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MAP/PH/1 Queue with Vacation, Customer Induced Interruption, Optional Service, Breakdown and Repair Completion

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Abstract

The paper considers a single server that provides consumers with both regular and optional services. The system's inter-arrival time is determined by a Markovian Arrival Process (MAP), the service time is determined by a phase type distribution, and the remaining random variables are distributed exponentially. This system was represented as a QBD process, with the block elements of the generated matrix having finite dimensions, to investigate steady state. Additionally, we addressed the busy period and waiting time distribution for our concept. The system's performance parameters are calculated and graphically shown.

Keywords: Markovian Arrival Process, PH distribution, Vacation, Optional service, Breakdown and Repair.

1. Introduction

Contribution of Nuets (1979) is immeasurable in the field of stochastic process. He pioneered the Markovian Point Process, which led to the development of the Markovian Arrival Process and the Batch Markovian Arrival Process. In his concept of communication and computer application, Lucantoni (1990) established these two arrival processes. One of the most important characteristics of MAP is that it can be used to solve stochastic models using matrix analytic solutions. Chakravarty (2010) in the Encyclopedia of Operations Research and Management Science streamlined this useful tool to make it easier to understand. The discrete and continuous cases of MAP are defined. The parameters utilised in MAP are D_0 and D_1 of dimension m in continuous time, where D_0 is a non-singular stable matrix that rules the transition corresponding to no arrival and D_1 governs the transition relating to arrival. The generator matrix $Q = D_0 + D_1$.The stationary distribution vector π satisfies the system $\pi(D_0 + D_1) = 0$ and $\pi e = 1$.The constant $\lambda = \pi D_1$ is called the fundamental rate of a MAP defined by Latuche at al (1999).

The concept of vacation in the queuing system was introduced by Levy and Yechiali (1975). Vacation is a time where the server is unavailable for service, for a short period of time due to many reasons like filing up the bills or document, verifying with other data etc or even a break. In this real world, there are many occasions where the server is busy or continues to work with low speed during his vacation. Servi (2002) and Finn classify this type of vacation as a working vacation. Doshi (1986), Takagi (1991) and Tian and Zhang (2006) has contributed an excellent survey on the vacation model. Ket et al (2010) and Tian et al (2009) have also recently contemplated on vacation and working vacation. This working vacation concept was introduced and further studied in a retail queue by Do (2010).

A detailed study has been performed by Doshi (1986) for the queueing model with vacations, breakdown and repair in his survey with demonstration. Ayyappan and Thamizhselvi (2018) have reviewed a priority retail model with vacation and also studied the time dependent PGF. Further, second optional service under mixed priority service was studied by K.Jeganathan (2015) in linear retail inventory system. Breakdown and repair process are unavoidable concepts in production unit, service stations etc. When the server gets breakdown then the server gets terminated and goes to the repair process. Depending on the model, the server begins serving the customer whose service was interrupted or begins serving a new customer after the repair process is completed. This concepts was studied in various queueing models by Gaver (1959) , Levy and Yechilai (1976) and many more are interested in this concept.

In reality, the service of the server can terminate for a short period of time. This phenomena is named as interruption which is one of the unavoidable aspects faced by both the server and the customer in the system due to many reasons like emergency call or work, the server/machine may get breakdown, external influence, get some suggestions/ ideas from the fellow workers etc. This interruption was studied in priority queue by Jaiswak(1961). Geramsimov(1973) came up with an idea for investigating an interrupted customer with an algorithm where another queue for interrupted customers were formed and served. This concept was developed in Computer and Communication System by Gelenbe and Derochette(1978). Takine and Sengupta (1997) developed this concept in a MAP process. Rakesh Kumar(2014) investigated discouraged arrivals and customer retention in a single server Markovian queueing system. Rakesh Kumar and Bhavneet Singh Soodam(2019) investigated linked imputes and reneging for a single server queueing model. El-Taha(2003) introduced two server in series where the customer gets interrupted by the set of proposed time threshold while getting service from the server one. Server two offer service only to the interrupted customer or else the customer leaves the system. Krishnamoorthy et al.(2009) developed the model in a single server queue in a level-dependent-quasi-birth and death (LDQBD) process. He further generalized this model in (2011) where a super clock is defined for his predetermined threshold time. Kilmenok and Dudin(2012) and Krishnamurthy et al.(2010) have also investigated further where interrupted customer service has been rejected. Varghese et al.(2010) introduced customer induced service interruption where the customer gets self interrupted while being served by the server. Further extension in this concept has been made by them in the year 2012.

2. Model Description

In this classical queuing model, a single server is considered with the infinite capacity queue where the customer arrives according to Markovian Arrival Process with the parameter matrix D_0 and D_1 are of dimension m. The customer in the service station can be self interrupted and moves onto the buffer 1 which is of maximum capacity K with an exponential distribution δ. After the completion of interruption, the customer moves onto the buffer 2 with an exponential distribution θ which is also of maximum capacity K where the customer gets served by the server. Optional service will be provided by the server whenever the customer needs it. The service time of the server offering service for the customer from the queue, buffer 2 and optional service follows phase type distribution $PH(\alpha_1, t_1), PH(\alpha_2, t_2), PH(\alpha_3, t_3)$ respectively of order n. The vector T_1^0, T_2^0, T_3^0 is given by $T_1^0 = -T_1 e, T_2^0 = -T_2 e, T_3^0 = -T_3 e$ respectively.

The interrupted customer will only enter buffer 1 if space is available; else, the customer will be lost indefinitely. The server follows non-preemptive priority for the customer in the buffer 2 over the customer in the queue. Thus a preference will be given to the customer in buffer 2 whenever the free server offers service. The customer in buffer 2 will be served by first in first service order. When the system is empty, the server avails vacation following exponential distribution with the parameter η. While the server is busy, breakdown of the server may occur. It follows exponential distribution with the parameter γ. Consequently, the repair process starts immediately with the phase type distribution $PH(\beta, R)$ of order r. The vector R^0 is given by

$R^0 = -Re$. After receiving the regular service, the customer can opt for optional service. During the optional service availed by the customer, if the server gets breakdown then the interrupted customer is considered to be lost forever from the system. The server is idle when there are no customers in the queue and buffer 2.

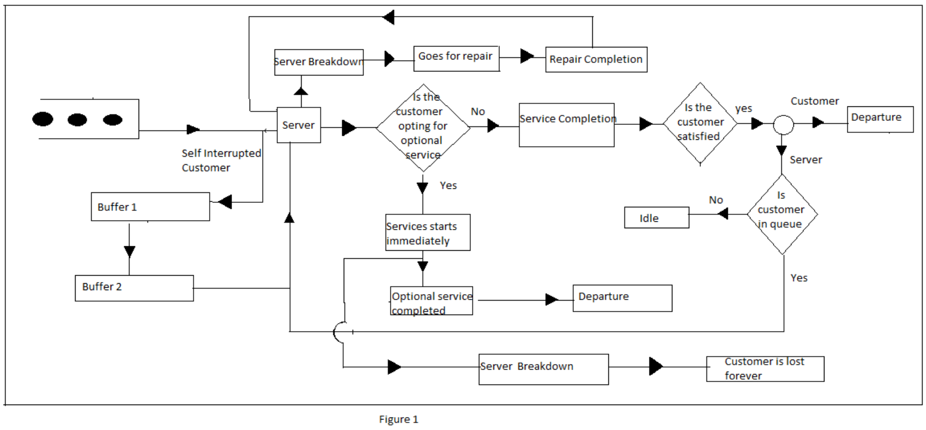

Figure 1: *Schematic Representation of Our Model*

To find a matrix-geometric type solution, this model is investigated as a QBD process. For a thorough examination of Matrix Analytic Methods, see Neuts (1981), Latouche and Ramaswami (1999). The state space under the considered QBD model is defined and the structure of the infinitesimal generator is also investigated using the following notations.

