three parameters are involved. The statistical properties of the new distribution such as survival function, hazard function, cumulative hazard function, reversed hazard function, quantile function, order statistics, moments, mean, median, variance, skewness, and kurtosis were derived. The Linear representation of the pdf of the newly developed distribution revealed that its probability density function is a weighted exponential distribution. Also, method of maximum likelihood was used in estimating the model's parameters. The simulation results revealed that as the sample sizes increased, the root mean squared errors decreased which showed that the parameters of Zech distribution are stable. The proposed distribution was applied to two real life data sets. The results showed that Zech distribution performs better than Gompertz Inverse Exponential distribution, Weibull Exponential distribution and Gompertz Exponential distribution.

Keywords: Zech distribution, Gompertz Inverse Exponential Distribution, maximum likelihood estimation, simulation studies, moments, linear representation.

1. Introduction

Probability distributions play a crucial role in modelling naturally occurring phenomena. In probability theory and statistics, an inverse distribution is the distribution of the reciprocal of a random variable. To model real life events, there is need for the extension of the classical forms of distributions so as to have a better fit to the real data. Several methods of extending distributions have been proposed in the literature. Among these is 'Inverse Distribution' which does not increase the number of parameter(s) of the parent distribution but provides a better fit. This is a strong motivation for studying inverse distribution as prescribed by the principle of parsimony. Eliwa [12] proposed Inverse Gompertz distribution which was found to out – perform other six competing distributions. The Gompertz Inverse Exponential distribution proposed by Pelumi [5] is good for modelling right – tailed data. Said [13] introduced Extended Inverse Weibull distribution whose density function can be expressed as a linear combination of the Inverse Weibull densities with increasing and decreasing hazard rates. Ogunsanya [11] developed Weibull Inverse Rayleigh distribution which is an extension of a one – parameter Inverse Rayleigh distribution that incorporated a transformation of the Weibull distribution and Log – logistic distribution as quantile functions. El – Gohari A [2] proposed Generalized Gompertz distribution which is a new generalization of the Exponential, Gompertz and Generalized Exponential distributions. The main advantage of this new distribution is that it has increasing or constant or decreasing or bathtub curve failure rate depending upon the shape parameter. It is this property that makes it suitable for survival analysis. Adewara [3] introduced Gompertz Exponential distribution which is an extension of Exponential distribution by using the Gompertz Generalized family of distributions proposed by Morad [4]. To increase the flexibility of Gompertz Exponential distribution, an extra shape parameter was added to it leading to the introduction of Exponentiated Gompertz Exponential distribution by Adewara [8].

The motivation for this study is to derive a distribution which will be more flexible for modelling heavy right – tailed data and to obtain interesting properties of the new model. Therefore, the inverse of '*Gompertz Inverse Exponential distribution*', which will henceforth be called *Zech* distribution is proposed. The adoption of the name '*Zech* distribution' is to avoid the confusion which might arise from using the name: Inverse Gompertz Inverse Exponential Distribution.

Given the cumulative distribution function (cdf) of a random variable Y, the distribution function of a random variable $X = \frac{1}{Y}$ is the reciprocal or inverse of the random variable Y. This implies that the cumulative distribution function $G(x)$ is the inverse function of $F(y)$. This is easier if Y is a continuous random variable and $F(y)$ is strictly on positive supports. The cumulative distribution function of inverse distribution is derived according to the method below:

$$\begin{aligned} G_X(x) &= P(X \le x) \\ &= P\left(\frac{1}{Y} \le X\right) \\ &= P\left(x \ge \frac{1}{Y}\right) \\ &= 1 - P\left(x \le \frac{1}{Y}\right) \\ &= 1 - F\left(\frac{1}{Y}\right) \end{aligned} \quad (1)$$

The cumulative distribution function (cdf) and probability density function (pdf) of Gompertz Inverse Exponential distribution are given in equations (2) and (3) respectively.

$$F(y) = 1 - e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\frac{\theta}{y}}\right]^{-\beta}\right\}} \quad ; y > 0, \alpha > 0, \beta > 0, \theta > 0 \quad (2)$$

$$f(y) = \alpha \frac{\theta}{y^2} e^{-\frac{\theta}{y}} \left[1 - e^{-\frac{\theta}{y}}\right]^{-\beta-1} e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\frac{\theta}{y}}\right]^{-\beta}\right\}} \; ; \; y > 0, \alpha > 0, \beta > 0, \theta > 0 \quad (3)$$

II. Zech Distribution

The Zech distribution is the reciprocal of Gompertz Inverse Exponential distribution. The cumulative distribution function of Zech distribution is stated in the following theorem.

Theorem 1: If a non – negative random variable Y follows the Gompertz inverse Exponential distribution expressed as Y~GIE$(y; \alpha, \theta, \beta)$. Assuming a new random variable $X = \frac{1}{y}$ is defined, then the random variable X follows *Zech* distribution, written as X~Zech$(x; \theta, \alpha, \beta)$ with the cdf in equation (4).

$$G(x) = e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}} \; ; \; x > 0, \alpha > 0, \beta > 0, \theta > 0 \quad (4)$$

Proof:

From equation (1), $G(x) = 1 - F\left(\frac{1}{y}\right)$

$$G(x) = 1 - \left[1 - e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\right]$$

$$G(x) = e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}$$

The result of the first derivative, with respect to x, of equation (4) is the probability density function of *Zech* distribution given by (5).

$$g(x) = \alpha\theta e^{-\theta x}\left[1 - e^{-\theta x}\right]^{-\beta-1} e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}; \quad x > 0, \alpha > 0, \beta > 0, \theta > 0 \tag{5}$$

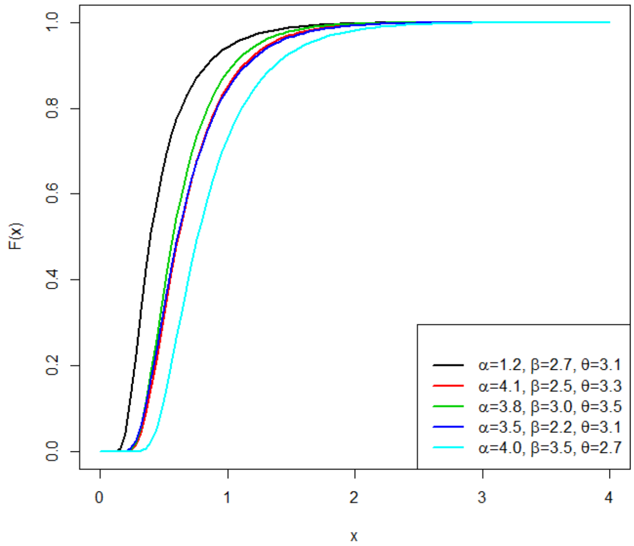

Figure 1: CDF Plots of Zech distribution.

