Weibull Inverse Power Rayleigh Distribution with Applications Related to Distinct Fields of Science

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Abstract

In this paper an extension of Weibull Power Rayleigh Distribution has been introduced, and named it is as Weibull Inverse Power Rayleigh Distribution. This distribution is obtained by adopting T-X family technique. Various Structural properties, Reliability measures and Characteristics have been calculated and discussed. The behaviour of Probability density function, Cumulative distribution function, Survival function, Hazard rate function and mean residual function are illustrated through different graphs. Various parameters are estimated through the technique of MLE. The versatility and flexibility of the new distribution is done by using real life data sets. To evaluate and compare the out effectiveness of estimators, a simulation analysis has also been carried out.

Keywords:- *Weibull distribution, Inverse Rayleigh distribution, Renyi entropy, maximum likelihood estimation, Order statistics.*

I. INTRODUCTION

Weibull distribution, although being first identified by Frechet [8] and first applied by Rosin and Rammler [9] to describe particle size distribution, was named after Swedish Mathematician Waloddi Weibull, who in 1951 described the distribution in detail in his paper "A Statistical Distribution Function of Wide Applicability". This distribution is now commonly used to assess product reliability, analyze life data and model failure times, see inverse Weibull-G by Amal S. Hassan et al [3]. This distribution has found its importance in the fields of biology, economics, engineering sciences and hydrology.

The pdf of Weibull distribution is given by

$$f(t;\alpha,\beta)=\alpha\beta t^{\beta-1}e^{-\alpha t^{\beta}} \quad \alpha,\beta>0, t\geq 0 \tag{1}$$

Where α,β and t are Location, Shape and Scale parameters respectively.

Rayleigh being a one-parameter continuous and simplest velocity probability distribution has a diverse range of applications. Various researchers have developed several extensions and modifications of the distribution, resulting in some flexible and more effective distributions by adding some more parameters or by compounding and thus showing its importance in various fields like

Social, Medical and Engineering Sciences. For instance transmuted Rayleigh distribution by Faton Merovci [6], Weibull-Rayleigh distribution by Faton Merovci et al [7], odd generalized exponential Rayleigh distribution by Albert Luguterah [1], Topp-Leone Rayleigh distribution by Fatoki Olayode [5], odd Lindley-Rayleigh distribution by Terna Godfrey Ieren [10], new generalisation of Rayleigh distribution by A.A Bhat et al [4].

The Inverse Rayleigh distribution finds its application in the field of reliability studies. Voda [11] worked out that the lifetime distributions of various types of experimental units can be approximated by Inverse Rayleigh distribution. In this article we use Weibull and Inverse Power Rayleigh Distributions to define a new model which generalizes the Inverse power Rayleigh distribution.

The cdf of Power inverse Rayleigh Distribution is given by

$$G(x,\theta,\lambda)=e^{-\frac{\theta}{2x^{2\lambda}}} \quad x>0,\theta,\lambda>0 \tag{2}$$

$$\overline{G}(x,\theta,\lambda)=1-e^{-\frac{\theta}{2x^{2\lambda}}} \tag{3}$$

And its associated pdf is given by

$$g(x,\theta,\lambda)=\frac{\theta\lambda}{x^{2\lambda+1}}e^{-\frac{\theta}{2x^{2\lambda}}} \qquad x>0,\theta,\lambda>0 \tag{4}$$

II. Materials and Methods

Transformed-transformer (T-X) family of distributions (Alzaatreh et al [2]) is given by

$$\mathrm{F(x)} = \int_0^{W[G(x)]} f(t)dt \tag{5}$$

Where f(t) is the probability density function of a random variable T and $W[G(x)]$ be a function of cumulative density function of random variable X .

Suppose $[G,\varsigma]$denotes the baseline cumulative distribution function, which depends on parameter vector ς. Now using T-X approach, the cumulative distribution function of Weibull Inverse Power Rayleigh distribution can be derived by replacing f(t) in equation(5) by equation (1) and $W[G(x)]=\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}$,where $\overline{G}(x,\varsigma)=1-G(x,\varsigma)$ which follows

$$\mathrm{F}(x,\alpha,\beta,\varsigma) = \int_0^{\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}} \alpha\beta t^{\beta-1}e^{-\alpha t^{\beta}}\,dt$$

$$F(x,\alpha,\beta,\varsigma)=1-e^{-\alpha\left[\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}\right]^{\beta}} \tag{6}$$

The corresponding pdf of (6) becomes

$$f(x,\alpha,\beta,\varsigma)=\alpha\beta g(x,\varsigma)\frac{\left[G(x,\varsigma)\right]^{\beta-1}}{\left[\overline{G}(x,\varsigma)\right]^{\beta-1}}e^{-\alpha\left[\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}\right]^{\beta}} \tag{7}$$

Where $G(x,\theta,\lambda)=e^{-\frac{\theta}{2x^{2\lambda}}}$ is known as Power Inverse Rayleigh Distribution. (8)

$$\overline{G}(x,\theta,\lambda)=1-G(x,\theta,\lambda)=1-e^{-\frac{\theta}{2x^{2\lambda}}} \tag{9}$$

And $g(x,\theta,\lambda)=\frac{\theta\lambda}{x^{2\lambda+1}}e^{-\frac{\theta}{2x^{2\lambda}}}\ ,\theta>0,\lambda>0$ (10)

III. Linear Transformation

Apply Taylor series expansion to the pdf in equation (7) we have

$$e^{-\alpha\left[\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}\right]^{\beta}}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\alpha^i\left[\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}\right]^{\beta i} \tag{11}$$

On substituting equation (11) in (7), we get the following expression

$$f(x,\varsigma)=\alpha\beta g(x,\varsigma)\frac{\left[G(x,\varsigma)\right]^{\beta-1}}{\left[\overline{G}(x,\varsigma)\right]^{\beta+1}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\alpha^i\left[\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}\right]^{\beta i}$$

$$f(x,\varsigma)=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\alpha^{(i+1)}\beta\, g(x,\varsigma)\left[G(x,\varsigma)\right]^{\beta(i+1)-1}\left[\overline{G}(x,\varsigma)\right]^{-[\beta(i+1)+1]} \tag{12}$$