Let

- N(t) be the number of customers in the system at time t
- S(t) be the server status at time t

where

$$S(t) = \begin{cases} 0, & \text{if server is idle} \\ 1, & \text{if server is offering service to the customer in the main queue} \\ 2, & \text{if server is offering service to the customer in the buffer 2} \\ 3, & \text{if server is offering optional service} \\ 4, & \text{if main server is availing vacation} \\ 5, & \text{if the server is under repair} \end{cases}$$

- $N_1(t)$ be the number of customers in the buffer 1 at time t
- $N_2(t)$ be the number of customers in the buffer 2 at time t
- $R(t)$ be the Repair phase at time t
- $C(t)$ be the service phase at time t
- $M(t)$ be the phase of the Markovian Arrival Process at time t
- $M_1 = \frac{K(K+1)}{2}$
- $M_2 = \frac{(K+1)(K+2)}{2}$, where K is the maximum capacity of buffer 1.

$\{N(t), S(t), N_1(t), N_2(t), C(t), R(t), M(t); t \geq 0\}$ is representation of this model by the Continuous Time Markov Process with the state space

$$\Omega = l(0) \cup l(i)$$

where

$$\begin{gathered} l(0) = \{(0,0,0,i_2) : 1 \le i_2 \le K\} \\ \cup \{(0,j,i_1,i_2,k,l) : j = 2; 0 \le i_1 \le K; 1 \le i_2 \le K; i_1 + i_2 \le K; 0 \le k \le n; 0 \le l \le m\} \\ \cup \{(0,j,i_1,i_2,k,l) : j = 3,5; 0 \le i_1, i_2 \le K; i_1 + i_2 \le K; 0 \le k \le n; 0 \le l \le m\} \\ \cup \{(0,4,0,0,l) : 0 \le l \le m\} \end{gathered}$$

for $i \ge 1$,

$$\begin{gathered} l(i) = \{(i,j,i_1,i_2,k,l) : j = 1,3,5; 0 \le i_1, i_2 \le K; i_1 + i_2 \le K; 0 \le k \le n; 0 \le l \le m\} \\ \cup \{(i,j,i_1,i_2,k,l) : j = 2; 0 \le i_1 \le K; 1 \le i_2 \le K; i_1 + i_2 \le K; 0 \le k \le n; 0 \le l \le m\} \\ \cup \{(i,4,0,0,l) : 0 \le l \le m\} \end{gathered}$$

The infinitesimal matrix generation of the QBD process is given by

$$Q = \begin{bmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0\cdots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0\cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0\cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0\cdots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \end{bmatrix}$$

where

$$B_{00} =$$
$$\begin{bmatrix} b^{(00)}_{(11)} & b^{(00)}_{(12)} & 0 & 0 & 0 \\ 0 & T_2 \oplus D_0 - \gamma I_m \otimes I_{M_1} & diag(a_K, a_{K-1}, ..., a_1) & e_{M_1} \otimes q T_2^0 \otimes I_m & diag(e_K, e_{K-1}, ..., e_1) \\ 0 & 0 & b^{(00)}_{(33)} & e_{M_2} \otimes T_3^0 \otimes I_m & \gamma I_{M_2 rm} \\ \gamma I_m & 0 & 0 & D_0 - \gamma I_m & 0 \\ b^{(00)}_{(51)} & diag(c_K, c_{K-1}, ..., c_1) & 0 & 0 & (R \oplus D_0) \otimes I_{M_2} \end{bmatrix}$$

$$b^{(00)}_{(11)} = \begin{bmatrix} D_0 & 0 \\ 0 & I_K \otimes (D_0 - \theta I_m) \end{bmatrix}$$

$$b^{(00)}_{(12)} = \begin{bmatrix} 0 \\ diag[\theta I_m \quad 0] \end{bmatrix}$$

$$b^{00}_{(33)} = \begin{bmatrix} T_3 \oplus (D_0 - \gamma I_m) \otimes I_{K+1} & 0 \\ 0 & T_3 \oplus (D_0 - (\gamma + \theta) I_m) \otimes I_{K+1} \end{bmatrix} + \begin{bmatrix} 0 \\ diag(b_K, b_{K-1}, ..., b_1) \end{bmatrix}$$

$$b^{(00)}_{(51)} = \begin{bmatrix} e_n \otimes R^0 \beta I_m & 0 & 0 & 0 & 0 \\ 0 & e_n \otimes R^0 \beta I_m & 0 & 0 & 0 \\ 0 & 0 & e_n \otimes R^0 \beta I_m & 0 & 0 \\ 0 & 0 & 0 & e_n \otimes R^0 \beta I_m & 0 \\ 0 & 0 & 0 & 0 & e_n \otimes R^0 \beta I_m \end{bmatrix}$$

$$B_{01} = \begin{bmatrix} b^{(0,1)}_{(1,1)} & 0 & 0 & 0 & 0 \\ 0 & I_{nM_1} \otimes D_1 & 0 & 0 & 0 \\ 0 & 0 & I_{nM_2} \otimes D_1 & 0 & 0 \\ 0 & 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 & I_{rM_2} \otimes D_1 \end{bmatrix}$$

$$b^{(01)}_{(11)} = \begin{bmatrix} D_1 & 0 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 & 0 \\ 0 & 0 & D_1 & 0 & 0 \\ 0 & 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 & D_1 \end{bmatrix}$$

$$B_{10} = \begin{bmatrix} b^{(10)}_{(11)} & diag(d_K, d_{K-1}, ..., d_1) & pT_1^0\alpha_1 \otimes I_{M_2} & e_n \otimes qT_1^0\alpha_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$b^{(10)}_{(11)} = \begin{bmatrix} 0 & e_n \otimes \delta I_{mn} & 0 & 0 & 0 \\ 0 & e_n \otimes pT_1^0\alpha_1 & e_n \otimes \delta I_{mn} & 0 & 0 \\ 0 & 0 & e_n \otimes pT_1^0\alpha_1 & e_n \otimes \delta I_{mn} & 0 \\ 0 & 0 & 0 & e_n \otimes pT_1^0\alpha_1 & 0 \\ 0 & 0 & 0 & 0 & e_n \otimes pT_1^0\alpha_1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} a^2_{(1,1)} & diag(d_K, d_{K-1}, ..., d_1) & pT_1^0\alpha_1 \otimes I_{mM_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$a^2_{(11)} = \begin{bmatrix} pT_1^0\alpha_1 \otimes Im & \delta I_{mn} & 0 & 0 & 0 \\ 0 & pT_1^0\alpha_1 \otimes Im & \delta I_{mn} & 0 & 0 \\ 0 & 0 & pT_1^0\alpha_1 \otimes Im & \delta I_{mn} & 0 \\ 0 & 0 & 0 & pT_1^0\alpha_1 \otimes Im & 0 \\ 0 & 0 & 0 & 0 & pT_1^0\alpha_1 \otimes Im \end{bmatrix}$$

$$A_1 = \begin{bmatrix} a^1_{(11)} & 0 & 0 & 0 & \gamma I_{rmM_2} \\ a^1_{(21)} & a^1_{(22)} & diag(a_K, a_{K-1}, ..., a_1) & 0 & diag(e_K, e_{K-1}, ..., e_1) \\ T_3^0\alpha_3 \otimes I_{M_2} & 0 & a^1_{(3,3)} & 0 & \gamma I_{rmM_2} \\ \eta I_{mM_2} & 0 & 0 & D_0 - \eta I_m & 0 \\ R^0\beta \otimes I_{mM_2} & 0 & 0 & 0 & (R + D_0) \otimes I_{M_2} \end{bmatrix}$$