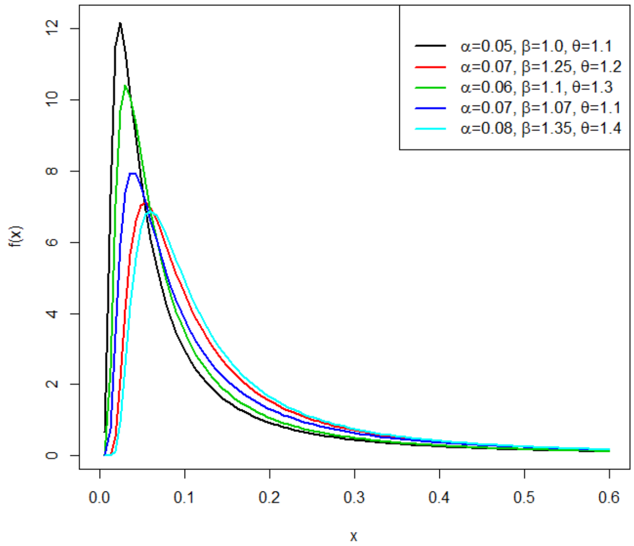

Figure 2: PDF Plots of Zech distribution.

Figures 1 and 2 illustrate some of possible shapes of the cumulative density function and probability density function respectively of Zech distribution.

III. Estimation of Parameters

The method of Maximum likelihood is used to estimate the parameters of *Zech* distribution.
Assuming each of the random samples $x_1, x_2, \ldots, x_n$ follows the pdf of *Zech* distribution, the likelihood function is given by

$$L(x_1, x_2, \ldots, x_n; \alpha, \beta, \theta) = \prod_{i=1}^{n}\left\{\alpha\theta e^{-\theta x_i}\left[1 - e^{-\theta x_i}\right]^{-\beta-1} e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x_i}\right]^{-\beta}\right\}}\right\} \tag{6}$$

Let l denote the log-likelihood function, that is, let

$$l = log\, L(x_1, x_2, \ldots, x_n; \alpha, \beta, \theta) \tag{7}$$

$$l = nlog\alpha + nlog\theta - \theta\sum_{i=1}^{n} x_i - (\beta+1)\sum_{i=1}^{n}\log\left(1 - e^{-\theta x}\right) + \frac{\alpha}{\beta}\sum_{i=1}^{n}\left\{1 - \left[1 - e^{-\theta x_i}\right]^{-\beta}\right\} \tag{8}$$

The solutions of simultaneous equations obtained from $\frac{dl}{d\alpha} = 0, \frac{dl}{d\beta} = 0 \; and \; \frac{dl}{d\theta} = 0$ are the maximum likelihood estimates of the parameters $\alpha, \beta \; and \; \theta$. Thus,

$$\frac{dl}{d\alpha}=\frac{n}{\alpha}+\frac{1}{\beta}\sum_{i=1}^{n}\left\{1-\left[1-e^{-\theta x_i}\right]^{-\beta}\right\} \tag{9}$$

$$\frac{dl}{d\beta}=\frac{\alpha}{\beta}\sum_{i=1}^{n}\left\{\left[1-e^{-\theta x_i}\right]^{-\beta}\ln\left(1-e^{-\theta x_i}\right)\right\}-\frac{\alpha}{\beta^2}\sum_{i=1}^{n}\left\{1-\left[1-e^{-\theta x_i}\right]^{-\beta}\right\}-\sum_{i=1}^{n}\ln\left(1-e^{-\theta x_i}\right)$$
$$-\frac{\alpha}{\beta^2}\sum_{i=1}^{n}\left\{1-\left[1-e^{-\theta x_i}\right]^{-\beta}\right\} \tag{10}$$

$$\frac{dl}{d\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}x_i-(\beta+1)\sum_{i=1}^{n}\left(x_ie^{-\theta x_i}\right)\left(1-e^{-\theta x_i}\right)^{-1}+\alpha\sum_{i=1}^{n}x_ie^{-\theta x_i}\left(1-e^{-\theta x_i}\right)^{-\beta-1} \tag{11}$$

Equating $\frac{dl}{d\alpha}=0$, $\frac{dl}{d\beta}=0$ and $\frac{dl}{d\theta}=0$, we have

$$\frac{n}{\alpha}+\frac{1}{\beta}\sum_{i=1}^{n}\left\{1-\left[1-e^{-\theta x_i}\right]^{-\beta}\right\}=0 \tag{12}$$

$$\frac{\alpha}{\beta}\sum_{i=1}^{n}\left\{\left[1-e^{-\theta x_i}\right]^{-\beta}\ln\left(1-e^{-\theta x_i}\right)\right\}-\frac{\alpha}{\beta^2}\sum_{i=1}^{n}\left\{1-\left[1-e^{-\theta x_i}\right]^{-\beta}\right\}-\sum_{i=1}^{n}\ln\left(1-e^{-\theta x_i}\right)$$
$$-\frac{\alpha}{\beta^2}\sum_{i=1}^{n}\left\{1-\left[1-e^{-\theta x_i}\right]^{-\beta}\right\}=0 \tag{13}$$

$$\frac{n}{\theta}+\sum_{i=1}^{n}x_i-(\beta+1)\sum_{i=1}^{n}\left(x_ie^{-\theta x_i}\right)\left(1-e^{-\theta x_i}\right)^{-1}+\alpha\sum_{i=1}^{n}x_ie^{-\theta x_i}\left(1-e^{-\theta x_i}\right)^{-\beta-1}=0 \tag{14}$$

The Maximum likelihood estimate for parameter α can be obtained from (12) in the form below for a given β and θ

$$\hat{\alpha}=\frac{-n\beta}{\sum_{i=1}^{n}\left\{1-\left[1-e^{-\theta x_i}\right]^{-\beta}\right\}} \tag{15}$$

To obtain the MLE of β and θ, equation (15) can be substituted into equations (13) and (14). The resulting system of non – linear equations can be solved numerically.

