On a wide plurimodal class of distributions suitable for asymmetric data sets

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Abstract

Asymmetric normal distributions have received much attention in the literature during the last three decades. But, plurimodal asymmetric normal distributions are not much studied in the literature even though it has much relevance in practical situations. Here we propose a new class of plurimodal, asymmetric normal distribution and investigate its several statistical properties, including certain reliability aspects. A location-scale extension of the proposed model is developed and studied their properties. The maximum likelihood estimation method is employed for estimating the parameters of the proposed extended class of distributions and conducted generalized likelihood ratio test procedure for testing the parameters of the distribution. Three real-life data sets are considered for illustrating the usefulness of the model and a brief simulation study is carried out for examining the performance of maximum likelihood estimators of the proposed model.

Keywords: Asymmetric distributions, Maximum likelihood estimation, Model selection, Plurimodality, Simulation

1. Introduction

The normal distribution is the most important and most widely used distribution in statistics. It is an inevitable tool for the analysis and interpretation of data. But in many practical applications it has been observed that real life data sets are not symmetric. So normal distribution is not an acceptable model for modeling such data sets. In order to overcome this drawback, [2] considered an asymmetric form of normal distribution by introducing a skewness parameter into its probability density function (p.d.f) and named it as "the skew normal distribution". The skew normal distribution defined by [2] as follows:

Let $\phi(.)$ and $\Phi(.)$ be the p.d.f and cumulative distribution function (c.d.f) of a standard normal variate. Then a random variable X is said to follow the skew normal distribution with parameter $\lambda \in R = (-\infty, \infty)$ if its p.d.f $g(x;\lambda)$, for $x \in R$, is given by

$$g(x;\lambda) = 2\phi(x)\,\Phi(\lambda x). \tag{1}$$

A distribution with p.d.f. (1) we denoted as $SND(\lambda)$ through out the manuscript. The $SND(\lambda)$ has been further studied by several authors such as [3],[4], [5], [6], [7], [8] and [10].

A generalized form of skew normal distribution is developed by [1] through the following p.d.f.

$$g_1(x;\lambda_1,\lambda_2) = 2\phi(x)\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right), \tag{2}$$

in which $x \in R$, $\lambda_1 \in R$, $\lambda_2 \geq 0$. A distribution with pdf (2) we denoted as $SGND(\lambda_1, \lambda_2)$.

The $SGND(\lambda_1, \lambda_2)$ of [1] is log-concave and hence it is not suitable for plurimodal data. To overcome this drawback, [11] considered an extended version of $SGND(\lambda_1, \lambda_2)$ through the name "extended skew generalized normal distribution $(ESGND(\lambda_1, \lambda_2, \alpha))$"which has the following p.d.f.

$$g_2(x; \lambda_1, \lambda_2, \alpha) = \frac{2}{\alpha + 2}\phi(x)\left[1 + \alpha\Phi\left(\frac{\lambda_1 x}{\sqrt{1 + \lambda_2 x^2}}\right)\right], \tag{3}$$

where $x \in R$, $\lambda_1 \in R$, $\lambda_2 \geq 0$ and $\alpha \geq -1$. Through the present work our intention is to propose a wide class of plurimodal asymmetric normal distributions as a modified version of the $ESGND(\lambda_1, \lambda_2, \alpha)$ and named it as the "modified skew generalized normal distribution(MSGND)". In section 2 we present the definition and properties of MSGND. In section 4 we present the characteristic function and moments of MSGND. In section 5 certain reliability measures such as reliability function, mean residual life function etc are derived along with some conditions for unimodal and plurimodal situations are obtained. In section 6 a location scale extension of the MSGND is defined and obtained its properties such as characteristic function, reliability measures etc. In section 7 maximum likelihood estimation of the parameters of the distribution is discussed and in section 8 we constructed a generalized likelihood ratio test (GLRT) procedure. Real life data applications are given for illustrating the usefulness in section 9, a brief simulation study is attempted in section 10. While modelling certain real life data sets ESGND will not give better fit, the MSGND gives better fits. For example see the illustrations given in section 9, where the MSGND is found to be suitable for modelling data sets arising from athletic as well as agricultural data sets.

2. Modified skew generalized normal distribution

Here we define a new class of skew normal distribution namely the "modified skew generalized normal distribution (MSGND)"and derive its distributional important properties.

Definition 2.1. A random variable X is said to follow modified skew generalized normal distribution if its p.d.f is of the following, in which $x \in R$, $\lambda_1 \in R$, $\lambda_2 \geq 0$, $\beta \in R$ and $\alpha \geq -1$.

$$f(x; \lambda_1, \lambda_2, \alpha, \beta) = \frac{\phi(x)}{\alpha + 2}\left[2 + \alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1 + \lambda_1^2} + \lambda(x)\right)\right], \tag{4}$$

where $\lambda(x) = \frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}$, $\phi(.)$ and $\Phi(.)$ are the p.d.f and c.d.f of a standard normal variate. A distribution with p.d.f (4) we denoted as MSGND$(\lambda_1, \lambda_2, \alpha, \beta)$. Note that

1. When $\alpha = 0$ or $\lambda_1 = 0$, $MSGND(\lambda_1, \lambda_2, \alpha, \beta)$ reduces to the standard normal distribution $N(0, 1)$.
2. When $\beta = 0$, MSGND$(\lambda_1, \lambda_2, \alpha, \beta)$ reduces to the $ESGND(\lambda_1, \lambda_2, \alpha)$.
3. When $\alpha = -1$ and $\beta = 0$, MSGND$(\lambda_1, \lambda_2, \alpha, \beta)$ reduces to the SGND(λ_1, λ_2).