$$a^1_{(11)} = \begin{bmatrix} T_1 \oplus D_0 - (\gamma+\delta) I_m \otimes I_K & 0 & 0 & 0 \\ 0 & T_1 \oplus D_0 - \gamma I_m & 0 & 0 \\ 0 & 0 & diag(f_K, f_{K-1}, ..., f_1) & 0 \\ 0 & 0 & 0 & T_1 \oplus D_0 - (\gamma+\theta) I_m \otimes I_{K-1} \end{bmatrix} + diag(b_K, b_{K-1}, ..., b_1)$$

$$a^1_{(21)} = diag \begin{bmatrix} I_K \otimes qT_2^0\alpha_2 \otimes I_n & 0 \end{bmatrix}$$

$$a^1_{(22)} = \begin{bmatrix} T_2 \oplus D_0 - \gamma I_m \otimes I_K & 0 \\ 0 & T_2 \oplus D_0 - (\gamma+\theta) I_m \otimes I_{M_1} \end{bmatrix} + \begin{bmatrix} 0 \\ diag(g_{K-1}, ..., g_1) \end{bmatrix} + diag \begin{bmatrix} 0 \\ qT_2^0\alpha_2 \otimes I_{mn(i-1)} \end{bmatrix}$$

$$a^1_{(33)} = \begin{bmatrix} T_3 \oplus (D_0 - \gamma) I_m \otimes I_{K+1} & 0 \\ 0 & diag(h_K, h_{K_1}, ..., h_1) \end{bmatrix} + \begin{bmatrix} 0 \\ diag(g_{K-1}, ..., g_1) \end{bmatrix}$$

$$A_0 = \begin{bmatrix} I_{nM_2} \otimes D_1 & 0 & 0 & 0 & 0 \\ 0 & I_{nM_1} \otimes D_1 & 0 & 0 & 0 \\ 0 & 0 & I_{nM_2} \otimes D_1 & 0 & 0 \\ 0 & 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 & I_{rM_2} \otimes D_1 \end{bmatrix}$$

$$a_i = pT_2^0\alpha_2 \otimes I_{im}$$

$$b_i = \theta I_{inm}$$

$$c_i = \begin{bmatrix} 0 \\ R^0\beta \otimes I_{im} \end{bmatrix}$$

$$d_i = \begin{bmatrix} 0 & 0 \\ pT_1^0\alpha_1 \otimes I_i & \delta I_{mn(i-1)} \end{bmatrix}$$

$$e_i = \begin{bmatrix} 0 & \gamma I_{irm} \end{bmatrix}$$

$$f_i = \begin{bmatrix} T_1 \oplus D_0 - (\gamma+\delta+\theta)I_m \otimes I_{i-1} & 0 \\ 0 & T_1 \oplus D_0 - (\gamma+\theta)I_m \end{bmatrix}$$

$$g_i = \begin{bmatrix} 0 & \theta I_{inm} \end{bmatrix}$$

$$h_i = T_3 \oplus D_0 - (\gamma+\theta)I_m \otimes I_K, \, K \le i \le 1.$$

3. Analysis of the Stability Condition

The square matrix $A = A_0 + A_1 + A_2$ is defined of order $m[M_2(2n+r) + (nM_1+1)]$ as an irreducible infinitesimal generator matrix. The invariant probability vector $\wp$ defined as $\wp = (\wp_0, \wp_1, \wp_2, \wp_3, \wp_4)$ satisfies the condition $\wp A = 0$ and $\wp e = 1$. The vector $\wp$ can be computed by solving the following equations.

$$\wp_0(I_{nM_2} \otimes D_1 + a^1_{(1,1)} + a^2_{(1,1)}) + \wp_1(a^1_{(2,1)}) + \wp_2(T_3^0\alpha_3 \otimes I_{M_2}) + \wp_3(\eta I_m) + \wp_4(R^0\beta \otimes I_{M_2}) = 0$$
$$\wp_0(diag(d_K, d_{K-1}, ..., d_1)) + \wp_1(I_{nM_1} \otimes D_1 + a^1_{(2,2)}) = 0$$
$$\wp_0(pT_1^0\alpha_1 \otimes I_{M_2}) + \wp_1(pT_2^0\alpha_2 \otimes I_{M_1}) + \wp_2(I_{nM_2} \otimes D_1 + a^1_{(3,3)}) = 0$$
$$\wp_3(D_1 + D_0 - \eta I_m) = 0$$
$$\wp_0(diag(\gamma I_{rmM_2}) + \wp_1(diag(e_K, e_{K-1}, ..., e_1)) + \wp_2(\gamma I_{rmM_2}) + \wp_4(I_{rM_2} \otimes D_1 + (R + D_0) \otimes I_{M_2}) = 0$$

For the system to attain stability, the necessary and sufficient condition is $\wp A_0 e \le \wp A_2 e$

$$(\wp_0 + \wp_2)(e_{nM_2} \otimes D_1 e_m) + \wp_1(e_{nM_1} \otimes D_1 e_m) + \wp_3(D_1 e_m) + \wp_4(e_{rM_2} \otimes D_1 e_m) < \wp_0(a^2_{(1,1)}) + \wp_1(diag(d_K, d_{K-1}, ..., d_1)) + \wp_2(pT_1^0\alpha_1 \otimes e_{M_2})$$

4. The Invariant Probability Vector

The unique solution to $XQ = 0$ and $Xe = 1$ is the transition probability vector of the infinitesimal generator Q. This X can be partitioned into $(X_0, X_1, X_2 ....)$ where each X_i is the row vector corresponding to the server status. The dimension of X_0 is $m[(K+1) + nM_1 + M_2(n+r) + 1]$, and the remaining probability vectors $X_1, X_2, X_3, ...$ are of equal dimension $m[M_2(2n+r) + (nM_1+1)]$. The steady state probability vector has a matrix geometric structure satisfying the condition for stability is as follows,

$$X_i = X_1 R^{i-1}, i = 2, 3, 4, ...,$$

where R, the rate matrix, is the minimal non-negative solution to the matrix quadratic equation

$$R^2 A_2 + R A_1 + A_0 = 0$$

and the boundary states X_0 and X_1 is the result of solving the equations

$$\begin{gathered} X_0 B_{00} + X_1 B_{10} = 0 \\ X_0 B_{01} + X_1 (A_1 + R A_2) = 0 \end{gathered}$$

subject to normalizing condition

$$X_0 e + X_1 (I - R)^{-1} e = 1$$

Lautouche and Ramaswamy(1999) have embellished the calculation of rate matrix R by developing Logarithmic Reduction Algorithm, which helps us to obtained R easily.

$$\text{Step 1} : H \leftarrow (-A_1)^{-1} A_0, L \leftarrow (-A_1)^{-1} A_2, G = L \text{and } T = H.$$

$$\begin{gathered} \text{Step 2} : U = HL + LH; \\ M = H^2; \\ H = (I - U)^{-1} M; \\ M = L^2; \\ L = (I - U)^{-1} M; \\ G = G + TL; \\ T = TH; \end{gathered}$$

continue Step 1 until $\|e - Ge\|_\infty < \epsilon$.

$$\text{Step 3} : R = -A_0 (A_1 + A_0 G)^{-1}.$$

5. Analysis of Busy Period

In a classical queueing model, the busy period is defined as the time between a customer arriving at an empty queue and the first epoch after that when the queue becomes empty again. Considering a QBD process, Latouche.G (1978) has coined the term fundamental period defined as the first passage time from level i to level $i-1$, $i \geq 2$. For the boundary states $i = 0, 1$ has to be dealt separately. We can also observe that for all level i, where $i \geq 2$ there are $m[M_2(2n+r) + (nM_1+1)]$ states.