IV. Linear Representation

Theorem 2: The pdf of *Zech* distribution is a weighted function of an exponential distribution with rate parameter $\theta(1+i)$.

Proof.

$$g(x)=\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{-\beta-1}e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}$$

Using the exponential expansion

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!} \tag{16}$$

$$g(x)=e^{-\theta x}\left[1-e^{-\theta x}\right]^{-\beta-1}\sum_{k=0}^{\infty}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}^k\frac{\alpha\theta}{k!}\left(\frac{\alpha}{\beta}\right)^k \tag{17}$$

From the mixture representation,

$$(1-z)^k=\sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma(k)j!}z^j(-1)^j \tag{18}$$

$$\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}^k=\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(k+1)}{\Gamma(k)j!}\left[1-e^{-\theta x}\right]^{-\beta j} \tag{19}$$

Inserting equation (18) into (16), we have

$$g(x)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma(k)j!}(-1)^j\frac{\alpha\theta}{k!}\left(\frac{\alpha}{\beta}\right)^k e^{-\theta x}\left[1-e^{-\theta x}\right]^{-\beta j}\left[1-e^{-\theta x}\right]^{-\beta-1} \tag{20}$$

Simplifying, $\left[1-e^{-\theta x}\right]^{-\beta j}\left[1-e^{-\theta x}\right]^{-\beta-1}=\left[1-e^{-\theta x}\right]^{-[\beta(j+1)+1]}$ (21)

Inserting equation (20) into equation (19), we have

$$g(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(k)j!}(-1)^{j}\frac{\alpha\theta}{k!}\left(\frac{\alpha}{\beta}\right)^{k} e^{-\theta x}\left[1-e^{-\theta x}\right]^{-[\beta(j+1)+1]} \tag{22}$$

By using mixture representation,

$$(1-z)^{-k} = \sum_{i=0}^{\infty}\frac{\Gamma(\mathrm{k}+\mathrm{i})}{\Gamma(k)i!}z^{i} \tag{23}$$

$$\left[1-e^{-\theta x}\right]^{-[\beta(j+1)+1]} = \sum_{i=0}^{\infty}\frac{\Gamma[(\beta(j+1)+1)+i]}{\Gamma[\beta(j+1)+1]i!}e^{-\theta i x} \tag{24}$$

$$g(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(k)j!}(-1)^{j}\frac{\alpha\theta}{k!}\left(\frac{\alpha}{\beta}\right)^{k}\frac{\Gamma[(\beta(j+1)+1)+i]}{\Gamma[\beta(j+1)+1]i!}e^{-\theta i x}e^{-\theta x} \tag{25}$$

$$g(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(k)j!}(-1)^{j}\frac{\alpha\theta}{k!}\left(\frac{\alpha}{\beta}\right)^{k}\frac{\Gamma[(\beta(j+1)+1)+i]}{\Gamma[\beta(j+1)+1]i!}e^{-\theta x(1+i)} \tag{26}$$

Let$(-1)^{j}\frac{\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(k)j!}\frac{\Gamma[(\beta(j+1)+1)+i]}{\Gamma[\beta(j+1)+1]i!} = w_{i,j,k}$ (27)

$$g(x) = \frac{\alpha\theta}{k!}\left(\frac{\alpha}{\beta}\right)^{k}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}w_{i,j,k}e^{-\theta(1+i)x} \tag{28}$$

$$g(x) = \frac{\theta\alpha^{k+1}}{\beta^{k}k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}w_{i,j,k}e^{-\theta(1+i)x} \tag{29}$$

$$g(x) = \frac{\theta(1+i)\alpha^{k+1}}{\beta^{k}k!\,(1+i)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}w_{i,j,k}e^{-\theta(1+i)x} \tag{30}$$

$$g(x) = \frac{\alpha^{k+1}}{\beta^{k}k!\,(1+i)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}w_{i,j,k}\left[\theta(1+i)e^{-\theta(1+i)x}\right] \tag{31}$$

V. Reliability Properties

The reliability function can be obtained from

$$S(x) = 1 - G(x) \tag{32}$$

Therefore, the survival function of *Zech* distribution is given as

$$S(x) = 1 - e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}} \;\; ; \;\; x>0, \alpha>0, \beta>0, \theta>0 \tag{33}$$

The hazard function of *Zech* distribution is obtained from

$$h(x) = \frac{g(x)}{S(x)} \tag{34}$$

The hazard function of *Zech* distribution is given by

$$h(x) = \frac{\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{-\beta-1}e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}}{1-e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}};\;\; x>0, \alpha>0, \beta>0, \theta>0 \tag{35}$$

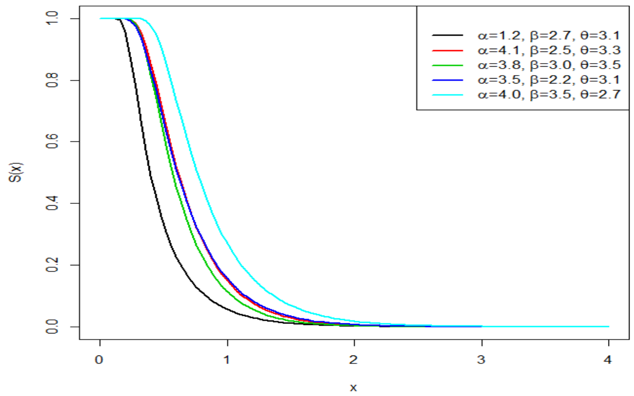

Figure 3: Survival Plots of Zech distribution.

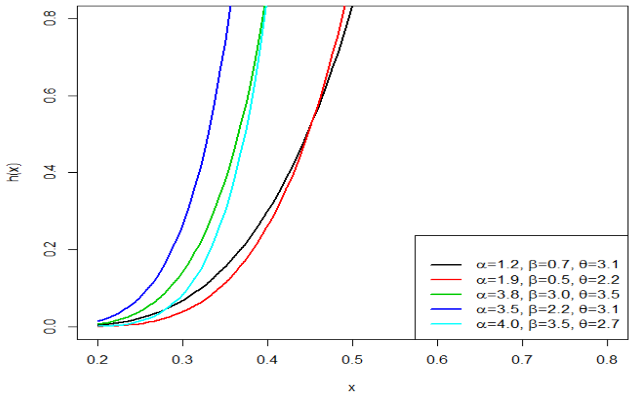

Figure 4: Hazard plots of Zech distribution.