For some particular choices of $\alpha, \lambda_1, \lambda_2$ and β, the p.d.f. $f(x; \lambda_1, \lambda_2, \alpha, \beta)$ given in (4) of $MSGND(\lambda_1, \lambda_2, \alpha, \beta)$ is plotted as given in Figure 1.

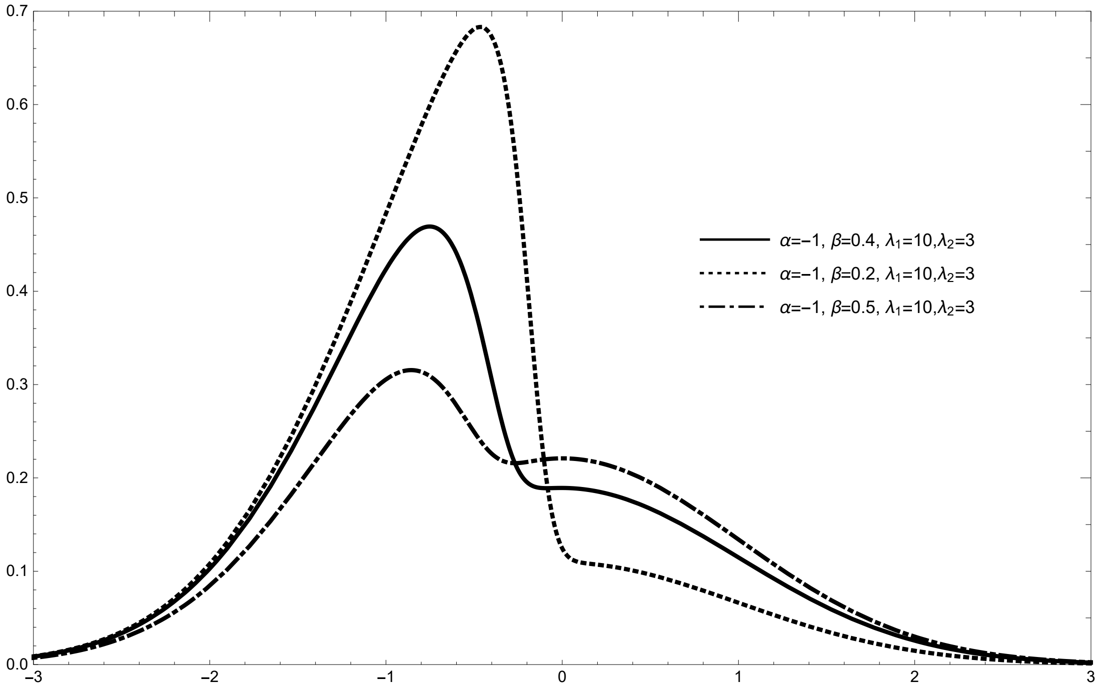

Figure 1: *Probability plots of MSGND($\lambda_1, \lambda_2, \alpha, \beta$) for fixed values of $\lambda_1, \lambda_2, \alpha$ and various values of β*

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3. Results

Result 3.1. If X has MSGND($\lambda_1, \lambda_2, \alpha, \beta$), then $Y_1 = -X$ has MSGND($-\lambda_1, \lambda_2, \alpha, \beta$).

Proof. The p.d.f $f_1(y)$ of $Y_1 = -X$ is the following, for $y \in R, \lambda_1 \in R$, $\lambda_2 \geq 0$, $\beta \in R$ and $\alpha \geq -1$.

$$\begin{aligned} f_1(y) &= f(-y; -\lambda_1, \lambda_2, \alpha, \beta)|\frac{dx}{dy}| \\ &= \frac{\phi(-y)}{\alpha + 2}\phi(-y)\left[2 + \alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1 + \lambda_1^2} + \lambda(-y)\right)\right] \\ &= f(y; -\lambda_1, \lambda_2, \alpha, \beta) \end{aligned}$$

■

Result 3.2. If X has MSGND($\lambda_1, \lambda_2, \alpha, \beta$) then $Y_2 = |X|$ has the p.d.f (5), in which $\Delta(y) = \Phi\left(\beta\sqrt{1 + \lambda_1^2} + \lambda(y)\right) + \Phi\left(\beta\sqrt{1 + \lambda_1^2} + \lambda(-y)\right)$.