Using generalized binomial expansion, we have

$$(1-x)^{-a}=\sum_{j=0}^{\infty}\binom{a+j-1}{j}x^j$$

$$f(x,\varsigma)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i}{i!}\binom{\beta(i+1)+j}{j}\alpha^{(i+1)}\beta\, g(x,\varsigma)\left[G(x,\varsigma)\right]^{j+\beta(i+1)-1} \tag{13}$$

Using (8) and (9) in (13) we get

$$f(x,\varsigma)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i}{i!}\binom{\beta(i+1)+j}{j}\alpha^{(i+1)}\beta\frac{\theta\lambda}{x^{2\lambda+1}}e^{\frac{-\theta}{2x^{2\lambda}}}\left[e^{\frac{-\theta}{2x^{2\lambda}}}\right]^{j+\beta(i+1)-1} \tag{14}$$

$$f(x,\varsigma)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{ij}\frac{\theta\lambda}{x^{2\lambda+1}}e^{\frac{-\theta}{2x^{2\lambda}}}\left[e^{\frac{-\theta}{2x^{2\lambda}}}\right]^{j+\beta(i+1)-1} \tag{15}$$

Where $\delta_{ij} = \frac{(-1)^i}{i!}\binom{\beta(i+1)+j}{j}\alpha^{(i+1)}\beta$

IV. Weibull Inverse Power Rayleigh Distribution

Let X be a random variable_following Weibull Inverse Power Rayleigh Distribution with its cdf given by

$$F(x;\alpha,\beta,\theta,\lambda) = 1 - e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}} \quad ,x>0,\alpha,\beta,\theta,\lambda>0 \tag{16}$$

And its pdf is given by

$$f(x;\alpha,\beta,\theta,\lambda) = \frac{\alpha\beta\theta\lambda}{x^{2\lambda+1}} e^{\frac{\theta}{2x^{2\lambda}}}\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta-1} e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}} \quad ,x>0,\alpha,\beta,\theta,\lambda>0 \tag{17}$$

V. Reliability measures

The Survival function, Hazard function, Cumulative Hazard function, Reverse Hazard function is given by:

$$s_x(x) = e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}}$$

$$h_x(x) = \frac{\alpha\beta\theta\lambda}{x^{2\lambda+1}} e^{\frac{\theta}{2x^{2\lambda}}}\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta-1}$$

$$H_x(x) = \alpha\left[e^{\frac{\theta}{2x^{2\lambda}}}-1\right]^{-\beta}$$

$$\tau_x = \frac{\frac{\alpha\beta\theta\lambda}{x^{2\lambda+1}} e^{\frac{\theta}{2x^{2\lambda}}}\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta-1} e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}}}{1-e^{-\alpha\left(e\frac{\theta}{2x^{2\lambda}}-1\right)^{-\beta}}}$$

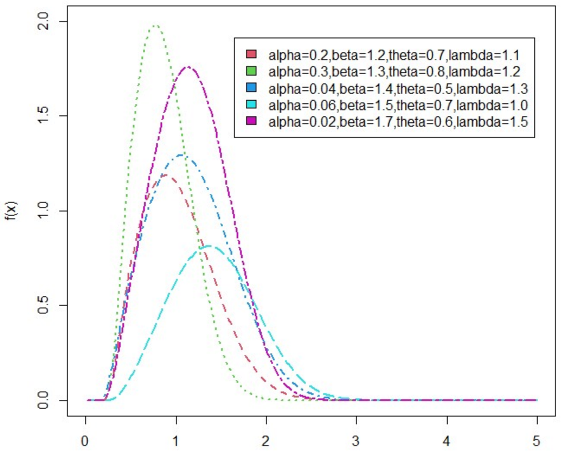


Fig a pdf of WPIRD under different values to parameters

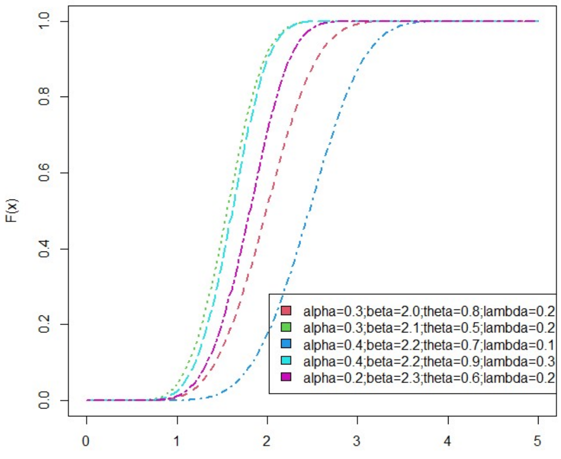


Fig b cdf of WIPRD under different values to parameters

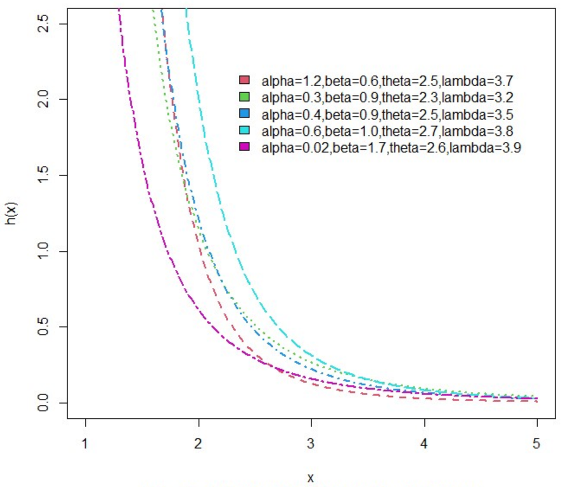


Fig c HRF of WPIRD under different values to parameters

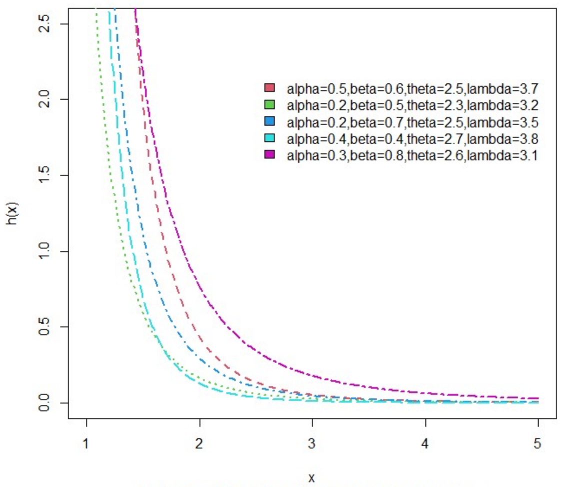


Fig d HRF of WPIRD under different values to parameters

Theorem 1: Show that the Quantile Function of Weibull Inverse Power Rayleigh Distribution is given by

$$x(u)=\left[\frac{\theta}{2\log\left[\left(\frac{-1}{\alpha}\log(1-u)\right)^{-\frac{1}{\beta}}+1\right]}\right]^{\frac{1}{2\lambda}}$$