Figures 3 and 4 illustrate some of possible shapes of the Survival function and Hazard function respectively of Zech distribution.

The cumulative hazard function, H(x) of a continuous random variable X from *Zech* distribution is derived from

$$H(x) = -\log\big(S(x)\big) \tag{36}$$

Substituting equation (33) into equation (36)

$$H(x)_{Zech} = -\log\left(1 - e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\right) \tag{37}$$

The reversed hazard function of a random variable x of *Zech* distribution is obtained from

$$r(x) = \frac{g(x)}{G(x)} \tag{38}$$

Therefore,

$$r(x)_{Zech} = \alpha\theta e^{-\theta x}\left[1 - e^{-\theta x}\right]^{-\beta-1} \tag{39}$$

VI. Quantile Function and Median of *Zech* distribution.

Quantile function is very important for generating random numbers which can be used for simulation studies. Aside that, it can also be used for finding quantiles i.e. quartiles, octiles, deciles and percentiles of a distribution which are necessary for deriving the measures of skewness and kurtosis.

The quantile function is derived by inverting the cdf

$$Q(u) = G^{-1}(u) \tag{40}$$

$$Q(u) = -\frac{1}{\theta}\left\{\ln\left[1 - \left(1 - \frac{\beta}{\alpha}\ln u\right)^{-\frac{1}{\beta}}\right]\right\} \tag{41}$$

Where $u \sim Uniform\ (0,1)$.

To generate random numbers from Zech distribution, it is sufficient that

$$x = -\frac{1}{\theta}\left\{\ln\left[1 - \left(1 - \frac{\beta}{\alpha}\ln u\right)^{-\frac{1}{\beta}}\right]\right\} \tag{42}$$

The median of *Zech* distribution can be obtained by substituting $u = 0.5$ in equation (41) as follows:

$$Median = -\frac{1}{\theta}\left\{\ln\left[1 - \left(1 - \frac{\beta}{\alpha}\ln 0.5\right)^{-\frac{1}{\beta}}\right]\right\} \tag{43}$$

Other quantiles can also be derived from (41) by substituting appropriate values of $''u''$.

VII. Quantile - based measures of skewness and kurtosis

The measure of Skewness(S) and Kurtosis (K) of Zech distribution using quantile function, defined by Galton [6] and Moors [7] are given by equations (44) and (45) respectively.

$$S = \frac{Q\left(\frac{6}{8}\right) - 2Q\left(\frac{4}{8}\right) + Q\left(\frac{2}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \quad (44)$$

$$K = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \quad (45)$$

$Q\left(\frac{1}{8}\right), Q\left(\frac{2}{8}\right), Q\left(\frac{3}{8}\right), Q\left(\frac{4}{8}\right), Q\left(\frac{5}{8}\right), Q\left(\frac{6}{8}\right)$, and $Q\left(\frac{7}{8}\right)$ can be obtained by substituting $\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, and\ \frac{7}{8}$ for u respectively in equation (41). Therefore,

$$Q\left(\frac{1}{8}\right) = -\frac{1}{\theta}\left\{\ln\left[1 - \left(1 - \frac{\beta}{\alpha} ln\left(\frac{1}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} \quad (46)$$

$$Q\left(\frac{2}{8}\right) = -\frac{1}{\theta}\left\{\ln\left[1 - \left(1 - \frac{\beta}{\alpha} ln\left(\frac{2}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} \quad (47)$$

$$Q\left(\frac{3}{8}\right) = -\frac{1}{\theta}\left\{\ln\left[1 - \left(1 - \frac{\beta}{\alpha} ln\left(\frac{3}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} \quad (48)$$

$$Q\left(\frac{4}{8}\right) = -\frac{1}{\theta}\left\{\ln\left[1 - \left(1 - \frac{\beta}{\alpha} ln\left(\frac{4}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} \quad (49)$$

$$Q\left(\frac{5}{8}\right) = -\frac{1}{\theta}\left\{\ln\left[1 - \left(1 - \frac{\beta}{\alpha} ln\left(\frac{5}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} \quad (50)$$

$$Q\left(\frac{6}{8}\right) = -\frac{1}{\theta}\left\{\ln\left[1 - \left(1 - \frac{\beta}{\alpha} ln\left(\frac{6}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} \quad (51)$$

$$Q\left(\frac{7}{8}\right) = -\frac{1}{\theta}\left\{\ln\left[1 - \left(1 - \frac{\beta}{\alpha} ln\left(\frac{7}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} \quad (52)$$

The skewness of *Zech* distribution is derived by substituting the values of $Q\left(\frac{6}{8}\right)$, $Q\left(\frac{4}{8}\right)$ and $Q\left(\frac{2}{8}\right)$ into equation (44).

Therefore,
$$S_{Zech} = \frac{\frac{2}{\theta}\left\{\ln\left[1-\left(1-\frac{\beta}{\alpha}ln\left(\frac{4}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} - \frac{1}{\theta}\left\{\ln\left[1-\left(1-\frac{\beta}{\alpha}ln\left(\frac{6}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} - \frac{1}{\theta}\left\{\ln\left[1-\left(1-\frac{\beta}{\alpha}ln\left(\frac{2}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}}{\frac{1}{\theta}\left\{\ln\left[1-\left(1-\frac{\beta}{\alpha}ln\left(\frac{2}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} - \frac{1}{\theta}\left\{\ln\left[1-\left(1-\frac{\beta}{\alpha}ln\left(\frac{6}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}} \quad (53)$$

Simplifying (53) by factoring out $\left(\frac{1}{\theta}\right)$, the result shows that symmetry or asymmetry of *Zech* distribution is independent of parameter θ.

$$S_{Zech} = \frac{2\left\{\ln\left[1-\left(1-\frac{\beta}{\alpha}ln\left(\frac{4}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} - \left\{\ln\left[1-\left(1-\frac{\beta}{\alpha}ln\left(\frac{6}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} - \left\{\ln\left[1-\left(1-\frac{\beta}{\alpha}ln\left(\frac{2}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}}{\left\{\ln\left[1-\left(1-\frac{\beta}{\alpha}ln\left(\frac{2}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\} - \left\{\ln\left[1-\left(1-\frac{\beta}{\alpha}ln\left(\frac{6}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}} \quad (54)$$