Proof. The p.d.f. $f_2(y) \, of \, Y_2 = |X|$ is the following, for $y > 0$.

$$f_2(y) = f(y; \lambda_1, \lambda_2, \alpha, \beta)|\frac{dx}{dy}| + f(-y; \lambda_1, \lambda_2, \alpha, \beta)|\frac{dx}{dy}|$$

in the light of Result 3.1 we have,

$$
\begin{aligned}
f_2(y) &= f(y;\lambda_1,\lambda_2,\alpha,\beta) + f(y;-\lambda_1,\lambda_2,\alpha,\beta) \\
&= \frac{\phi(y)}{\alpha+2}\left[2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda(y)\right)\right] + \\
&\quad \frac{\phi(y)}{\alpha+2}\left[2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda(-y)\right)\right] \\
&= \frac{\phi(y)}{\alpha+2}\left[4+\alpha[\Phi(\beta)]^{-1}\left\{\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda(y)\right)\right.\right. \\
&\quad \left.\left. +\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda(-y)\right)\right\}\right] \\
&= \frac{\phi(y)}{\alpha+2}\left[4+\alpha[\Phi(\beta)]^{-1}\Delta(y)\right]. \qquad (5)
\end{aligned}
$$

■

Result 3.3. If X has MSGND$(\lambda_1,\lambda_2,\alpha,\beta)$ then $Y_3 = X^2$ has pdf (6), in which $\Delta(y)$ is as defined in Result 3.2.

Proof. For $y > 0$, the p.d.f of $f_3(y)$ of $Y_3 = X^2$ is

$$
\begin{aligned}
f_3(y) &= f(\sqrt{y};\lambda_1,\lambda_2,\alpha,\beta)|\frac{dx}{dy}| + f(-\sqrt{y};\lambda_1,\lambda_2,\alpha,\beta)|\frac{dx}{dy}| \\
&= \frac{\phi(\sqrt{y})}{\alpha+2}\left[2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\lambda\left(\sqrt{(y)}\right)\right)\right]\frac{1}{2\sqrt{y}} + \\
&\quad \frac{\phi(-\sqrt{y}}{\alpha+2}\left[2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\lambda(-\sqrt{(y)})\right)\right]\frac{1}{2\sqrt{y}} \\
&= \frac{\phi(\sqrt{y})}{(\alpha+2)2\sqrt{y}}\left[4+\alpha[\Phi(\beta)]^{-1}\Delta(\sqrt{y})\right]. \qquad (6)
\end{aligned}
$$

■

Result 3.4. The c.d.f of MSGND$(\lambda_1,\lambda_2,\alpha,\beta)$ with p.d.f (4) is the following, for $x \in R$.

$$
F(x) = \frac{\Phi(x)}{\alpha+2}\left[2+\alpha\frac{[\Phi(\beta)]^{-1}}{2}\right] - \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\xi_\beta(x;\lambda(v)), \qquad (7)
$$

where

$$
\xi_\beta(x,\lambda(v)) = \int_x^\infty \int_0^{\beta\sqrt{1+\lambda^2}+\lambda(v)} \phi(v)\phi(u)\,dvdu, \qquad (8)
$$

with $\lambda(v) = \frac{\lambda_1 v}{\sqrt{1+\lambda_2 v^2}}$. For particular values of $\lambda_1, \lambda_2, \beta$ and x, we can evaluate (8) by using the mathematical softwares such as MATHCAD, MATHEMATICA, etc.

Proof.

$$
\begin{aligned}
F(x) &= \int_{-\infty}^x f(v;\lambda_1,\lambda_2,\alpha,\beta)dv \\
&= \frac{2}{\alpha+2}\Phi(x) + \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\int_{-\infty}^x \phi(v)\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda(v)\right)dv \\
&= \frac{2\Phi(x)}{\alpha+2} + \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\left[\frac{1}{2}\Phi(x) - \xi_\beta(x,\lambda(v))\right] \\
&= \frac{\Phi(x)}{\alpha+2}\left[2+\frac{\alpha[\Phi(\beta)]^{-1}}{2}\right] - \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\xi_\beta(x,\lambda(v)).
\end{aligned}
$$

■

4. Characteristic function and Moments

In this section we obtain the characteristic function and moments of MSGND.

Result 4.1. The characteristic function $\psi_X(t)$ of MSGND($\lambda_1, \lambda_2, \alpha, \beta$) with p.d.f (4) is the following, for any $t \in R$ and $i = \sqrt{-1}$.

$$\psi_X(t) = \frac{e^{\frac{-t^2}{2}}}{\alpha+2}\left\{2 + \alpha[\Phi(\beta)]^{-1} E\left[\Phi\left(\beta\sqrt{1+\lambda_1^2} + \lambda(u+it)\right)\right]\right\} \tag{9}$$

where $\lambda(u+it) = \frac{\lambda_1(u+it)}{\sqrt{1+\lambda_2(u+it)^2}}$.