Proof: Let x be a random variable following Weibull Inverse Power Rayleigh Distribution with parameters α,β,θ and λ, then we derive its Quantile Function from the corresponding cdf as given below

We know that cdf of WIPRD is given by

$$1-e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}}$$

$$\text{Put } 1-e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}}=u$$

Apply log on both sides

$$-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)=\log(1-u)$$

After solving we get

$$x(u)=\left[\frac{\theta}{2\log\left[\left(\frac{-1}{\alpha}\log(1-u)\right)^{-\frac{1}{\beta}}+1\right]}\right]^{\frac{1}{2\lambda}} \tag{18}$$

Median:

The median for the new WIPRD can be derived from the quantile function in equation (18) by putting u=0.5 as below

$$M_e=Q_x(0.5)=\left[\frac{\theta}{2\log\left[\left(\frac{-1}{\alpha}\log(\frac{1}{2})\right)^{-\frac{1}{\beta}}+1\right]}\right]^{\frac{1}{2\lambda}}$$

Theorem 2: If $X\sim WIPRD(\alpha,\beta,\theta,\lambda)$ then its r^th moment is given by.

$$E(X)^r=\mu_r{}'=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i}{i!}\binom{\beta(i+1)+j}{j}\alpha^{(i+1)}\beta\left(\frac{\theta}{2}\right)^{\frac{r}{2\lambda}}\frac{\Gamma\left(1-\frac{r}{2\lambda}\right)}{[j+\beta(i+1)]^{1-\frac{r}{2\lambda}}}$$

Proof: We know that r^th moment about origin is given by

$$E(X)^r = \mu_r' = \int_0^\infty x^r f(x;\alpha,\beta,\theta,\lambda)dx$$

Using (14) we get the following equation

$$\mu_r' = \int_0^\infty x^r \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i}{i!}\binom{\beta(i+1)+j}{j}\alpha^{(i+1)}\beta \frac{\theta\lambda}{x^{2\lambda+1}} e^{\frac{-\theta}{2x^{2\lambda}}}\left[e^{\frac{-\theta}{2x^{2\lambda}}}\right]^{j+\beta(i+1)-1} dx$$

$$\mu_r' = \sum_{i=0}^\infty \sum_{j=0}^\infty \delta_{ij} \int_0^\infty x^r \frac{\theta\lambda}{x^{2\lambda+1}} e^{\frac{-[j+\beta(i+1)]\theta}{2x^{2\lambda}}} dx$$

Where $\delta_{ij} = \frac{(-1)^i}{i!}\binom{\beta(i+1)+j}{j}\alpha^{(i+1)}\beta$

$$\mu_r' = \sum_{i=0}^\infty \sum_{j=0}^\infty \delta_{ij}\, \theta\lambda \int_0^\infty x^{r-2\lambda-1} e^{\frac{-[j+\beta(i+1)]\theta}{2x^{2\lambda}}} dx$$

Put $\frac{-[j+\beta(i+1)]\theta}{2x^{2\lambda}} = z,$ we get

$$E(X) = \mu_r' = \sum_{i=0}^\infty \sum_{j=0}^\infty \delta_{ij}\left(\frac{\theta}{2}\right)^{\frac{r}{2\lambda}} \frac{\Gamma\left(1-\frac{r}{2\lambda}\right)}{[j+\beta(i+1)]^{1-\frac{r}{2\lambda}}}$$

On substituting r=1,2,3,4 we get first four moments about origin.

Theorem 3. Show that the Moment generating function of Weibull Inverse Power Rayleigh Distribution is given by

$$M_x(t) = \sum_{r=0}^\infty \frac{t^r}{r!} \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i}{i!}\binom{\beta(i+1)+j}{j}\alpha^{(i+1)}\beta\left(\frac{\theta}{2}\right)^{\frac{r}{2\lambda}} \frac{\Gamma\left(1-\frac{r}{2\lambda}\right)}{[j+\beta(i+1)]^{1-\frac{r}{2\lambda}}}$$

Proof: We know that Moment generating function is given by

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x;\alpha,\beta,\theta,\lambda)dx$$

$$M_x(t) = E(e^{tx}) = \int_0^\infty \left\{1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots\right\} f(x;\alpha,\beta,\theta,\lambda)dx$$

$$M_x(t) = E(e^{tx}) = \int_0^\infty \sum_{r=0}^\infty \frac{(tx)^r}{r!} f(x;\alpha,\beta,\theta,\lambda)dx$$

$$M_x(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x;\alpha,\beta,\theta,\lambda)dx$$

After solving the above equation we get,

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \delta_{ij} \left(\frac{\theta}{2}\right)^{\frac{r}{2\lambda}} \frac{\Gamma\left(1-\frac{r}{2\lambda}\right)}{[j+\beta(i+1)]^{1-\frac{r}{2\lambda}}}$$

Theorem 4. Show that the Characteristic function of Weibull Inverse Power Rayleigh Distribution is given by

$$\phi_x(it) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i}{i!} \binom{\beta(i+1)+j}{j} \alpha^{(i+1)} \beta \left(\frac{\theta}{2}\right)^{\frac{r}{2\lambda}} \frac{\Gamma\left(1-\frac{r}{2\lambda}\right)}{[j+\beta(i+1)]^{1-\frac{r}{2\lambda}}}$$