Notations:

- $G_{vv'}(k,x)$: The conditional probability that the QBD process enters the level $u-1$ at time $t=0$, by making merely k transition to the left and also by entering the state (u,v') conditioned that it only started in the state (u,v) at time $t=0$.
- The transition matrix $\bar{G}_{vv'}(z,s)=\sum_0^\infty z^k\int_0^\infty e^{-sx}dG_{vv'}(k,x):|z|\leq 1, Re(s)\geq 0$
- $\bar{G}(z,s)$: The matrix $(G_{vv'}(z,s))$, satisfying $\bar{G}(z,s)=z\left[sI-A_1\right]^{-1}A_2+\left[sI-A_1\right]^{-1}A_0\bar{G}^2(z,s)$
- $G=G_{vv'}=\bar{G}(0,1)$ is the first passage time without the boundary states.
- $\bar{G}^{(1,0)}_{(vv')}(k,x)$ is the conditional probability that enters the level 0 from 1 at time $t=0$.
- $\bar{G}^{(0,0)}_{(vv')}(k,x)$ is the first conditional probability returning to level 0.
- $\Re_{1v}$ is the expected first passage time between the levels u and $u-1$, the process in the state (u,v),at time $t=0$.
- $\bar{\Re}_1$ is the column vector $\Re_{1v}$ as its entries.
- $\Re_{2v}$ is the average number of customer who received service in the first passage time between the levels u and $u-1$, begins in the state (u,v), at time $t=0$.
- $\bar{\Re}_2$ is the column vector $\Re_{2v}$ as its entries.
- $\bar{\Re}_1^{(1,0)}$ is the average first passage times from the level 1 to 0.
- $\bar{\Re}_2^{(1,0)}$ is the average number of service completions during the first passage time from the level 1 to 0.
- $\bar{\Re}_1^{(0,0)}$ is the average first return time to level 0.
- $\bar{\Re}_2^{(0,0)}$ is the average number of completed services in the initial return time to level 0.

G matrix can be computed with the help of the result $G=-[A_1+RA_2]^{-1}A_2$ where the rate matrix R is already evaluated using Logarithmic Reduction Algorithmic technique. For the boundary states namely 1 and 0 we have the equations satisfied by $\bar{G}^{(1,0)}(z,s)$ and $\bar{G}^{(0,0)}(z,s)$ respectively.

$$\bar{G}^{(1,0)}(z,s)=z\left[sI-A_1\right]^{-1}B_{10}+\left[sI-A_1\right]^{-1}A_0\bar{G}(z,s)\bar{G}^{(1,0)}(z,s)$$
$$\bar{G}^{(0,0)}(z,s)=z\left[sI-B_{00}\right]^{-1}B_{01}\bar{G}^{(1,0)}(z,s).$$

Since G, $\bar{G}^{(1,0)}(z,s)$, $\bar{G}^{(0,0)}(z,s)$ are stochastic moments can be easily evaluate as follows.

$$\begin{aligned}
\Re_1 &= -\tfrac{\partial}{\partial s}\bar{G}(z,s)|_{s=0,z=1}=-\left[A_0(G+1)+A_1\right]^{-1}e\\
\Re_2 &= \tfrac{\partial}{\partial z}\bar{G}(z,s)|_{s=0,z=1}=-\left[A_0(G+1)+A_1\right]^{-1}A_2e\\
\Re_1^{(1,0)} &= -\tfrac{\partial}{\partial s}\bar{G}^{(1,0)}(z,s)|_{s=0,z=1}=-\left[A_1+A_0G\right]^{-1}\left[A_0\Re_1+e\right]\\
\Re_2^{(1,0)} &= \tfrac{\partial}{\partial z}\bar{G}^{(1,0)}(z,s)|_{s=0,z=1}=-\left[A_1+A_0G\right]^{-1}\left[B_{10}e+A_0\Re_2\right]\\
\Re_1^{(0,0)} &= -\tfrac{\partial}{\partial s}\bar{G}^{(0,0)}(z,s)|_{s=0,z=1}=-B_{00}^{-1}\left[e+B_{01}\Re_1^{(1,0)}\right]\\
\Re_2^{(0,0)} &= \tfrac{\partial}{\partial z}\bar{G}^{(0,0)}(z,s)|_{s=0,z=1}=-B_{00}^{-1}B_{01}\Re_2^{(1,0)}.
\end{aligned}$$

6. Analysis of Waiting Time Distribution

Analysis of distribution of waiting time period of the customer in the queue has been performed in this section by using the first passage time analysis. Let $W(t)$, where $t \geq 0$, denote the distribution function of the waiting time of the tagged incoming customer in the system. The customer in the system has to wait in order to get service from the server if the server is busy or availing vacation or under repair. Otherwise, the customer in the system can get immediate service without any delay when the server is idle. The state space of absorbing time in a Markov chain is given by

$$(*) \cup \overline{0}, \overline{1}, \overline{2}, \overline{3}, ...$$

where $(*)$ denotes the absorbing state, in which the tagged customer gets service from the server without delay and it is defined as

$$(*) = (0,0,0,i_2) : 1 \leq i_2 \leq K$$

The state space of level 0 in a Markov chain is given by

$$\begin{gathered}\overline{0} = \{(0,j,i_1,i_2,k,l) : j = 2; 0 \leq i_1 \leq K; 1 \leq i_2 \leq K; i_1 + i_2 \leq K; 0 \leq k \leq n; 0 \leq l \leq m\} \\ \cup \{(0,j,i_1,i_2,k,l) : j = 3,5; 0 \leq i_1, i_2 \leq K; i_1 + i_2 \leq K; 0 \leq k \leq n; 0 \leq l \leq m\} \\ \cup \{(0,4,0,0,l) : 0 \leq l \leq m\}\end{gathered}$$

The state space of level $i \geq 1$ in a Markov chain is given by

$$\begin{gathered}\overline{i} = \{(i,j,i_1,i_2,k,l) : j = 1,3,5; 0 \leq i_1, i_2 \leq K; i_1 + i_2 \leq K; 0 \leq k \leq n; 0 \leq l \leq m\} \\ \cup \{(0,j,i_1,i_2,k,l) : j = 2; 0 \leq i_1 \leq K; 1 \leq i_2 \leq K; i_1 + i_2 \leq K; 0 \leq k \leq n; 0 \leq l \leq m\} \\ \cup \{(i,4,0,0,l) : 0 \leq l \leq m\}\end{gathered}$$

The transition matrix $\bar{Q}$ is given by

$$\bar{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \cdots \\ H_0 & F_0 & 0 & 0 & 0 \cdots \\ H_1 & F_{10} & F & 0 & 0 \cdots \\ 0 & 0 & F_2 & F & 0 \cdots \\ \cdots & \cdots & \ddots & \ddots & \cdots \end{bmatrix}$$

where the block matrix are as follows.