Proof. Let X follows MSGND($\lambda_1, \lambda_2, \alpha, \beta$) with p.d.f (4). Then by the definition of characteristic function, we have the following for any $t \in R$ and $i = \sqrt{-1}$.

$$\begin{aligned}
\psi_X(t) &= E(e^{itX}) \\
&= \frac{2}{\alpha+2}\int_{-\infty}^{\infty} e^{itx}\phi(x)dx + \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\int_{-\infty}^{\infty} e^{itx}\phi(x)\Phi\left(\beta\sqrt{1+\lambda^2} + \lambda(x)\right)dx \\
&= \frac{e^{\frac{-t^2}{2}}}{\alpha+2}\left\{2 + \alpha[\Phi(\beta)]^{-1}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-(x-it)^2}{2}}\Phi(\beta\sqrt{1+\lambda^2} + \lambda(x))dx\right\} \qquad (10)
\end{aligned}$$

On substituting $x - it = u$ in (10), we obtain

$$\psi_X(t) = \frac{e^{\frac{-t^2}{2}}}{\alpha+2}\left\{2 + \alpha[\Phi(\beta)]^{-1} E\left[\Phi\left(\beta\sqrt{1+\lambda_1^2} + \lambda(u+it)\right)\right]\right\}, \tag{11}$$

which implies (9). ■

The expression for even moments and odd moments of $MSGND(\lambda_1, \lambda_2, \alpha, \beta)$ are obtained through the following results.

Result 4.2. If X follows $MSGND(\lambda_1, \lambda_2, \alpha, \beta)$, then for k=1,2,...,

$$E(X^{2k}) = \frac{2^{k+\frac{1}{2}}}{(\alpha+2)\sqrt{2\pi}}\Gamma(k+\frac{1}{2}) + \frac{\alpha[\Phi(\beta)]^{-1}}{2(\alpha+2)}A_k(\beta, \lambda_1, \lambda_2), \tag{12}$$

in which

$$A_k(\lambda_1, \lambda_2, \beta) = \int_0^{\infty} u^{k-\frac{1}{2}}\phi(\sqrt{u})\Phi\left(\beta\sqrt{1+\lambda_1^2} + \lambda(\sqrt{u})\right)du,$$

where $\lambda(\sqrt{u}) = \frac{\lambda_1\sqrt{u}}{\sqrt{1+\lambda_2 u}}$, $\lambda_1 \in R$, $\lambda_2 \geq 0$, $\beta \in R$ which can be easily evaluated by using the softwares MATHCAD and MATHEMATICA.

Proof. By the definition of raw moments, for any $k \geq 0$, integer,

$$E(X^{2k}) = \int_{-\infty}^{\infty} x^{2k} f(x; \lambda_1, \lambda_2, \alpha, \beta)dx. \tag{13}$$

On substituting $x^2 = u$ in (13) we obtain the following in the light of (4) we have,

$$\begin{aligned}
E(X^{2k}) &= \frac{1}{\alpha+2}\int_0^{\infty} u^k\phi(\sqrt{u})\frac{1}{\sqrt{u}}du + \frac{\alpha[\Phi(\beta)]^{-1}}{2(\alpha+2)} \\
&\quad \int_0^{\infty} u^k\phi(\sqrt{u})\Phi\left(\beta\sqrt{1+\lambda_1^2} + \lambda(\sqrt{u})\right)\frac{1}{\sqrt{u}}du \\
&= \frac{1}{(\alpha+2)}\left[\int_0^{\infty} u^{k-\frac{1}{2}}\phi(\sqrt{u})du + \frac{\alpha[\Phi(\beta)]^{-1}}{2}\right. \\
&\quad \left. u^{k-\frac{1}{2}}\phi(\sqrt{u})\Phi\left(\beta\sqrt{1+\lambda_1^2} + \lambda(\sqrt{u})\right)\right]du,
\end{aligned}$$

which leads to (12). ■

Result 4.3. If X follows $MSGND(\lambda_1, \lambda_2, \alpha, \beta)$, then for k=0,1,2,...,

$$E(X^{2k+1}) = \frac{2^{k+1}}{(\alpha+2)\sqrt{2\pi}}\Gamma(k+1) + \frac{\alpha[\Phi(\beta)]^{-1}}{2(\alpha+2)}A_{k+\frac{1}{2}}(\lambda_1, \lambda_2, \beta), \tag{14}$$

in which

$$A_{k+\frac{1}{2}}(\lambda_1, \lambda_2, \beta) = \int_0^\infty u^k \phi(\sqrt{u})\Phi\left(\beta\sqrt{1+\lambda_1^2} + \lambda(?\sqrt{u})\right) du,$$

for $\lambda_1 \in R$, $\lambda_2 \geq 0$, $\beta \in R$ which can be easily evaluated using the softwares MATHCAD and MATHEMATICA.

Proof. By definition of raw moments,

$$E(X^{2k+1}) = \int_{-\infty}^{\infty} x^{2k+1} f(x; \lambda_1, \lambda_2, \alpha, \beta)dx. \tag{15}$$

On substituting $x^2 = u$ in (15) in the light of (4), we get

$$\begin{aligned} E(X^{2k+1}) &= \frac{1}{\alpha+2}\int_0^\infty u^{k+\frac{1}{2}}\phi(\sqrt{u})\frac{1}{\sqrt{u}}du + \frac{\alpha[\Phi(\beta)]^{-1}}{2(\alpha+2)} \\ &\quad \int_0^\infty u^{k+\frac{1}{2}}\phi(\sqrt{u})\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda(\sqrt{u})\right)\frac{1}{\sqrt{u}}du \\ &= \frac{1}{(\alpha+2)}\left[\int_0^\infty u^k\phi(\sqrt{u})du + \frac{\alpha[\Phi(\beta)]^{-1}}{2}\right. \\ &\quad \left.\int_0^\infty u^k\phi(\sqrt{u})\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda(\sqrt{u})\right)\right] du, \end{aligned}$$

which leads to (14). ∎

5. Reliability measures and mode

Here we obtain some properties of MSGND($\lambda_1, \lambda_2, \alpha, \beta$) with p.d.f. (4) useful in reliability studies. Let X follows MSGND($\lambda_1, \lambda_2, \alpha, \beta$) with p.d.f (4). Now, from the definition of reliability function $R(t)$, failure rate $r(t)$ and mean residual life function $\mu(t)$ of X, we obtain the following results.