Proof: We know that Characteristic function is given by

$$\phi_x(it) = E(e^{itx}) = \int_0^{\infty} e^{itx} f(x;\alpha,\beta,\theta,\lambda)dx$$

$$\phi_x(it) = E(e^{itx}) = \int_0^{\infty} \sum_{r=0}^{\infty} \frac{(itx)^r}{r!} f(x;\alpha,\beta,\theta,\lambda)dx$$

$$\phi_x(it) = E(e^{itx}) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_0^{\infty} x^r f(x;\alpha,\beta,\theta,\lambda)dx$$

After solving the above equation we get

$$\phi_x(it) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \delta_{ij} \left(\frac{\theta}{2}\right)^{\frac{r}{2\lambda}} \frac{\Gamma\left(1-\frac{r}{2\lambda}\right)}{[j+\beta(i+1)]^{1-\frac{r}{2\lambda}}}$$

VI. Incomplete Moments

We know that

$$T_q(S) = \int_0^{s} x^{s} f(x;\alpha,\beta,\theta,\lambda)dx$$

Put (14) in the above equation, we get

$$T_q(S) = \int_0^{s} x^s \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i}{i!} \binom{\beta(i+1)+j}{j} \alpha^{(i+1)} \beta \frac{\theta\lambda}{x^{2\lambda+1}} e^{\frac{-\theta}{2x^{2\lambda}}} \left[e^{\frac{-\theta}{2x^{2\lambda}}}\right]^{j+\beta(i+1)-1} dx$$

$$T_q(S)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{ij}\theta\lambda\int_0^s \frac{x^s}{x^{2\lambda+1}}\,e^{\frac{-(j+\beta(i+1))\theta}{2x^{2\lambda}}}dx$$

$$T_q(S)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}2\delta_{ij}\left[\frac{(j+\beta(i+1))\theta}{2}\right]^{\frac{s-2\lambda}{2\lambda}}\Gamma\left(1-\frac{s}{2\lambda},M\right)$$

Where $M=\dfrac{(j+\beta(i+1))\theta}{2x^{2\lambda}}$

VII. Renyi Entropy

If X is a continuous random variable following WIPRD with pdf $f(x;\alpha,\beta,\theta,\lambda)$, then

$$T_R(\rho)=\frac{1}{1-\rho}\log\left\{\int_0^{\infty} f^{\rho}(x,\varsigma)dx\right\}$$

$$T_R(\rho)=\frac{1}{1-\rho}\log\left\{\int_0^{\infty}\left(\alpha\beta g(x,\varsigma)\frac{[G(x,\varsigma)]^{\beta-1}}{[\bar{G}(x,\varsigma)]^{\beta+1}}e^{-\alpha\left[\frac{G(x,\varsigma)}{\bar{G}(x,\varsigma)}\right]^{\beta}}\right)^{\rho}dx\right\}$$

$$T_R(\rho)=\frac{1}{1-\rho}\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\upsilon_{ij}\int_0^{\infty}[g(x,\varsigma)]^{\rho}[G(x,\varsigma)]^{j+(\rho+i)\beta-\rho}dx\right\} \tag{19}$$

Where $\upsilon_{ij}=\dfrac{(-1)^i}{i!}(\rho\alpha)^i(\alpha\beta)^{\rho}\dbinom{\beta(\rho+1)+\rho+j-1}{j}$

Now put (10) and (8) in equation (19) we get

$$T_R(\rho)=\frac{1}{1-\rho}\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\upsilon_{ij}(\theta\lambda)^{\rho}\int_0^{\infty}x^{-\rho(2\lambda+1)}e^{\frac{-[j+(\rho+i)\beta]\theta}{2x^{2\lambda}}}dx\right\}$$

$$Put\ \frac{[j+(\rho+i)\beta]\theta}{2x^{2\lambda}}=z\ \Rightarrow\ dx=\frac{\left\{\frac{[j+(\rho+i)\beta]\theta}{2z}\right\}^{\frac{2\lambda+1}{2\lambda}}}{[j+(\rho+i)\beta]\theta\lambda}dz$$

$$T_R(\rho)=\frac{1}{1-\rho}\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\upsilon_{ij}\frac{(\theta\lambda)^{\rho-1}}{[j+(\rho+i)\beta]}\left[\frac{j+(\rho+i)\beta\theta}{2}\right]^{\frac{-\rho(2\lambda+1)+2\lambda+1}{2\lambda}}\Gamma\left(\frac{\rho(2\lambda+1)-1}{2\lambda}\right)\right\}$$

VIII. Tsallis Entropy of WIPRD

We know that Tsallis Entropy is defined as

$$S_\rho = \frac{1}{\rho-1}\left\{1-\int_0^\infty f^\rho(x,\varsigma)dx\right\}$$

$$S_\rho = \frac{1}{\rho-1}\left\{1-\int_0^\infty\left(\alpha\beta g(x,\varsigma)\frac{[G(x,\varsigma)]^{\beta-1}}{[\overline{G}(x,\varsigma)]^{\beta+1}}e^{-\alpha\left[\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}\right]^\beta}\right)^\rho dx\right\}$$

$$S_\rho = \frac{1}{\rho-1}\left\{1-\left(\sum_{i=0}^\infty\sum_{j=0}^\infty \upsilon_{ij}\int_0^\infty [g(x,\varsigma)]^\rho [G(x,\varsigma)]^{j+(\rho+i)\beta-\rho}dx\right)\right\} \tag{20}$$

Now put (10) and (8) in equation (20) we get

$$S_\rho = \frac{1}{\rho-1}\left\{1-\left(\sum_{i=0}^\infty\sum_{j=0}^\infty \upsilon_{ij}\int_0^\infty\left[\frac{\theta\lambda}{x^{2\lambda+1}}e^{\frac{-\theta}{2x^{2\lambda}}}\right]^\rho\left(e^{\frac{-\theta}{2x^{2\lambda}}}\right)^{j+(\rho+i)\beta-\rho}dx\right)\right\}$$

After solving the above equation we get

$$S_\rho = \frac{1}{\rho-1}\left\{1-\left[\sum_{i=0}^\infty\sum_{j=0}^\infty \upsilon_{ij}\frac{(\theta\lambda)^{\rho-1}}{[j+(\rho+i)\beta]}\left[\frac{j+(\rho+i)\beta\theta}{2}\right]^{\frac{-\rho(2\lambda+1)+2\lambda+1}{2\lambda}}\Gamma\left(\frac{\rho(2\lambda+1)-1}{2\lambda}\right)\right]\right\}$$