$$H_0 = \begin{bmatrix} 0 \\ 0 \\ \gamma \\ R^0 \beta \otimes e_n \end{bmatrix}$$

$$F_0 = \begin{bmatrix} f^0_{(1,1)} & 0 & 0 & 0 \\ T_2 \oplus D_0 - \gamma I_N \otimes I_{M_1} & diag(a_K, a_{K-1}, ..., a_1) & e_{M_1} \otimes q T^0_2 & f^0_{(2,5)} \\ 0 & f^0_{(3,3)} & e_{M_2} \otimes T^0_3 & \gamma I_{M_2 r} \\ 0 & 0 & D_0 - \gamma I_n & 0 \\ diag(c_K, c_{K-1}, ..., c_1) & 0 & 0 & (R \oplus D_0) \otimes I_{M_2} \end{bmatrix}$$

where

$$f^0_{(1,1)} = \begin{bmatrix} 0 \\ diag[I_K \otimes \theta \quad 0] \end{bmatrix}$$

$$f^0_{(1,1)} = \begin{bmatrix} D_0 & 0 \\ 0 & I_K \otimes (D_0 - \theta I_n) \end{bmatrix}$$

$$f^0_{(2,5)} = diag\begin{bmatrix}0 & \gamma I_{rM_1}\end{bmatrix}$$

$$f^0_{(3,3)} = \begin{bmatrix} T_3 \oplus (D_0 - \gamma I_m) \otimes I_{K+1} & 0 \\ 0 & T_3 \oplus (D_0 - (\gamma+\theta)) \otimes I_{K+1} \end{bmatrix} + \begin{bmatrix} 0 \\ diag(b_K, b_{K-1}, ..., b_1) \end{bmatrix}$$

$$H_1 = \begin{bmatrix} h^1_{(1,1)} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$h^1_{(1,1)} = \begin{bmatrix} 0 & \delta I_n & 0 & 0 & 0 \\ 0 & pT_1^0\alpha_1 & \delta I_n & 0 & 0 \\ 0 & 0 & pT_1^0\alpha_1 & \delta I_n & 0 \\ 0 & 0 & 0 & pT_1^0\alpha_1 & 0 \\ 0 & 0 & 0 & 0 & pT_1^0\alpha_1 \end{bmatrix}$$

$$F_{10} = \begin{bmatrix} diag(d_K, d_{K-1}, ..., d_1) & pT_1^0\alpha_1 \otimes I_{M_2} & qT_1^0\alpha_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} f_{(1,1)} & 0 & 0 & 0 & \gamma I_{rM_2} \\ f_{(2,1)} & f_{(2,2)} & diag(a_K, a_{K-1}, ..., a_1) & 0 & diag(e_K, e_{K-1}, ..., e_1) \\ T_3^0\alpha_3 \otimes I_{M_2} & 0 & f_{(3,3)} & 0 & \gamma I_{rM_2} \\ \eta & 0 & 0 & D_0 - \eta I_n & 0 \\ R^0\beta \otimes I_{M_2} & 0 & 0 & 0 & (R + D_0) \otimes I_{M_2} \end{bmatrix}$$

$$f_{(1,1)} =$$

$$\begin{bmatrix} T_1 \oplus D_0 - (\gamma+\delta) I_n \otimes I_K & 0 & 0 & 0 \\ 0 & T_1 \oplus (D_0 - \gamma I_n) & 0 & 0 \\ 0 & 0 & diag(f_K, f_{K-1}, ..., f_1) & 0 \\ 0 & 0 & 0 & T_1 \oplus D_0 - (\gamma+\theta) I_n \otimes I_{K-1} \end{bmatrix}$$

$$+ \, diag(b_K, b_{K-1}, ..., b_1)$$

$$f_{(2,1)} = diag\begin{bmatrix} qT_2^0\alpha_2 \otimes I_n & 0 \end{bmatrix}$$

$$f_{(2,2)} = \begin{bmatrix} T_2 \oplus (D_0 - \gamma) \otimes I_K & 0 \\ 0 & T_2 \oplus D_0 - (\gamma+\theta) I_n \otimes I_{M_1} \end{bmatrix} + \begin{bmatrix} 0 \\ diag(g_{K-1}, ..., g_1) \end{bmatrix} + diag\begin{bmatrix} 0 \\ qT_2^0\alpha_2 \otimes I_{n(K-1)} \end{bmatrix}$$

$$f_{(3,3)} = \begin{bmatrix} T_3 \oplus (D_0 - \gamma) \otimes I_{K+1} & 0 \\ 0 & diag(h_K, h_{K_1}, ..., h_1) \end{bmatrix} + \begin{bmatrix} 0 \\ diag(b_{K-1}, ..., b_1) \end{bmatrix}$$

$$F_2 = \begin{bmatrix} f^2_{(1,1)} & diag(d_K, d_{K-1}, ..., d_1) & pT_1^0\alpha_1 \otimes I_{M_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$f^2_{(1,1)} = \begin{bmatrix} pT_1^0\alpha_1 \otimes I_n & \delta I_n & 0 & 0 & 0 \\ 0 & pT_1^0\alpha_1 \otimes I_n & \delta I_n & 0 & 0 \\ 0 & 0 & pT_1^0\alpha_1 \otimes I_n & \delta I_n & 0 \\ 0 & 0 & 0 & pT_1^0\alpha_1 \otimes I_n & 0 \\ 0 & 0 & 0 & 0 & pT_1^0\alpha_1 \otimes I_n \end{bmatrix}$$

$$a_i = pT_2^0\alpha_2 \otimes I_i$$

$$b_i = \theta I_{in}$$

$$c_i = \begin{bmatrix} 0 \\ R^0\beta \otimes I_i \end{bmatrix}$$

$$d_i = \begin{bmatrix} 0 & 0 \\ pT_1^0\alpha_1 \otimes I_i & \delta I_{n(i-1)} \end{bmatrix}$$

$$e_i = \begin{bmatrix} 0 & \gamma I_{ir} \end{bmatrix}$$

$$f_i = \begin{bmatrix} T_1 \oplus D_0 - (\gamma + I + \theta)I_n \otimes I_{i-1} & 0 \\ 0 & T_1 \oplus D_0 - (\gamma+\theta)I_n \end{bmatrix}$$

$$g_i = \begin{bmatrix} 0 & \theta I_{in} \end{bmatrix}$$

$$h_i = T_3 \oplus D_0 - (\gamma+\theta)I_n \otimes I_K,\, K \leq i \leq 1.$$

Let $z(0) = (z_0(0), z_1(0), z_2(0), z_3(0), ...)$ be defined as a conditional probability distribution of the system at the arrival time of the tagged customer is given by

$$z_0(0) = x_0 \left[\frac{D_1 e_(K+1)}{\lambda}\right]$$
$$z_i(0) = x_i \left[I_{[M_2(2n+r)+(nM_1+1)]}\frac{D_1 e_(K+1)}{\lambda}\right], \text{ for } i \geq 1$$

where λ is the fundamental arrival rate of the Markov Arrival Process. Now, on defining $z(t) = (z_*(t), z_0(t), z_1(t), z_2(t), ...)$, where

$$z_0(t) : \text{a } 1 \times 1 \text{ vector}$$
$$z_i(t) : \text{a row vector of order } 1 \times (M_2(2n+r) + (nM_1+1))$$

The chance that the continuous time Markov chain with the generator matrix $\bar{Q}$ is in the corresponding state of level i at instant t is given by their entries. Since $z*(t)$ denotes the likelihood that the tagged customer is in the absorbing state at time t, we have $W(t) = z_*(t)$, where $t \geq 0$.

The differential equation $z'(t) = z(t)\bar{Q}$ for $t \geq 0$ becomes

$$z_*^{'}(t) = z_0(t)H_0 + z_1(t)H_1;$$
$$z_0^{'}(t) = z_0(t)F_0 + z_1(t)F_{10};$$
$$z_i^{'}(t) = z_i(t)F + z_{i+1}(t)F_2; i \geq 1$$

where $'$ denotes the derivative with respect to t. Let us calculate the Laplace-Stieltjes Transform for $W(t)$ using the Neuts et al. (1990) technique, where the initial probability row vector $w(s)$ denotes the first passage time to level 1 as follows

$$w(s) = \sum_{i=1}^{\infty} z_i(0)[(sI-F)^{-1}F_2]^{i-1} \tag{1}$$

Let $\varphi(i,s)$ be the LST of the time it takes to get absorbed into the state $(*)$, with the constraint that the process starts at level $i = 0, 1$. We have

$$\varphi(0,s) = [sI - F_0]^{-1} H_0 \tag{2}$$

$$\varphi(1,s) = [sI - F]^{-1} F_{10} \varphi(0,s) + [sI - F]^{-1} H_1. \tag{3}$$