Result 5.1. The reliability function $R(t)$ of X is the following, in which $\xi_\beta(t, \lambda(x))$ is as defined in Result 3.4.

$$R(t) = \frac{[1-\Phi(t)]}{\alpha+2}\left\{2 + \frac{\alpha[\Phi(\beta)]^{-1}}{2}\right\} + \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\xi_\beta(t, \lambda(x))$$

Result 5.2. The failure rate $r(t)$ of X is given by

$$r(t) = \frac{\phi(t)[2+\alpha[\Phi(\beta)]^{-1}\Phi(\beta\sqrt{1+\lambda^2}+\lambda(x))]}{(1-\Phi(t))[2+\frac{\alpha[\Phi(\beta)]^{-1}}{2}] + \alpha[\Phi(\beta)]^{-1}\xi_\beta(t, \lambda(x))}.$$

Result 5.3. The mean residual life function of MSGND($\lambda_1, \lambda_2, \alpha, \beta$) is

$$\begin{aligned} M(t) &= \frac{2\phi(t)}{(\alpha+2)R(t)} + \frac{\alpha[\Phi(\beta)]^{-1}}{(\alpha+2)R(t)}\left[\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda(x)\right)\phi(t)\right. \\ &\quad \left.+\Lambda_\beta(t; \lambda_1, \lambda_2)\right] - t \end{aligned} \tag{16}$$

where

$$\Lambda_\beta(t; \lambda_1, \lambda_2) = \int_t^\infty \phi(x)\left[\frac{d}{dx}\left(\int_0^{\beta\sqrt{1+\lambda_1^2}+\lambda(x)}\phi(u)du\right)\right]dx.$$

Proof. By definition, the mean residual life function (MRLF) of X is given by

$$\begin{aligned} M(t) &= E(X-t|X>t) \\ &= E(X|X>t)-t, \end{aligned} \tag{17}$$

where

$$\begin{aligned} E(X|X>t) &= \frac{2}{R(t)(\alpha+2)}\int_t^{\infty} x\phi(x)dx \\ &+\frac{\alpha[\Phi(\beta)]^{-1}}{R(t)}\int_t^{\infty} x\phi(x)\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda(x)\right)dx. \end{aligned} \tag{18}$$

Since $\phi(.)$ is the p.d.f of standard normal variate $\phi^{'}(x)=-x\phi(x)$. Therefore (18) becomes,

$$\begin{aligned} E(X|X>t) &= \frac{2}{(\alpha+2)R(t)}\int_t^{\infty} -\phi^{'}(x)dx \\ &+\frac{\alpha[\Phi(\beta)]^{-1}}{(\alpha+2)R(t)}\int_t^{\infty} -\phi^{'}(x)\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda(x)\right)dx. \end{aligned} \tag{19}$$

On integrating (19), we obtain the following

$$\begin{aligned} E(X|X>t) &= \frac{2}{(\alpha+2)R(t)}\phi(t)+\frac{\alpha[\Phi(\beta)]^{-1}}{(\alpha+2)R(t)}\left(-\Phi(\lambda(x)+\beta\sqrt{1+\lambda^2})\phi(x)\right)_t^{\infty} \\ &-\frac{\alpha[\Phi(\beta)]^{-1}}{R(t)(\alpha+2)}\int_t^{\infty} -\phi(x)\left[\frac{d}{dx}\left(\int_{-\infty}^{\beta\sqrt{1+\lambda_1^2}+\lambda(x)}\phi(u)du\right)\right]dx. \end{aligned} \tag{20}$$

On solving (20) and substituting in (17), we get (16).

The functions $R(t), r(t)$ and $\mu(t)$ are equivalent in the sense that if one of them is given, the other two can be uniquely determined. ■

Next, through the following result we derive certain conditions under which the MSGND($\lambda_1, \lambda_2, \alpha, \beta$) is log-concave.

Result 5.4. The p.d.f of MSGND($\lambda_1, \lambda_2, \alpha, \beta$) is log-concave under the following two cases.
Case 1: For $x > 0$,

(i) when $\lambda_1 < 0$ provided for all $\alpha > 0$ and $\beta > 0$ and

(ii) when $\lambda_1 > 0$ provided $|\frac{3\lambda_1\lambda_2^2x^3}{(1+\lambda_2x^2)^{\frac{5}{2}}}| < |\frac{3\lambda_1\lambda_2 x}{(1+\lambda_2x^2)^{\frac{3}{2}}}|$

Case 2: For $x < 0$, the p.d.f of MSGND($\lambda_1, \lambda_2, \alpha, \beta$) is log concave

(i) when $\lambda_1 > 0$ provided for all $\alpha > 0$ and $\beta > 0$ and

(i) when $\lambda_1 < 0$ provided $|\frac{3\lambda_1\lambda_2^2x^3}{(1+\lambda_2x^2)^{\frac{5}{2}}}| < |\frac{3\lambda_1\lambda_2 x}{(1+\lambda_2x^2)^{\frac{3}{2}}}|$.