The kurtosis of *Zech* distribution is derived by substituting the values of $Q\left(\frac{7}{8}\right), Q\left(\frac{5}{8}\right), Q\left(\frac{3}{8}\right), Q\left(\frac{1}{8}\right), Q\left(\frac{1}{8}\right)\ and\ Q\left(\frac{2}{8}\right)$ into equation (45)

$$K_{Zech} = \frac{\frac{1}{\theta}\left\{\ln\left[1-\left(1-\frac{\alpha}{\beta}ln\left(\frac{5}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}-\frac{1}{\theta}\left\{\ln\left[1-\left(1-\frac{\alpha}{\beta}ln\left(\frac{7}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}-\frac{1}{\theta}\left\{\ln\left[1-\left(1-\frac{\alpha}{\beta}ln\left(\frac{3}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}-\frac{1}{\theta}\left\{\ln\left[1-\left(1-\frac{\alpha}{\beta}ln\left(\frac{1}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}}{\frac{1}{\theta}\left\{\ln\left[1-\left(1-\frac{\alpha}{\beta}ln\left(\frac{2}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}-\frac{1}{\theta}\left\{\ln\left[1-\left(1-\frac{\alpha}{\beta}ln\left(\frac{6}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}} \quad (55)$$

Simplifying equation (55) by factoring out $\left(\frac{1}{\theta}\right)$, the result shows that the peakedness or otherwise of *Zech* distribution is independent of parameter θ.

$$K_{Zech} = \frac{\left\{\ln\left[1-\left(1-\frac{\alpha}{\beta}ln\left(\frac{5}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}-\left\{\ln\left[1-\left(1-\frac{\alpha}{\beta}ln\left(\frac{7}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}-\left\{\ln\left[1-\left(1-\frac{\alpha}{\beta}ln\left(\frac{3}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}-\left\{\ln\left[1-\left(1-\frac{\alpha}{\beta}ln\left(\frac{1}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}}{\left\{\ln\left[1-\left(1-\frac{\alpha}{\beta}ln\left(\frac{2}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}-\left\{\ln\left[1-\left(1-\frac{\alpha}{\beta}ln\left(\frac{6}{8}\right)\right)^{-\frac{1}{\beta}}\right]\right\}} \quad (56)$$

VIII. Distribution of Order Statistics

Let $x_1, x_2, \dots, x_n$ be a random sample from a cdf and pdf of *Zech* distribution as defined in (4) and (5) respectively. The pdf of jth order statistics of any random variable X is given by:

$$f_{j:n}(x) = \frac{n!}{(j-1)!\,(n-j)!}g(x)G(x)^{j-1}[1-G(x)]^{n-j} \quad (57)$$

From (56), putting the pdf of the jth order statistics of *Zech* distribution,

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!}\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{-\beta-1}e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}.\left\{e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\right\}^{j-1}\left\{1-\left(e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\right)\right\}^{n-j} \quad (58)$$

Simplifying equation (58),

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!}\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{-\beta-1}.\left\{e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\right\}^{j}\left\{1-\left(e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\right)\right\}^{n-j}$$

(59)

Therefore, the distribution of minimum and maximum order statistics for the *Zech* distribution is given by $f_{1:n}(x)$ $i.e\ when\ j = 1$ and $f_{n:n}(x)$ respectively in equations (60) and (61) respectively.

$$f_{1:n}(x) = \frac{n!}{(j-1)!\,(n-1)!}\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{-\beta-1}.e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\left\{1-\left(e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\right)\right\}^{n-1}$$

After some simplifications,

$$f_{1:n}(x) = n\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{-\beta-1}.e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\left\{1-\left(e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\right)\right\}^{n-1} \quad 60)$$

Also,

$$f_{n:n}(x) = \frac{n!}{(n-1)!\,(n-n)!}\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{-\beta-1}.\left\{e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\right\}^{n}\left\{1-\left(e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\right)\right\}^{n-n}$$

After some simplifications,

$$f_{n:n}(x) = n\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{-\beta-1}.\left\{e^{\frac{\alpha}{\beta}\left\{1-\left[1-e^{-\theta x}\right]^{-\beta}\right\}}\right\}^{n} \quad (61)$$

IX. Moments of Zech distribution

The moment about the Origin of *Zech* distribution is derived as follows: recall from the linear expansion of the pdf of *Zech* distribution,

$$g(x) = \frac{\alpha^{k+1}}{\beta^k k!(1+i)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}w_{i,j,k}\left[\theta(1+i)e^{-\theta(1+i)x}\right]$$

The rth moment about the origin of a random variable X is given by

$$E(X^r) = \int x^r f(x) dx \tag{62}$$

$$E(X^r) = \int_0^\infty x^r \frac{\alpha^{k+1}}{\beta^k k!\,(1+i)} \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty w_{i,j,k} [\theta(1+i)e^{-\theta(1+i)x}]\, dx \tag{63}$$

$$E(X^r) = \frac{\theta\alpha^{k+1}}{\beta^k k!} \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty w_{i,j,k} \int_0^\infty x^r e^{-\theta(1+i)x}\, dx \tag{64}$$

From gamma expansion,

$$\frac{\Gamma a}{b^a} = \int_0^\infty x^{a-1} e^{-bx}\, dx \tag{65}$$

$a - 1 = r, a = r + 1\,, b = \theta(1+i)$

$$\int_0^\infty x^r e^{-\theta(1+i)x}\, dx = \frac{\Gamma(r+1)}{[\theta(1+i)]^{r+1}} \tag{66}$$

$$E(X^r) = \frac{\theta\alpha^{k+1}}{\beta^k k!} \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty w_{i,j,k} \frac{\Gamma(r+1)}{[\theta(1+i)]^{r+1}} \tag{67}$$

The first moment about the origin represents the mean of *Zech* distribution. This can be done by setting $r = 1$,

$$E(X) = \frac{\theta\alpha^{k+1}}{\beta^k k!} \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty w_{i,j,k} \frac{\Gamma(2)}{[\theta(1+i)]^2} \tag{68}$$

The second moment about the origin of *Zech* distribution is

$$E(X^2) = \frac{\theta\alpha^{k+1}}{\beta^k k!} \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty w_{i,j,k} \frac{\Gamma(3)}{[\theta(1+i)]^3} \tag{69}$$

The variance of *Zech* distribution is obtained from

$$Var(X) = E(X^2) - [E(X)]^2 \tag{70}$$

$$Var(X) = \frac{\theta\alpha^{k+1}}{\beta^k k!} \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty w_{i,j,k} \frac{\Gamma(3)}{[\theta(1+i)]^3} - \left[\frac{\theta\alpha^{k+1}}{\beta^k k!} \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty w_{i,j,k} \frac{\Gamma(2)}{[\theta(1+i)]^2}\right]^2 \tag{71}$$

The Moment about the Mean of Zech distribution is thus derived.