Thus, it can easily seen that the Laplace-Stieltjes Transform for the distribution of sojourn time is as

$$\bar{W}(s) = z_0(0)\varphi(0,s) + w(s)\varphi(1,s). \tag{4}$$

The Expected waiting time is

$$E(W) = -\bar(W')(0) = -z_0(0)\varphi(0,0) - w'(0)e_{(M_2(2n+r)+(nM_1+1))} - w(0)\varphi'(1,0) \tag{5}$$

The first term in the preceding equation denotes the average time to enter the absorption state (*) assuming the system is in the level state i=0. On differentiating both the equation (2) and (3) , and setting s=0 , we have,

$$\varphi'(0,0) = (-1)[-F_0]^{-2} H_0 \tag{6}$$

$$\varphi'(1,0) = (-1)[-F_0]^{-2} F_{10}\varphi(0,0) + [-F_1]^{-1} F_{10}\varphi'(0,0) - [-F_1]^{-2} H_1 \tag{7}$$

By using equation (6) and (7) along with the primary conditions $z(t) = (z_0(0), z_1(0), z_2(0), ...)$, it can be easily evaluated the initial terms of (5). From (1) we have

$$w(s) = \sum_{i=1}^{\infty} z_i(0) U^{i-1} \tag{8}$$

where the stochastic matrix $U = [-F]^{-1} F_2$. We have

$$w(0)e_{(M_2(2n+r)+(nM_1+1))} = 1 - z_0(0). \tag{9}$$

Along with the primary conditions $z(t) = (z_0(0), z_1(0), z_2(0), ...)$, using (7) and (8), the last term of equation (5) can be evaluated. Differentiating (1) and substituting $s = 0$, we get,

$$w'(0) = (-1)\sum_{i=1}^{\infty} z_{i+1}(0) \sum_{j=0}^{i-1} U^j [-F]^{-1} U^{i-j} \tag{10}$$

by the condition U is stochastic , we have

$$(-1)w'(0)e_{(M_2(2n+r)+(nM_1+1))} = \sum_{i=1}^{\infty} z_{i+1}(0) \sum_{j=0}^{i-1} U^j [-F]^{-1} e_{(M_2(2n+r)+(nM_1+1))} \tag{11}$$

Defining an irreducible matrix U_2 satisfying two conditions such that $1 - U + U_2$ is non singular and the generalized inverse is of the form $(I - K_1)$. Then the matrix $U_2 = u_0 e_{(M_2(2n+r)+(nM_1+1))}$ where u_0 represents the stationary probability vector of U such that $u_0 U = u_0$ and $u_0 e_{(M_2(2n+r)+(nM_1+1))} = 1$. Moreover U_2 satisfies the property $UU_2 = U_2 U = U_2$. Then we have,

$$\sum_{j=0}^{i-1} U^j (I - U + U_2) = 1 - U^i + iU_2, for \;\; i \geq 1 \tag{12}$$

substituting (12) in (11) and simplifying we get the following

$$\begin{aligned}(-1)w'(0)e_{(M_2(2n+r)+(nM_1+1))} = \{ & x_1 (I - R)^{-1} \left\{ I_{(M_2(2n+r)+(nM_1+1))} \otimes \frac{D_1 e_n}{\lambda} \right\} \\ & - w(0) + x_1 R (I - R)^{-2} \left\{ I_{(M_2(2n+r)+(nM_1+1))} \otimes \frac{D_1 e_n}{\lambda} \right\} \} \\ & \times [I - U + U_2]^{-1} [-F]^{-1} e_{(M_2(2n+r)+(nM_1+1))}. \end{aligned} \tag{13}$$

As a result, we have obtained all of the terms in (5), which aids in determining the expected waiting time.

7. Performance Measures

To investigate the behaviour of our model under a steady state condition, a few performance measure of the system are computed.

- Probability that the server is idle
  $P_i = \sum_{k=0}^{K} x_{00k}$
- Probability that the server is busy with the main customer
  $P_{BM} = \sum_{i=1}^{\infty} \sum_{k=0}^{K} x_{i1k}$
- Probability that the server is busy with the customer in buffer 2
  $P_{BB2} = \sum_{i=0}^{\infty} \sum_{k=0}^{K} x_{i2k}$
- Probability that the server is busy with optional service
  $P_{BO} = \sum_{i=0}^{\infty} \sum_{k=0}^{K} x_{i3k}$
- Probability that the server is on vacation
  $P_V = \sum_{i=0}^{\infty} x_{i40}$
- Probability that the server is under repair
  $P_R = \sum_{i=0}^{\infty} \sum_{k=0}^{K} x_{i5k}$
- Expected system size
  $E_{system} = \sum_{p=1}^{\infty} \sum_{i=1}^{j} \sum_{k=0}^{K} p x_{pik} = x_1(1-R)^{-2}e$

8. Numerical Results

The qualitative behaviour of this model will be understood in this section with the help of a few illustrations, both numerically and graphically, by changing various model parameters such as the arrival process and service time distribution. For both the arrival process and the service time distribution, three sets of values from the literature are used as input.

Erlang of order 2 (ERL-A)

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}; D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

Exponential (Exp-A)

$$D_0 = \begin{bmatrix} -1 \end{bmatrix}; D_1 = \begin{bmatrix} 1 \end{bmatrix}$$

Hyperexponential (HYP-EXP-A)

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}; D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$$

Considering three phase type distributions for the service process which was suggested by Chakravarthy() **Erlang of order 2 (ERL-S)**

$$\alpha_1 = \alpha_2 = \alpha_3 = \beta = (1,0); T_1 = T_2 = T_3 = R = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

Exponential (Exp-A)

$$\alpha_1 = (1); T_1 = \begin{bmatrix} -44 \end{bmatrix}$$

$$\alpha_2 = (1); T_2 = \begin{bmatrix} -46 \end{bmatrix}$$

$$\alpha_3 = (1); T_3 = \begin{bmatrix} -34 \end{bmatrix}$$

$$\beta = (1);\ R = \begin{bmatrix} -12 \end{bmatrix}$$

Hyperexponential (HYP-EXP-A)

$$\alpha_1 = (0.3, 0.7);\ T_1 = \begin{bmatrix} -9 & 3 \\ 2 & -8 \end{bmatrix}$$

$$\alpha_2 = (0.4, 0.6);\ T_2 = \begin{bmatrix} -12 & 6 \\ 5 & -10 \end{bmatrix}$$

$$\alpha_3 = (0.4, 0.6);\ T_3 = \begin{bmatrix} -6 & 4 \\ 3 & -4 \end{bmatrix}$$

$$\beta = (0.5, 0.5);\ R = \begin{bmatrix} -12 & 3 \\ 3 & -12 \end{bmatrix}$$

Illustration 1

With the aid of 2D graphs, figure 2 to 10 represents the vacation rate versus probability of the server is idle for all possible ordering of arrival and service time by fixing $\eta = 5$, $\gamma = 7$, $\theta = 5$, $\delta = 2$, $P = 0.3$, $q = 0.7$, $K = 4$. An increase in vacation rate implies that the server will pay more attention to serve the customer which has a direct proportion on probability of the server is idle.

Illustration 2

The effect of the customer entering buffer 2 from buffer 1 after completion of interruption with rate θ and breakdown rate verses the expected system size has been investigated by fixing $\eta = 5$, $\gamma = 7$, $\theta = 5$, $\delta = 2$, $P = 0.3$, $q = 0.7$, $K = 4$. In figure 38 to 46, an increase in both the breakdown rate and self interrupted customer moving onto buffer 2 at the rate θ along with the expected system size with distinct group of arrival and service time has been observed briefly.

While there is an increase in self interrupted customer moving onto buffer 2 at the rate θ implies that arrival of customer to buffer 2 increases rapidly and an increase in breakdown rate implies an increase in the waiting time of the customer both in the main queue as well as buffer 2. In both the scenario it is obvious that the expected system size increases due to the minimal availability of the server.