Proof. To prove $log[f(x;\lambda_1,\lambda_2,\alpha,\beta)]$ is a concave function of x, it is enough to show that its second derivative is negative for all x. Thus

$$\frac{d}{dx}log[f(x;\lambda_1,\lambda_2,\alpha,\beta)] = -x+\frac{\alpha[F(\beta)]^{-1}f(\eta)\eta^{'}}{2+\alpha[F(\beta)]^{-1}F(\eta)}$$

and

$$\frac{d^2}{dx^2}log[f(x;\lambda_1,\lambda_2,\alpha,\beta)] = -1-\Delta_1-\Delta_2+\Delta_3$$

in which,

$$\Delta_1 = \frac{\alpha[F(\beta)]^{-1}\eta'^2 f(\eta)\eta}{2+\alpha[F(\beta)]^{-1}F(\eta)} \quad (21)$$

$$\Delta_2 = \frac{\alpha^2[F(\beta)]^{-2}(f(\eta))^2\eta'^2}{[2+\alpha[F(\beta)]^{-1}F(\eta)]^2} \quad (22)$$

and

$$\Delta_3 = \frac{\alpha[F(\beta)]^{-1}f(\eta)\eta''}{2+\alpha[F(\beta)]^{-1}F(\eta)}, \quad (23)$$

where

$$\eta = \lambda(x) + \beta\sqrt{1+\lambda_1^2}$$
$$\eta' = \frac{\lambda_1}{\sqrt{1+\lambda_2 x^2}} - \frac{\lambda_1\lambda_2 x^2}{(1+\lambda_2 x^2)^{\frac{3}{2}}}$$
$$\eta'' = \frac{3\lambda_1\lambda_2^2 x^3}{(1+\lambda_2 x^2)^{\frac{5}{2}}} - \frac{3\lambda_1\lambda_2 x}{(1+\lambda_2 x^2)^{\frac{3}{2}}}.$$

Note that $\Delta_1 > 0$, for $\alpha > 0$ and $\eta > 0$. Here $\eta > 0$ for all values of $\lambda_1 > 0$ and $\beta > 0$. Consequently $\Delta_2 > 0$ for all values of $\alpha > 0, \beta > 0$ and $\lambda_1 > 0$. Also, $\Delta_3 < 0$ for either when $\alpha < 0$ and $\eta'' > 0$ or when $\alpha > 0$ and $\eta'' < 0$. Hence (4) is log-concave in these situations. ■

As a consequence of the above result, we have the following result.

Result 5.5. MSGND($\lambda_1, \lambda_2, \alpha, \beta$) density is strongly unimodal under the following two cases. Case 1: For $x > 0$,

(i) if $\lambda_1 < 0$ provided for all $\alpha > 0$ and $\beta > 0$ and

(ii) if $\lambda_1 > 0$ provided $|\frac{3\lambda_1\lambda_2^2 x^3}{(1+\lambda_2 x^2)^{\frac{5}{2}}}| < |\frac{3\lambda_1\lambda_2 x}{(1+\lambda_2 x^2)^{\frac{3}{2}}}|$

Case 2: For $x < 0$,

(i) if $\lambda_1 > 0$ provided for all $\alpha > 0$ and $\beta > 0$ and

(i) if $\lambda_1 < 0$ provided $|\frac{3\lambda_1\lambda_2^2 x^3}{(1+\lambda_2 x^2)^{\frac{5}{2}}}| < |\frac{3\lambda_1\lambda_2 x}{(1+\lambda_2 x^2)^{\frac{3}{2}}}|$.

Result 5.6. MSGND($\lambda_1, \lambda_2, \alpha, \beta$) density is plurimodal under the following two cases. Case 1: For $x > 0$,

(i) if $\lambda_1 < 0$ provided for all $\alpha < 0$ and $\beta > 0$ and

(ii) if $\lambda_1 > 0$ provided $|\frac{3\lambda_1\lambda_2^2 x^3}{(1+\lambda_2 x^2)^{\frac{5}{2}}}| > |\frac{3\lambda_1\lambda_2 x}{(1+\lambda_2 x^2)^{\frac{3}{2}}}|$

Case 2: For $x < 0$,

(i) if $\lambda_1 > 0$ provided for all $\alpha < 0$ and $\beta > 0$ and

(i) if $\lambda_1 < 0$ provided $|\frac{3\lambda_1\lambda_2^2 x^3}{(1+\lambda_2 x^2)^{\frac{5}{2}}}| > |\frac{3\lambda_1\lambda_2 x}{(1+\lambda_2 x^2)^{\frac{3}{2}}}|$.