The rth central moment of a random variable X having *Zech* distribution is given by

$$E((x-\mu)^r) = \int_0^\infty (x-\mu)^r f(x) dx \tag{72}$$

$$E((x-\mu)^r) = \int_0^\infty (x-\mu)^r \frac{\alpha^{k+1}}{\beta^k k!\,(1+i)} \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty w_{i,j,k} \left[\theta(1+i)e^{-\theta(1+i)x}\right] dx \tag{73}$$

$$E((x-\mu)^r) = \frac{\theta\alpha^{k+1}}{\beta^k k!} \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty w_{i,j,k} \int_0^\infty (x-\mu)^r \left[e^{-\theta(1+i)x}\right] dx \tag{74}$$

By setting $y = x - \mu$, $\frac{dy}{dx} = 1$, $dx = dy$, $x = y + \mu$

$$E((x-\mu)^r) = \frac{\theta\alpha^{k+1}}{B^k k!} \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty w_{i,j,k} \int_0^\infty y^r\, e^{-\theta(1+i)(y+\mu)} dy \tag{75}$$

$$E(X^r) = \frac{\theta\alpha^{k+1}}{B^k k!} \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty w_{i,j,k} \int_0^\infty y^r\, e^{-\theta(1+i)y} . e^{-\theta(1+i)\mu} dy \tag{76}$$

$$E((x-\mu)^r) = \frac{\theta\alpha^{k+1}}{B^k k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{i,j,k}\, e^{-\theta(1+i)\mu}\int_0^{\infty} y^r\, e^{-\theta(1+i)y} dy \tag{77}$$

Using the Gamma function expansion,

$$\frac{\Gamma a}{b^a} = \int_0^{\infty} x^{a-1} e^{-bx}\, dx$$

$$a-1=r,\quad a=r+1,\quad b=\theta(1+i)$$

$$E((x-\mu)^r) = \frac{\theta\alpha^{k+1}}{B^k k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{i,j,k}\, e^{-\theta(1+i)\mu}\frac{\Gamma_{(r+1)}}{[\theta(1+i)]^{(r+1)}} \tag{78}$$

X. Moment - based measures of Skewness and Kurtosis

The skewness of *Zech* distribution based on central moment is given as

$$\text{Skewness} = \frac{\mu_3^2}{\mu_2^3} \tag{79}$$

Where $\mu_3 = Third\ central\ momen$

$\mu_2 = Second\ central\ moment$

$$\mu_3 = \frac{6\theta\alpha^{k+1}}{B^k k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{i,j,k}\frac{e^{-\theta(1+i)\mu}}{[\theta(1+i)]^4} \tag{80}$$

$$\mu_2 = \frac{2\theta\alpha^{k+1}}{B^k k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{i,j,k}\frac{e^{-\theta(1+i)\mu}}{[\theta(1+i)]^3} \tag{81}$$

$$Skewness = \frac{\left[\frac{6\theta\alpha^{k+1}}{B^k k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{i,j,k}\frac{e^{-\theta(1+i)\mu}}{[\theta(1+i)]^4}\right]^2}{\left[\frac{2\theta\alpha^{k+1}}{B^k k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{i,j,k}\frac{e^{-\theta(1+i)\mu}}{[\theta(1+i)]^3}\right]^3} \tag{82}$$

$$\text{Kurtosis} = \frac{\mu_4}{\mu_2^2} \tag{83}$$

But $\mu_4 = \frac{24\theta\alpha^{k+1}}{B^k k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{i,j,k}\frac{e^{-\theta(1+i)\mu}}{[\theta(1+i)]^5}$ (84)

The Kurtosis, based on central moment of *Zech* distribution is given by

$$Kurtosis = \frac{\frac{24\theta\alpha^{k+1}}{B^k k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{i,j,k}\frac{e^{-\theta(1+i)\mu}}{[\theta(1+i)]^5}}{\left[\frac{2\theta\alpha^{k+1}}{B^k k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_{i,j,k}\frac{e^{-\theta(1+i)\mu}}{[\theta(1+i)]^3}\right]^2} \tag{85}$$

XI. Results

I. Simulation Studies

The behavior of the parameters of Zech distribution was investigated through simulation studies using R statistical software. Data were replicated 1000 times. A random sample of sizes 50, 100, 150 and 200 were selected. The parameters were varied as follows: $\alpha = 0.5$, $\theta = 0.5$, and $\beta = 0.5$; and $\alpha = 1$, $\theta = 1$, and $\beta = 1$; and $\alpha = 1.5$, $\theta = 1.5$, and $\beta = 1.5$ respectively. The maximum likelihood estimates of the true parameters, the bias, standard error and Root Mean Square Error were obtained from the simulation. The results are shown in Tables 1, 2 and 3.