IX. Mean Deviation from Mean

We know that

$$D(\mu) = E(|x-\mu|)$$

$$D(\mu) = \int_0^\infty |x-\mu|\, f(x)dx$$

$$D(\mu) = 2\mu F(\mu) - 2\int_0^\mu xf(x)dx \tag{21}$$

$$Now\, we\, have \int_0^\mu xf(x)dx = \int_0^\mu x\sum_{i=0}^\infty\sum_{j=0}^\infty \frac{(-1)^i}{i!}\binom{\beta(i+1)+j}{j}\alpha^{(i+1)}\beta\frac{\theta\lambda}{x^{2\lambda+1}}e^{\frac{-\theta}{2x^{2\lambda}}}\left[e^{\frac{-\theta}{2x^{2\lambda}}}\right]^{j+\beta(i+1)-1}dx$$

$$\int_0^\mu xf(x)dx = \sum_{i=0}^\infty\sum_{j=0}^\infty \delta_{ij}\left(\frac{\theta}{2}\right)^{\frac{1}{2\lambda}}\Gamma\left\{\left((1-\frac{1}{2\lambda}\right),\frac{(j+\beta(i+1))\theta}{2\mu^{2\lambda}}\right\} \tag{22}$$

Substituting (22) in (21) we get

$$D(\mu)=2\mu\left[1-e^{-\alpha\left(e^{\frac{\theta}{2\mu^{2\lambda}}}-1\right)^{-\beta}}\right]-2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{ij}\left(\frac{\theta}{2}\right)^{\frac{1}{2\lambda}}\Gamma\left\{\left((1-\frac{1}{2\lambda}\right),\frac{(j+\beta(i+1))\theta}{2\mu^{2\lambda}}\right\}$$

X. Mean Deviation from Median

We know that

$$D(M)=E(|x-M|)$$

$$D(M)=\int_0^{\infty}|x-M|\,f(x)dx$$

$$D(M)=\mu-2\int_0^{M}xf(x)dx \tag{23}$$

Now we have

$$\int_0^{M}xf(x)dx=\int_0^{M}x\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i}{i!}\binom{\beta(i+1)+j}{j}\alpha^{(i+1)}\beta\frac{\theta\lambda}{x^{2\lambda+1}}e^{\frac{-\theta}{2x^{2\lambda}}}\left[e^{\frac{-\theta}{2x^{2\lambda}}}\right]^{j+\beta(i+1)-1}dx$$

$$\int_0^{M}xf(x)dx=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{ij}\left(\frac{\theta}{2}\right)^{\frac{1}{2\lambda}}\Gamma\left\{\left((1-\frac{1}{2\lambda}\right),\frac{(j+\beta(i+1))\theta}{2M^{2\lambda}}\right\} \tag{24}$$

Substituting (24) in (23) we get

$$D(M)=\mu-2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\delta_{ij}\left(\frac{\theta}{2}\right)^{\frac{1}{2\lambda}}\Gamma\left\{\left((1-\frac{1}{2\lambda}\right),\frac{(j+\beta(i+1))\theta}{2M^{2\lambda}}\right\}$$

XI. Maximum Likelihood Estimation

Let $x_1, x_2, x_3, ..., x_n$ be n random samples from Weibull Inverse Power Rayleigh Distribution, and then its likelihood function is given by

$$l=\prod_{i=1}^{n}f(x;\alpha,\beta,\theta,\lambda)$$

$$l=(\alpha\beta\theta\lambda)^n\prod_{i=1}^{n}x_i^{-(2\lambda+1)}e^{\frac{\theta}{2}\sum_{i=1}^{n}x_i^{-2\lambda}}\left(e^{\frac{\theta}{2x^{2x}}}-1\right)^{-\beta-1}e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}}$$

Its Log Likelihood function is given by

$$\log l = n\log\alpha + n\log\beta + n\log\theta + n\log\lambda - (2\lambda+1)\sum_{i=1}^{n}\log x_i + \frac{\theta}{2}\sum_{i=1}^{n} x_i^{-2\lambda}$$

$$-(\beta+1)\sum_{i=1}^{n}\log\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right) - \alpha\sum_{i=1}^{n}\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}$$

Differentiating the above w.r.t $\alpha, \beta, \theta\, and\, \lambda$, we will get the following equations

$$\frac{\partial \log l}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}$$

$$\frac{\partial \log l}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\log\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right) + \alpha\sum_{i=1}^{n}\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}\log\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)$$

$$\frac{\partial \log l}{\partial\theta} = \frac{n}{\theta} + \frac{1}{2}\sum_{i=1}^{n} x_i^{-2\lambda} - (\beta+1)\sum_{i=1}^{n}\frac{1}{\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)}\frac{1}{2x_i^{2\lambda}}e^{\frac{\theta}{2x_i^{2\lambda}}} + \alpha\beta\sum_{i=1}^{n}\frac{1}{2x_i^{2\lambda}}e^{\frac{\theta}{2x_i^{2\lambda}}}\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta-1}$$

$$\frac{\partial \log l}{\partial\lambda} = \frac{n}{\lambda} - 2\sum_{i=1}^{n}\log x_i - \theta\sum_{i=1}^{n} x_i^{-2\lambda}\log x_i + (\beta+1)\theta\sum_{i=1}^{n}\frac{1}{\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)}e^{\frac{\theta}{2x_i^{2\lambda}}} x_i^{-2\lambda}\log x_i$$

$$-\alpha\beta\theta\sum_{i=1}^{n} e^{\frac{\theta}{2x_i^{2\lambda}}}\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta-1} x_i^{-2\lambda}\log x_i$$

The above equations are non-linear equations which cannot be expressed in compact form and it is difficult to solve these equations explicitly for α, β, θ and λ. By applying the iterative methods such as Newton–Raphson method, secant method, Regula-Falsi method etc. the MLE of the parameters denoted as $\hat{\eta}(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda})$ of $\eta(\alpha, \beta, \theta, \lambda)$ can be obtained by using the above methods.