6. Extended form of MSGND

In this section we discuss an extended form of MSGND$(\lambda_1, \lambda_2, \alpha, \beta)$ by introducing the location parameter μ and scale parameter σ.

Definition 6.1. Let $X \sim MSGND(\lambda_1, \lambda_2, \alpha, \beta)$ with p.d.f given in (4). Then $Y = \mu + \sigma X$ is said to have an extended MSGND with the following p.d.f.

$$f^*(y, \mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta) = \frac{1}{\sigma(\alpha+2)}\phi(\frac{y-\mu}{\sigma}) \Big[2 + \alpha[\Phi(\beta)]^{-1} \Phi(\beta\sqrt{1+\lambda_1^2} + \lambda^*(y))\Big], \tag{24}$$

in which $\lambda^*(y) = \frac{\lambda_1(y-\mu)}{\sqrt{\sigma^2+\lambda_2(y-\mu)^2}}$, $y \in R$, $\mu \in R$, $\lambda_1 \in R$, $\beta \in R$, $\sigma > 0$, $\lambda_2 \geq 0$ and $\alpha \geq -1$. A distribution with p.d.f (24) is denoted as EMSGND$(\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta)$. Clearly when

(i) When β=0, the EMSGND $(\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta)$ reduces to ESGND $(\mu,\sigma;\lambda_1, \lambda_2, \alpha)$ of [11].

(ii) β=0 and $\lambda_1 = 0$, the EMSGND$(\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta)$ reduces to the p.d.f of normal distribution.

(iii) When $\beta = 0$ and $\lambda_2 = 0$, the EMSGND$(\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta)$ reduces to EGMNSN ($\mu, \sigma; \alpha, \lambda$) of [9].

Now, we obtain the following results of EMSGND$(\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta)$, in a similar way as we defined in section 2 and 4.

Result 6.1. The cumulative distribution function (c.d.f) $F^*(y)$ of EMSGND$(\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta)$ with p.d.f (24) is the following, for $y \in R$.

$$F^*(y) = \left[2 + \frac{\alpha\Phi[(\beta)]^{-1}}{2}\right] \frac{\Phi(\frac{y-\mu}{\sigma})}{\sigma(\alpha+2)} - \frac{\alpha[\Phi(\beta)]^{-1}}{\sigma(\alpha+2)}\xi_\beta\left(y, \lambda^*(y)\right)$$

where $\xi_\beta\left(y, \lambda^*(y)\right)$, is as defined in Result 3.4.

Result 6.2. The characteristic function of EMSGND$(\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta)$ is given by

$$\psi_Y^*(t) = \frac{e^{it\mu - \frac{t^2\sigma^2}{2}}}{\alpha+2}\left\{2 + \alpha[\Phi(\beta)]^{-1} E\left[\Phi\left(\beta\sqrt{1+\lambda_1^2} + \lambda^*(z+\sigma^2 it)\right)\right]\right\},$$

where $\lambda^*(z + \sigma^2 it) = \frac{\lambda_1(z+\sigma^2 it)}{\sqrt{\sigma^2+\lambda_2(z+\sigma^2 it)^2}}$.

Result 6.3. The reliability function $R^*(t)$ of Y is the following, in which $\xi_\beta(t, \lambda^*(y))$ is as defined in Result 3.4, with $\lambda^*(y) = \frac{\lambda_1(y-\mu)}{\sqrt{\sigma^2+\lambda_2(y-\mu)^2}}$

$$R^*(t) = \frac{1}{\sigma(\alpha+2)}\left[1 - F(\frac{t-\mu}{\sigma})\right]\left\{2 + \frac{\alpha}{2}[F(\beta)]^{-1}\right\} + \frac{\alpha[F(\beta)]^{-1}}{\sigma(\alpha+2)} \xi_\beta\left(t, \lambda^*(y)\right).$$

Result 6.4. The failure rate $r^*(t)$ of Y is given by

$$r^*(t) = \frac{f(\frac{t-\mu}{\sigma})\left[2 + \alpha[F(\beta)]^{-1} F\left(\beta\sqrt{1+\lambda_1^2} + \lambda^*(t)\right)\right]}{\frac{1}{\sigma(\alpha+2)}\left[1 - F(\frac{t-\mu}{\sigma})\right]\left\{2 + \frac{\alpha}{2}[F(\beta)]^{-1}\right\} + \frac{\alpha[F(\beta)]^{-1}}{\sigma(\alpha+2)}\xi_\beta\left(t, \lambda^*(t)\right)},$$

where $\lambda^*(t) = \frac{\lambda_1(t-\mu)}{\sqrt{\sigma^2+\lambda_2(t-\mu)^2}}$.