Illustration 3

Table 1 to 3 represents the vacation rate versus expected system size by fixing $\eta = 5$, $\gamma = 7$, $\theta = 5$, $\delta = 2$, $p = 0.3$, $q = 0.7$, $K = 4$. As long as the vacation rate increases it is evident that the expected system size reduces gradually. The effect of increasing the vacation rate leads to more availability of the server in the system which in turn reduces the expected system size in the table. It is obviously that the expected system size decreases rapidly for hyper exponential service compared to a Erlang service which is pretty gradual.

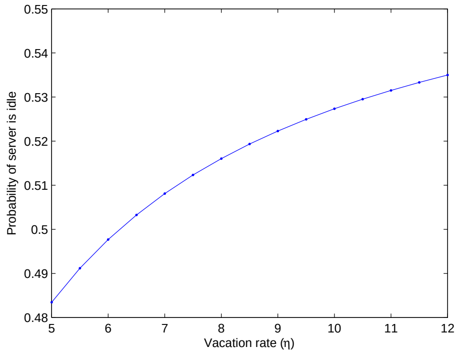

Figure 2: *Vacation rate* (η) *vs Probability of server is Idle - M/M/1*

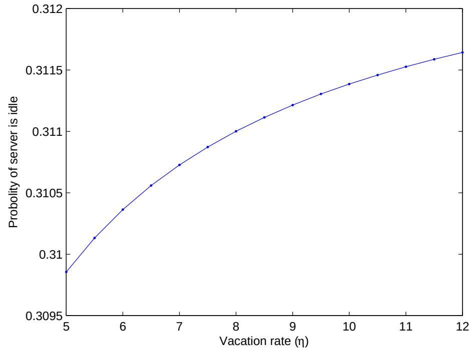

Figure 3: *Vacation rate* (η) *vs Probability of server is Idle - M/Ek/1*

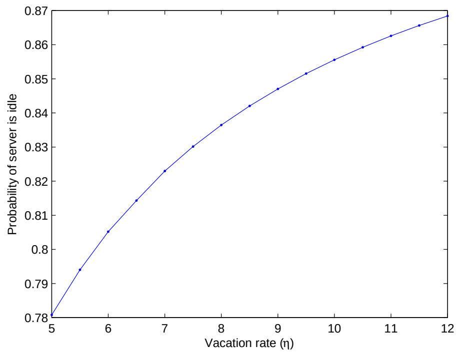

Figure 4: *Vacation rate* (η) *vs Probability of server is Idle - M/Hk/1*

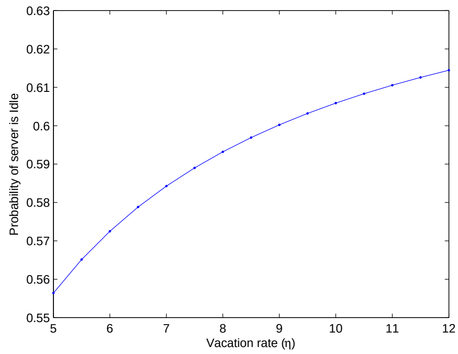

Figure 5: *Vacation rate* (η) *vs Probability of server is Idle - Ek/M/1*

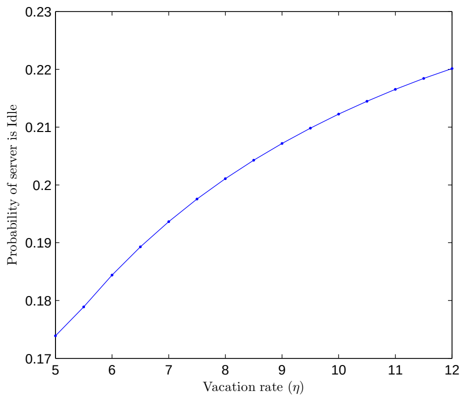

Figure 6: *Vacation rate* (η) *vs Probability of server is Idle - Ek/Ek/1*

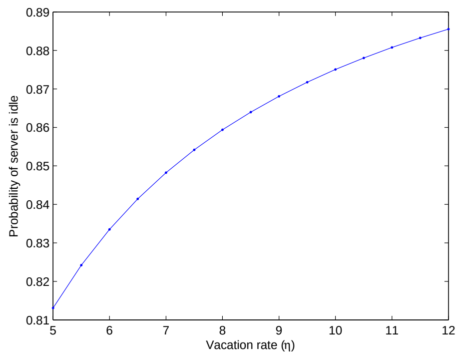

Figure 7: *Vacation rate* (η) *vs Probability of server is Idle - Ek/Hk/1*

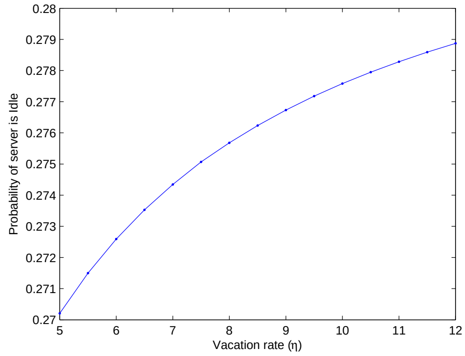

Figure 8: *Vacation rate (η) vs Probability of server is Idle - Hk/M/1*

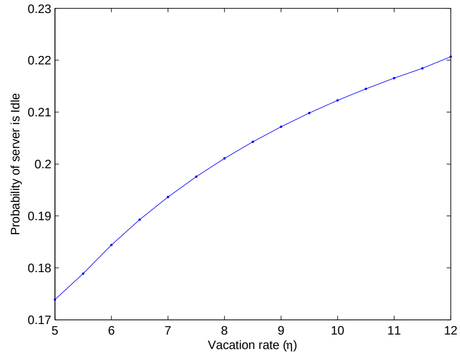

Figure 9: *Vacation rate (η) vs Probability of server is Idle - Hk/Ek/1*

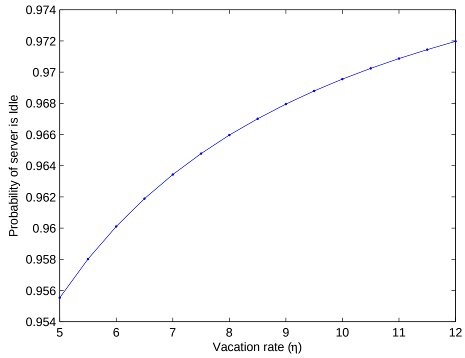

Figure 10: *Vacation rate (η) vs Probability of server is Idle - Hk/Hk/1*

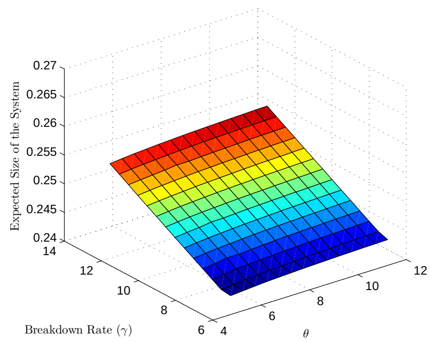

Figure 11: *The customer moving from buffer 1 to buffer 2 after interruption with the rate θ and Breakdown rate (γ) vs Expected Size of the System - M/M/1*

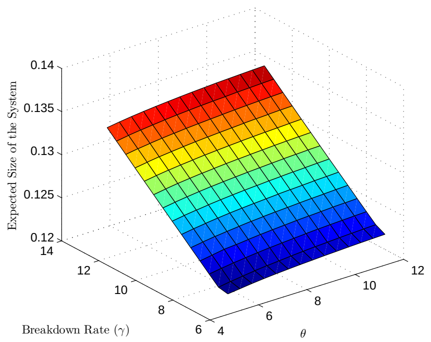

Figure 12: *The customer moving from buffer 1 to buffer 2 after interruption with the rate θ and Breakdown rate (γ) vs Expected Size of the System - M/Ek/1*