Results of Simulation Studies

Table 1: *Simulation study at* $\alpha = 0.5$, $\theta = 0.5$, *and* $\beta = 0.5$

N	Parameters	Means	Bias	Std. Error	RMSE
50	$\alpha = 0.5$	0.0339	0.4661	0.0152	0.0174
	$\theta = 0.5$	0.6382	- 0.1382	0.1931	0.0621
	$\beta = 0.5$	0.3763	0.1237	0.2300	0.0678
100	$\alpha = 0.5$	0.0466	0.4534	0.0170	0.0130
	$\theta = 0.5$	0.5295	- 0.0295	0.1348	0.0367
	$\beta = 0.5$	0.5720	-0.0720	0.1457	0.0382
150	$\alpha = 0.5$	0.0667	0.4333	0.0185	0.0111
	$\theta = 0.5$	0.4228	0.0772	0.0849	0.0238
	$\beta = 0.5$	0.5170	- 0.0170	0.1100	0.0271
200	$\alpha = 0.5$	0.0818	0.4182	0.0221	0.0105
	$\theta = 0.5$	0.4815	0.0185	0.0927	0.0215
	$\beta = 0.5$	0.6211	- 0.1211	0.0987	0.0222

Table 2: *Simulation study at* $\alpha = 1.0$, $\theta = 1.0$, *and* $\beta = 1.0$

N	Parameters	Means	Bias	Std. Error	RMSE
50	$\alpha = 1.0$	0.0627	0.9373	0.0446	0.0299
	$\theta = 1.0$	0.2452	0.7548	0.4291	0.0926
	$\beta = 1.0$	0.1369	0.8631	0.4911	0.0991
100	$\alpha = 1.0$	0.0834	0.9166	0.0363	0.0191
	$\theta = 1.0$	0.8306	0.1694	0.1951	0.0442
	$\beta = 1.0$	0.9365	0.0635	0.2667	0.0516
150	$\alpha = 1.0$	0.1469	0.8531	0.0536	0.0189
	$\theta = 1.0$	1.0107	- 0. 0107	0.1898	0.0356
	$\beta = 1.0$	0.9909	0.0091	0.2452	0.0404
200	$\alpha = 1.0$	0.1609	0.8391	0.0515	0.0160
	$\theta = 1.0$	0.8606	0.1394	0.1484	0.0272
	$\beta = 1.0$	0.9995	0.0005	0.1901	0.0308

Table 3: *Simulation study at* $\alpha = 1.5$, $\theta = 1.5$, *and* $\beta = 1.5$

N	Parameters	Means	Bias	Std. Error	RMSE
50	$\alpha = 1.5$	0.0636	1.4364	0.5580	0.0334
	$\theta = 1.5$	1.5227	- 0.0227	0.5782	0.1075
	$\beta = 1.5$	1.9951	- 0.4951	0.5450	0.1044
100	$\alpha = 1.5$	0.0821	1.4179	0.0517	0.0227
	$\theta = 1.5$	1.2049	0.2951	0.3506	0.0592
	$\beta = 1.5$	1.9077	-0.4074	0.3208	0.0566
150	$\alpha = 1.5$	0.1249	1.3751	0.0621	0.02035
	$\theta = 1.5$	1.0912	0.4088	0.2543	0.04120
	$\beta = 1.5$	0.1713	1.3287	0.2802	0.04320
200	$\alpha = 1.5$	0.2188	1.2812	0.0825	0.02031
	$\theta = 1.5$	1.3248	0.1752	0.2275	0.03370
	$\beta = 1.5$	1.6958	- 0.1958	0.2539	0.03560

The tables 1, 2 and 3, for each of the selected true parameter values show that as the sample sizes increase, the root mean square errors decrease. This implies that the parameters of Zech distribution are stable.

II. Applications to Real Life Data Sets

The performance of Zech distribution, as well as goodness of fit tests when fitted to the real life data sets is hereby compared with other three – parameter distributions such as Gompertz Inverse Exponential (GIE) distribution, Weibull Exponential (WE) distribution and Gompertz Exponential (GE) distribution are provided in tables 6 and 7 respectively.
To select the best among the competing distributions, the following statistical criteria are used: Negative Log-likelihood, Akaike Information criterion (AIC) and Bayesian Information criterion (BIC). The distribution having the least value of the criteria above is adjudged to be the best. Also, the goodness of fit tests like Kolmogorov-Smirnov Statistic (KS) and Anderson-Darling Statistic (ADS) are also computed to select the best fit. The best distribution has the least value of the statistics above.

Data 1: The first data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli observed and reported by Bjerkedal [9] and used by Adewara [1].
10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100,100,102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115,116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146,153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196,197, 202, 213, 215, 216, 222, 230, 231, 240, 245, 251, 253,254, 254, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555

Table 4: *Descriptive Statistics for data 1.*

Min	1st Quartile	Median	Mean	3rd Quartile	Max.	Standard Deviation	Skewness	Kurtosis
10.0	108.0	149.5	176.8	224.0	555.0	103.4549	1.341869	4.991056

Table 5: *Performance rating for the fitted models using data 1.*

Distributions	Estimates	-LL	AIC	BIC	KS	ADS
Zech	$\hat{\alpha} = 5.1884384$ $\hat{\beta} = -0.6383047$ $\hat{\theta} = 0.0128380$	424.8790	855.7579	862.5879	0.0835	0.4925
GIE	$\hat{\alpha} = 0.02683774$ $\hat{\beta} = 1.88823061$ $\hat{\theta} = 22.03993914$	427.6661	861.3322	868.1622	0.1095	1.0777
WE	$\hat{\alpha} = 1.157801294$ $\hat{\beta} = 1.340608545$ $\hat{\theta} = 0.002981762$	431.3887	868.7774	875.6074	0.1195	1.9894
GE	$\hat{\alpha} = 0.004089507$ $\hat{\beta} = 1.08290306$ $\hat{\theta} = 0.002751085$	434.3901	874.7802	881.6102	0.1759	2.6168

The distributions tested showed the performances of each. The results revealed that the distribution with the lowest value of –LL, AIC, BIC, KS and ADS is considered to be the best. From Table 5, *Zech* distribution had the least value of 424.8790 for –LL, AIC= 855.7579, BIC = 862.5879, KS = 0.0835 and ADS = 0.4925 hence, it was considered the best fitted distribution among other distributions.