Since the MLE of $\hat{\eta}$ follows asymptotically normal distribution as given as follows $\sqrt{n}(\hat{\eta}-\eta) \rightarrow N(0, I(\eta))$

Where $I^{-1}(\eta)$ is the limiting variance covariance matrix $\hat{\eta}$ and $I(\eta)$ is a 4x4 Fisher Information matrix i.e

$$I(\eta) = -\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2 \log l}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \log l}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 \log l}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log l}{\partial \alpha \partial \lambda}\right) \\ E\left(\frac{\partial^2 \log l}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial \beta^2}\right) & E\left(\frac{\partial^2 \log l}{\partial \beta \partial \theta}\right) & E\left(\frac{\partial^2 \log l}{\partial \beta \partial \lambda}\right) \\ E\left(\frac{\partial^2 \log l}{\partial \theta \partial \alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial \theta \partial \beta}\right) & E\left(\frac{\partial^2 \log l}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log l}{\partial \theta \partial \lambda}\right) \\ E\left(\frac{\partial^2 \log l}{\partial \lambda \partial \alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial \lambda \partial \beta}\right) & E\left(\frac{\partial^2 \log l}{\partial \lambda \partial \theta}\right) & E\left(\frac{\partial^2 \log l}{\partial \lambda^2}\right) \end{bmatrix}$$

Hence the approximate $100(1-\psi)\%$ confidence interval for α, β, θ and λ are respectively given by

$$\hat{\alpha} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\eta})} \qquad \hat{\beta} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\beta\beta}^{-1}(\hat{\eta})} \qquad \theta \pm Z_{\frac{\psi}{2}} \sqrt{I_{\theta\theta}^{-1}(\hat{\eta})} \qquad \lambda \pm Z_{\frac{\psi}{2}} \sqrt{I_{\lambda\lambda}^{-1}(\hat{\eta})}$$

Where $Z_{\frac{\psi}{2}}$ is the ψ^{th} percentile of the standard distribution.

XII. Order Statistics

Let $x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(n)}$ denotes the order statistics of n random samples drawn from Weibull Inverse Power Rayleigh Distribution, then the pdf of $X_{(k)}$ is given by

$$f_{x(k)}(x;\theta) = \frac{n!}{(k-1)!(n-k)!} f_X(x)\left[F_X(x)\right]^{k-1}\left[1-F_X(x)\right]^{n-k}$$

$$f_{x(k)}(x;\theta) = \frac{n!}{(k-1)!(n-k)!} \frac{\alpha\beta\theta\lambda}{x^{2\lambda+1}} e^{\frac{\theta}{2x^{2\lambda}}} \left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta-1} e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}} \left\{1-e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}}\right\}^{k-1} \left(e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}}\right)^{n-k}$$

Then the pdf of first order $X_{(1)}$ Weibull Inverse Power Rayleigh Distribution is given by

$$f_{x(1)}(x;\theta) = \frac{n\alpha\beta\theta\lambda}{x^{2\lambda+1}} e^{\frac{\theta}{2x^{2\lambda}}} \left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta-1} e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}} \left(e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}}\right)^{n-1}$$

and the pdf of nth order $X_{(n)}$ Weibull Inverse Power Rayleigh Distribution is given by

$$f_{x(n)}(x;\theta) = \frac{n\alpha\beta\theta\lambda}{x^{2\lambda+1}} e^{\frac{\theta}{2x^{2\lambda}}} \left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta-1} e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}} \left\{1-e^{-\alpha\left(e^{\frac{\theta}{2x^{2\lambda}}}-1\right)^{-\beta}}\right\}^{n-1}$$

XIII. Simulation Study

In this section, we study the performance of ML estimators for different sample sizes (n=, 50,150, 250,500). We have employed the inverse CDF technique for data simulation for WIPRD distribution using R software. The process was repeated 1000 times for calculation of bias, variance and MSE.

Table 1: *The Mean values, Average bias and MSEs of 1,000 simulations of WIPRD for parameter values*

Sample Size n	parameters	$\alpha = 1.0 \quad \beta = 0.9 \quad \theta = 2.3 \; and \; \lambda = 0.3$			
		Average	Bias	Variance	MSE
50	α	0.9493	-0.050	0.0001	0.0027
	β	0.7717	-0.128	0.0001	0.0166
	θ	2.0815	-0.218	0.0246	0.0723
	λ	2.0510	1.7510	0.0768	3.1429
150	α	0.9537	-0.046	8.41e-05	0.0022
	β	0.7760	-0.123	8.11e-05	0.0154
	θ	2.0854	-0.214	7.07e-03	0.0531
	λ	1.9675	1.6675	2.55e-02	2.8063
250	α	0.9545	-0.045	6.23e-05	0.0021
	β	0.7767	-0.123	5.81e-05	0.0152
	θ	2.0866	-0.213	4.24e-05	0.0497
	λ	1.9479	1.6479	1.56e-05	2.7313
500	α	0.9556	-0.044	3.43e-05	0.0020
	β	0.7778	-0.122	3.11e-05	0.0149
	θ	2.0905	-0.209	2.15e-05	0.0460
	λ	1.9261	1.6261	8.36e-05	2.6528
		$\alpha = 1.1 \quad \beta = 0.8 \quad \theta = 2.0 \; and \; \lambda = 0.5$			
50	α	0.9465	-0.153	0.0001	0.0236
	β	0.7688	-0.031	0.0002	0.0011
	θ	1.7951	-0.204	0.0216	0.0635
	λ	1.1839	0.6839	0.0248	0.4926
150	α	0.9496	-0.150	6.99e-o5	0.0226
	β	0.7716	-0.028	7.04e-05	0.0008
	θ	1.8038	-0.196	6.59e-03	0.0450
	λ	1.1476	0.6476	8.11e-03	0.4275
250	α	0.9508	-0.149	5.19e-05	0.0223
	β	0.7728	-0.027	5.04e-05	0.0007
	θ	1.8116	-0.188	4.33e-03	0.0398
	λ	1.1355	0.6355	4.83e-03	0.4087
500	α	0.9513	-0.148	2.83e-05	0.0221
	β	0.7733	-0.026	2.69e-05	0.0007
	θ	1.8128	-0.187	2.08e-03	0.0371
	λ	1.1277	0.6277	2.59e-03	0.3967

As is clear from table 1, decreasing trend is being observed in average bias, variance and MSE as we increase the sample size. Hence, the performance of ML estimators is quite well, consistent in case of Weibull Inverse Power Rayleigh Distribution.