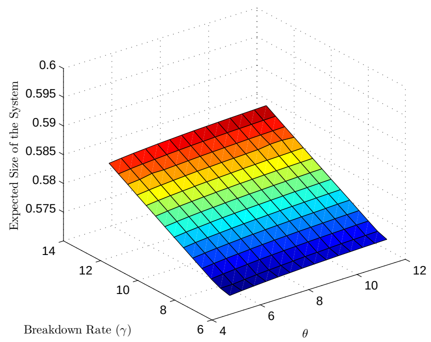

Figure 13: *The customer moving from buffer 1 to buffer 2 after interruption with the rate θ and Breakdown rate (γ) vs Expected Size of the System - M/Hk/1*

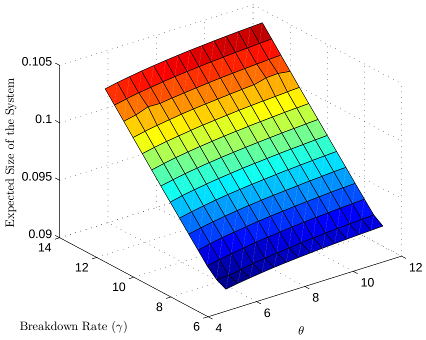

Figure 14: *The customer moving from buffer 1 to buffer 2 after interruption with the rate θ and Breakdown rate (γ) vs Expected Size of the System - Ek/M/1*

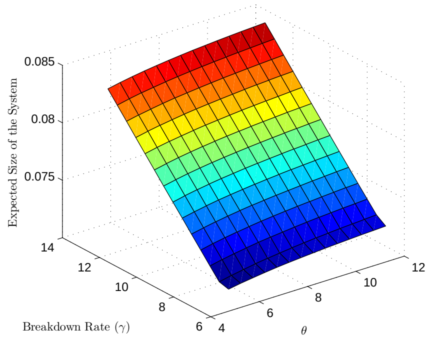

Figure 15: *The customer moving from buffer 1 to buffer 2 after interruption with the rate θ and Breakdown rate (γ) vs Expected Size of the System - Ek/Ek/1*

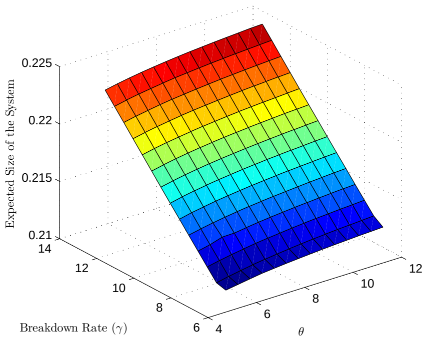

Figure 16: *The customer moving from buffer 1 to buffer 2 after interruption with the rate θ and Breakdown rate (γ) vs Expected Size of the System - Ek/Hk/1*

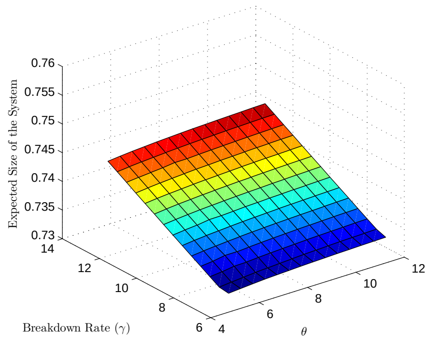

Figure 17: *The customer moving from buffer 1 to buffer 2 after interruption with the rate θ and Breakdown rate (γ) vs Expected Size of the System - Hk/M/1*

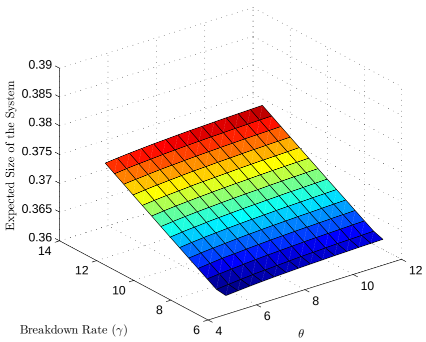

Figure 18: *The customer moving from buffer 1 to buffer 2 after interruption with the rate θ and Breakdown rate (γ) vs Expected Size of the System - Hk/Ek/1*

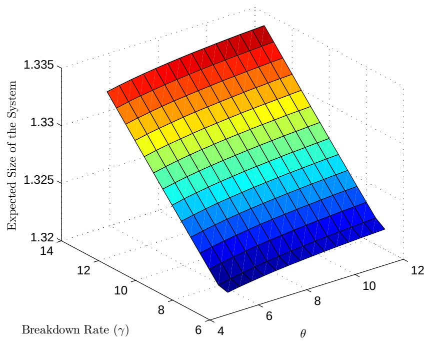

Figure 19: *The customer moving from buffer 1 to buffer 2 after interruption with the rate θ and Breakdown rate (γ) vs Expected Size of the System - Hk/Hk/1*

Table 1 Vacation rate (η) vs expected system size - Exponential-A

	service		
η	Exponential	Erlang	Hyperexponential
5	0.120659907	0.335551494	0.798654237
5.5	0.117283553	0.331603054	0.790156984
6	0.114733332	0.328581189	0.785684522
6.5	0.112762892	0.326217823	0.778725485
7	0.111210648	0.324335072	0.776131839
7.5	0.109967234	0.322811193	0.773832593
8	0.108956588	0.321560623	0.77193052
8.5	0.108124546	0.320521817	0.770586385
9	0.10743173	0.319649605	0.768578623
9.5	0.106848976	0.318910231	0.767878243
10	0.106354332	0.318278064	0.766823475
10.5	0.105931023	0.317733363	0.765928795
11	0.105566063	0.317260725	0.765248458
11.5	0.105249277	0.316847992	0.764572653
12	0.104972601	0.316485461	0.764568136

Table 2 Vacation rate (η) vs Expected system size - Erlang-A

	service		
η	Exponential	Erlang	Hyperexponential
5	0.057197966	0.070334672	0.09364885
5.5	0.054411688	0.064934106	0.091947737
6	0.052360653	0.060794972	0.090637074
6.5	0.05081441	0.057554181	0.109606071
7	0.049624632	0.054970138	0.108780591
7.5	0.048692757	0.052877138	0.108109511
8	0.04795146	0.051158507	0.107556655
8.5	0.04735363	0.049730205	0.107095843
9	0.046865592	0.048530471	0.106707752
9.5	0.046462815	0.047513107	0.106377871
10	0.046127126	0.046643004	0.106095132
10.5	0.045844857	0.045893101	0.105850975
11	0.045605584	0.045242268	0.105638696
11.5	0.045401257	0.044673818	0.105452986
12	0.045225591	0.04417443	0.105289594

Table 3 Vacation rate (η) vs Expected system size - Hyperexponential-A

	service		
η	Exponential	Erlang	Hyperexponential
5	0.675602876	0.695049695	0.983254166
5.5	0.67458552	0.695038233	0.974170849
6	0.67380935	0.695029517	0.972452185
6.5	0.673203901	0.695022735	0.968948456
7	0.67272264	0.695017355	0.966131839
7.5	0.672015304	0.695013015	0.963832593
8	0.672015304	0.695009464	0.96193052
8.5	0.671751078	0.695006521	0.960338541
9	0.671529505	0.695004055	0.958992295
9.5	0.671341885	0.695001969	0.957843366
10	0.671181629	0.695012002	0.956854752
10.5	0.671043669	0.684998654	0.955997773
11	0.670924057	0.684997326	0.955249916
11.5	0.670819681	0.684996166	0.954593302
12	0.670728061	0.684995149	0.954013588

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