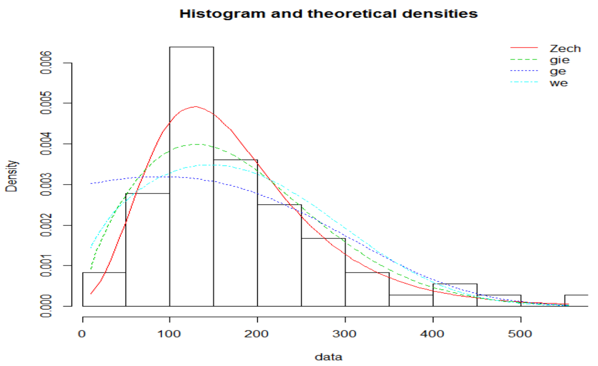

Figure 5: *Histogram and theoretical densities for data 1*

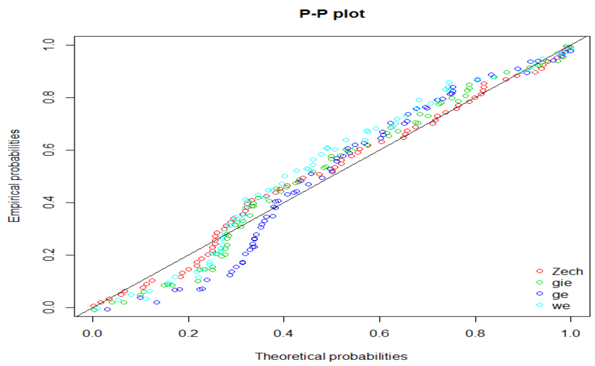

Figure 6: *P – P plot for data 1*

Figures 5 and 6 respectively depict the performance of the new distribution with survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli observed, this is compared to other distributions mentioned in the research.

Data 2: The second dataset represents the gauge of length of 10mm observed by Mohammed [10]
1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

Table 6: *Descriptive Statistics for data 2.*

Min	1st Quartile	Median	Mean	3rd Quartile	Max.	Standard Deviation	Skewness	Kurtosis
1.901	2.554	2.996	3.059	3.422	5.020	0.6209216	0.6178407	3.286345

Table 7: *Performance rating for the fitted models using data 2.*

Distributions	Estimates	-LL	AIC	BIC	KS	ADS
Zech	$\hat{\alpha} = 238.223663$ $\hat{\beta} = -5.177056$ $\hat{\theta} = 1.971310$	56.5097	119.0194	125.4488	0.0885	0.3571
GIE	$\hat{\alpha} = 330.740054$ $\hat{\beta} = -41.32409$ $\hat{\theta} = 18.50223$	57.28159	120.5632	126.9926	0.0903	0.3846
WE	$\hat{\alpha} = 1.8379671$ $\hat{\beta} = 3.7232674$ $\hat{\theta} = 0.1842052$	63.6584	133.3168	139.7462	0.0986	1.1866
GE	$\hat{\alpha} = 0.944957015$ $\hat{\beta} = 1.480487938$ $\hat{\theta} = 0.008481201$	69.1480	144.2960	150.7254	0.1389	1.9840

The distributions tested showed the performances of each. The results revealed that Zech distribution had the lowest value of –LL, AIC, BIC, KS and ADS and is considered to be the best. From Table 7, *Zech* distribution had the least value of = 56.5097 for –LL, AIC=119.0194, BIC =125.4488, KS = 0.0885 and ADS= 0.3571 hence, it was considered the best fitted distribution among other distributions.

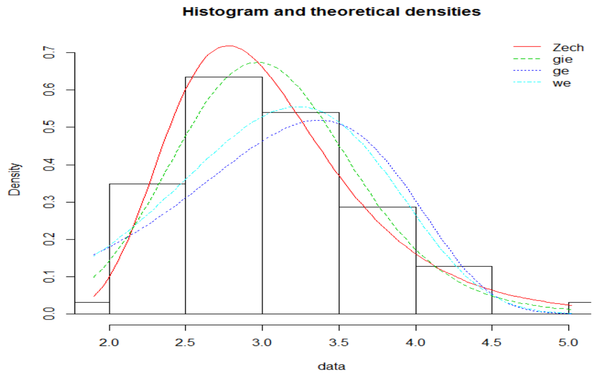

Figure 7: *Histogram and theoretical densities for data 2*

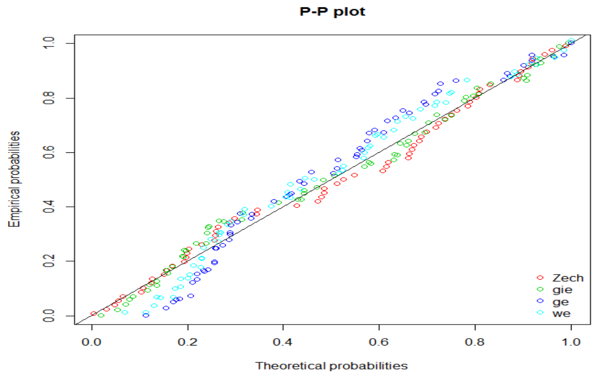

Figure 8: *P – P plot for data 2*

Figures 7 and 8 respectively depict the performance of the new distribution with gauge of length of 10mm data, and compared to other distributions mentioned in the research

XII. Discussion

The histogram and theoretical densities plot shows that *Zech* distribution fits data 1 and 2 best. Also, the probability plot i.e. (the PP plot) which compares the empirical cdf of data sample with specified theoretical cumulative distribution, reveals that *Zech* distribution is closer to the line than the remaining three fitted models. Tables 1, 2 and 3, for each of the selected true parameter values show that as the sample sizes increase, the Root Mean Square errors decrease which shows that the parameters of Zech distribution are stable.

Tables 4 and 6 show that the data is skewed to the right. Interestingly, the shape of the pdf graph of *Zech* distribution is also positively skewed. Also, the Kurtosis values of 4.991056 and 3.2866345 suggest that the data is leptokurtic. Data 1 and 2 have a kurtosis of 1.991056 and 0.2866345 respectively above that of normal distribution which is 3.0

Tables 5 and 7, show that Zech distribution has the lowest value of –LL, AIC, BIC, KS and ADS and is considered to be the best when compared with the competing distributions such as Gompertz Inverse Exponential distribution, Weibull Exponential distribution and Gompertz Exponential distribution.

XIII. Conclusion

In this paper, a new three – parameter continuous distribution named Zech distribution was proposed from Gompertz Inverse Exponential distribution. Its probability density function was plotted and the result revealed a heavy positively skewed distribution which is suitable for modelling heavily right-tailed data. Several statistical and mathematical properties of the new distribution were derived. The results of the simulation studies revealed that the parameters of the new distribution are stabled and as the sample sizes increased, the Root Mean Square (RMS) errors decreased. The applications to two real life data sets showed that Zech distribution has the lowest -LL, AIC, BIC, KS and ADS when compared with other competing distributions such as Gompertz Inverse Exponential distribution, Weibull Exponential distribution and Gompertz Exponential distribution used in this research paper.

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