XIV. Application

In this segment, the efficacy of the developed distribution has been assessed using two realistic sets of data. As the new distribution is compared to New Modified Weibull distribution (NMWD), Additive Weibull distribution (AWD), Power Gompertz distribution (PGD), Inverse power Rayleigh distribution (IPRD), Weibull distribution (WD), Lindley distribution (LD) and Hamza distribution (HD). It is revealed that the new developed distribution offers an appropriate fit.

Various criterion including the AIC (Akaike information criterion), CAIC (Consistent Akaike information criterion), BIC (Bayesian information criterion), HQIC (Hannan-Quinn information criteria) and KS (Kolmogorov-Smirnov) are used to compare the fitted models. The p-value of each model is also recorded. A distribution having lesser AIC, CAIC, BIC, HQIC and KS values and with large p-value is considered better one.

$$AIC = 2k - 2\ln l \qquad CAIC = \frac{2k}{n-k-1} - 2\ln l$$

$$BIC = k\ln n - 2\ln l \qquad HQIC = 2k\ln(\ln(n)) - 2\ln l$$

Data set 1:- The following represents the dataset of 63 0bservations of the tensile strength measurements on 1000 carbon fiber-impregnated tows at four different gauge lengths. The data reported by Bader and Priest (1982) as follows:

1.901, 2.132, 2.203, 2.228, 2.257,2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886 ,3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

Data set 2: The second data represents COVID-19 mortality rates data belongs to Italy of 59 days that is recorded from 27 February to 27 April 2020. The data is as follows:

4.571, 7.201, 3.606, 8.479, 11.410, 8.961, 10.919, 10.908, 6.503, 18.474, 11.010 ,17.337, 16.561, 13.226, 15.137, 8.697, 15.787,13.333, 11.822, 14.242, 11.273, 14.330, 16.046, 11.950, 10.282, 11.775, 10.138, 9.037, 12.396, 10.644, 8.646, 8.905, 8.906, 7.407, 7.445 ,7.214, 6.194, 4.640, 5.452, 5.073 ,4.416 ,4.859 ,4.408, 4.639, 3.148, 4.040, 4.253, 4.011, 3.564, 3.827, 3.134, 2.780, 2.881, 3.341, 2.686, 2.814, 2.508, 2.450, 1.518.

The fitted models are compared using empirical goodness of fit measures such as the AIC (Akaike information criterion), CAIC (Consistent Akaike information criterion), BIC (Bayesian information criterion), HQIC (Hannan-Quinn information criteria), and KS (Kolmogorov- Smirnov). Each model's p-value is also displayed. A distribution with a lower AIC, CAIC, BIC, and HQIC together with a higher p value is rated as the top distribution

Table 2 and 5 shows the descriptive statistics for data set 1 and data set 2 respectively. Table 3 and 6 displays the parameter estimates for the data set 1 and data set 2 respectively. Table 4 and 7 displays the log-likelihood, Akaike information criteria (AIC), BIC (Bayesian information criterion) etc. details and some other statistics for the data set 1 and data set 2.

Table 2. *Descriptive Statistics for data set 1*

Mean	Min.	Max.	Q_1	Q_3	Median	S.D	Skew.	Kurt.
3.059	1.901	5.020	2.554	3.421	2.996	0.620	0.632	3.286

Table 3. *The ML Estimates and standard error of the unknown parameters*

Model		WIPRD	NMWD	AWD	PGD	IPRD	WD	LD	HD
$\hat{\alpha}$		11.4370	0.01469	0.00292	0.00438	---------	0.003775	---------	---------
$\hat{\beta}$		0.7586	0.00100	0.00100	2.72725	---------	4.69909	---------	0.00100
$\hat{\theta}$		60.8587	3.14576	4.68182	0.71363	15.1809	-----------	0.53923	2.28724
$\hat{\lambda}$		0.91032	0.43441	4.54990	---------	0.91031	-----------	---------	---------
Stan-	$\hat{\alpha}$	5.880	0.0063	0.00081	0.00165	---------	0.00105	---------	---------
dard	$\hat{\beta}$	0.5707	0.00034	0.00058	0.51536	---------	0.22866	---------	0.62231
Error	$\hat{\theta}$	53.0940	0.29681	0.21350	0.09294	2.34246	-----------	0.04958	0.05084
	$\hat{\lambda}$	0.0810	0.26257	0.26577	---------	0.08109	-----------	---------	---------

Table 4. *Performance of distributions*

Model	WIPRD	NMWD	AWD	PGD	IPRD	WD	LD	HD
-2logl	111.612	138.816	124.704	134.466	185.528	124.546	242.714	150.514
AIC	119.612	146.816	132.705	140.466	189.528	128.546	244.715	154.515
AICC	120.301	147.505	133.395	146.896	193.814	128.746	244.780	154.715
HQIC	122.983	150.187	136.077	140.873	189.728	130.232	245.558	156.201
BIC	128.184	155.388	141.278	142.995	191.214	132.832	246.858	158.802
K-S Value	0.08215	0.1448	0.11151	0 .143	0.35439	0.11139	0.4308	0.21545
P Value	0.7888	0.1424	0.4137	0.152	2.68e-07	0.4151	1.39e-10	0.00576

Table 5. *Descriptive Statistics for data set 2*

Mean	Min.	Max.	Q_1	Q_3	Median	S.D	Skew.	Kurt.
8.156	1.518	18.474	4.146	11.341	7.445	4.5267	0.4523	2.1281

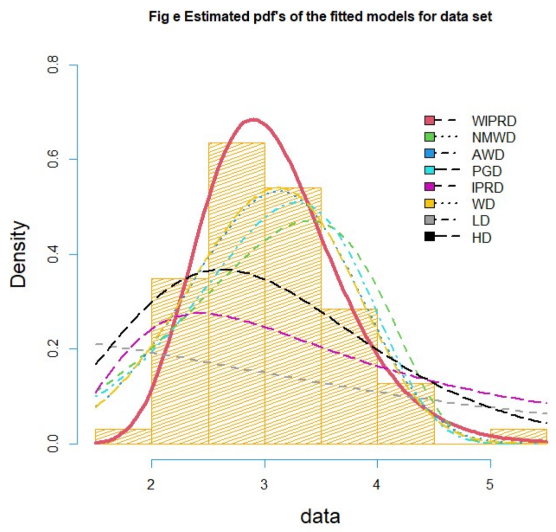
Table 6. *The ML Estimates and standard error of the unknown parameters*

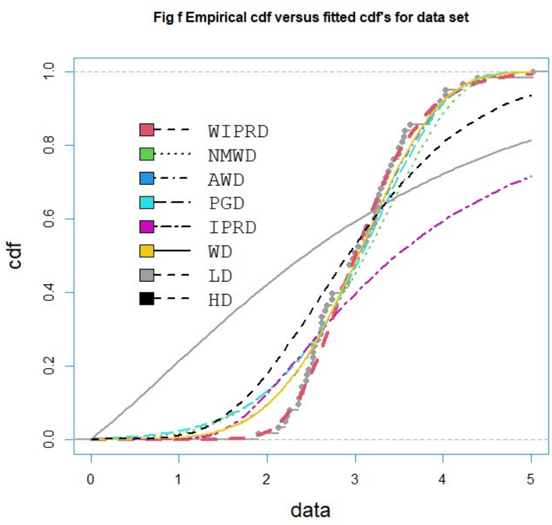
Model		WIPRD	NMWD	AWD	PGD	IPRD	WD	LD	HD
$\hat{\alpha}$		0.2065	0.02612	0.02612	0.01746	---------	0.014105	---------	---------
$\hat{\beta}$		0.82124	0.00100	0.00100	0.00387	---------	1.918049	---------	21.5900
$\hat{\theta}$		11.26308	1.434910	1.43491	1.75839	22.8479	-----------	0.222869	0.81384
$\hat{\lambda}$		0.51936	0.293231	0.29323	---------	0.51936	-----------	---------	---------
Standard Error	$\hat{\alpha}$	0.16217	0.01111	0.01111	0.01006	---------	0.006603	---------	---------
	$\hat{\beta}$	0.12048	0.013344	0.01334	0.00864	---------	0.18601	---------	28.7503
	$\hat{\theta}$	7.0582	0.15826	0.15826	0.31366	3.6430	-----------	0.02068	0.05698
	$\hat{\lambda}$	0.04781	0.23093	0.230931	---------	0.04781	-----------	---------	---------

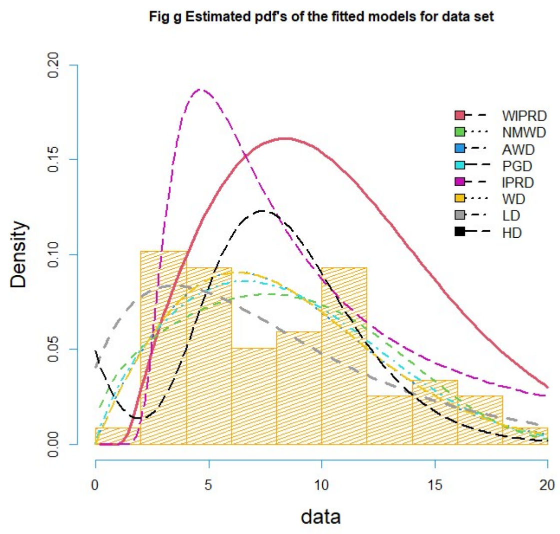
Table 7. *Performance of distributions*

Model	WIPRD	NMWD	AWD	PGD	IPRD	WD	LD	HD
-2logl	267.932	336.232	336.232	334.882	289.302	335.404	346.598	362.508
AIC	275.932	344.232	344.232	340.882	293.302	339.404	348.599	366.508
AICC	276.673	344.973	344.973	341.319	293.516	339.618	348.669	366.722
HQIC	279.176	347.476	347.476	343.315	294.924	341.026	349.410	368.129
BIC	284.242	352.543	352.543	347.115	297.457	343.559	350.677	370.663
K-S Value	0.10235	0.12345	0.21187	0.12154	0.22795	0.121	0.14516	0.21409
P Value	0.5331	0.3041	0.00839	0.3216	0.00354	0.3268	0.1506	0.00748

As it is obvious from table 4 and table 7 that the Weibull inverse power Rayleigh distribution has smaller values for AIC, AICC, BIC, HQIC and K-S statistics as compared with its sub models. Accordingly we arrive at the conclusion that Weibull inverse power Rayleigh distribution provides an adequate fit than the compared ones.







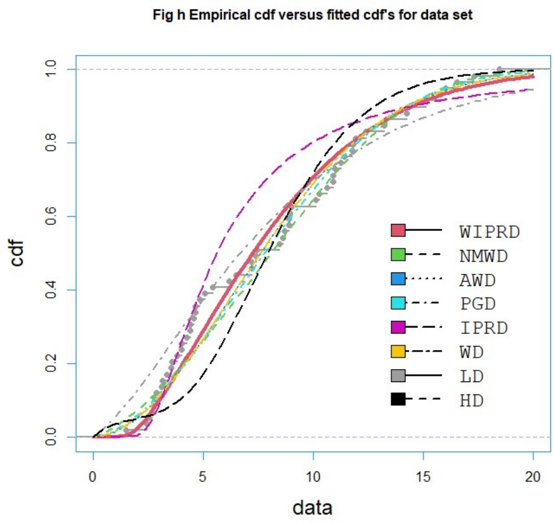


Figure e, f, g and h represents the estimated densities and cdfs of the fitted distributions to data set 1st and 2nd.

XV. Conclusion

This newly introduced distribution "Weibull inverse power Rayleigh distribution" which is obtained by T-X method. Several mathematical quantities for the newly developed distribution are derived including moments, moment generating function, incomplete moments, order statistics, different measure of entropies etc. To show the behavior of p.d.f, c.d.f and other related measures different plots have been drawn. The parameters are obtained by MLE technique. Lastly by carrying out through two real life data sets to show that the formulated distribution leads an improved fit than the compared ones.

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