7. Maximum likelihood estimation

The log-likelihood function, $\ln L$ of the random sample of size n from a population following EMSGND$(\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta)$ is the following.

$$\begin{aligned} ln\, L &= n \ln\left(\frac{1}{\sqrt{2\pi}}\right) - n \ln \sigma - n \ln(\alpha+2) - \frac{1}{2}\sum_{i=1}^{n} \frac{(y_i-\mu)^2}{\sigma^2} \\ &+ \sum_{i=1}^{n} \ln\left(2 + \alpha\, [\Phi(\beta)]^{-1} \Phi\left(\beta\sqrt{1+\lambda_1^2} + \lambda^*(y)\right)\right). \end{aligned} \tag{25}$$

On differentiating (25) with respect to parameters $\mu, \sigma, \lambda_1, \lambda_2, \alpha$ and β and then equating to zero, we obtain the following normal equations.

$$\begin{aligned} \sum_{i=1}^{n} \frac{(y_i-\mu)}{\sigma^2} &- \sum_{i=1}^{n} \frac{\alpha[\Phi(\beta)]^{-1}\phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)\left(\frac{\lambda_1}{\sqrt{\sigma^2+\lambda_2(y_i-\mu)^2}}\right)}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)} \\ &+ \sum_{i=1}^{n} \frac{\alpha[\Phi(\beta)]^{-1}\phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)\left(\frac{\lambda_1\lambda_2(y_i-\mu)^2}{[\sigma^2+\lambda_2(y_i-\mu)^2]^{\frac{3}{2}}}\right)}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)} = 0, \end{aligned} \tag{26}$$

$$\frac{n}{\sigma} = \sum_{i=1}^{n} \frac{(y_i-\mu)^2}{\sigma^3} + \sum_{i=1}^{n} \frac{\alpha\lambda_1\Phi[(\beta)]^{-1}\phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)\left(\frac{(y_i-\mu)\sigma}{[\sigma^2+\lambda_2(y_i-\mu)^2]^{\frac{3}{2}}}\right)}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)}, \tag{27}$$

$$\sum_{i=1}^{n} \frac{\alpha[\Phi(\beta)]^{-1}\phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)\left[\frac{\lambda^*(y)}{\lambda_1}+\frac{\beta\lambda_1}{\sqrt{1+\lambda_1^2}}\right]}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)} = 0, \tag{28}$$

$$\sum_{i=1}^{n} \frac{\alpha[\Phi(\beta)]^{-1}\phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)\left[\frac{\lambda_1(y_i-\mu)^3}{[\sigma^2+\lambda_2(y_i-\mu)^2]^{\frac{3}{2}}}\right]}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)} = 0, \tag{29}$$

$$\sum_{i=1}^{n} \frac{[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)} = 0 \tag{30}$$

and

$$\begin{aligned} &\sum_{i=1}^{n} \frac{\alpha[\Phi(\beta)]^{-1}\phi(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y))\sqrt{1+\lambda_1^2}}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)} - \\ &\sum_{i=1}^{n} \frac{\alpha[\Phi(\beta)]^{-2}\phi(\beta)\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda_1^2}+\lambda^*(y)\right)} = 0. \end{aligned} \tag{31}$$

On solving the equations (26) to (31), we get the maximum likelihood estimate of the parameters of EMSGND$(\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta)$.

8. Generalized likelihood ratio test

In this section we discuss a test procedure for testing the parameter β of $EMSGND$. For testing the null hypothesis $H_0 : \beta = 0$ against the alternative hypothesis $H_1 : \beta \neq 0$ by using the generalized likelihood ratio test, the test statistic is

$$-2ln\lambda(x) = 2[lnL(\hat{\Theta}; x) - lnL(\hat{\Theta}^*; x)],$$

where $\hat{\Theta}$ is the maximum likelihood estimator of $\Theta = (\mu, \sigma, \lambda_1, \lambda_2, \alpha, \beta)$ with no restriction, and $\hat{\Theta}^*$ is the maximum likelihood estimator of Θ when $\beta = 0$. The test statistic given is asymptotically distributed as χ^2 with 1 degrees of freedom. For further details see [12].

9. Applications

In this section we consider three real life data applications of the EMSGND. The first data is taken from [9]. The data gives the Otis IQ scores for 52 non-white males hired by a large insurance company in 1971. The observed data is given below:
Data set 1:
91, 102, 100, 117, 122, 115, 97, 109, 108, 104, 108, 118, 103, 123, 123, 103, 106, 102, 118, 100, 103, 107, 108, 107, 97, 95, 119, 102, 108, 103, 102, 112, 99, 116, 114, 102, 111, 104, 122, 103, 111, 101, 91, 99, 121, 97, 109, 106, 102, 104, 107, 95.
The second data is taken from [8]. This data is related to the milk production of 28 cows in which the variable under study is the daily milk production in kilogram and the variable recorded for three times milking cows. Data set 2:
34.6, 27.7, 29.2, 25.3, 27.6, 37.9, 32.6, 32, 30.7, 29.6, 38.3, 32.9, 30.8, 32.2, 32.9, 28.1, 33.9, 28.6, 28.1, 35.9, 34.8, 40.3, 30.9, 34.4, 19.8, 25.8, 37.3, 32.4.
The third data is taken from [8]. The data includes 100 females and 102 males with 13 variables such as height, weight, body mass index (BMI) etc. We choose for the variable under study is the BMI values for the second 50 females. The data is given below: Data set 3:
24.47, 23.99, 26.24, 20.04, 25.72, 25.64, 19.87, 23.35, 22.42, 20.42, 22.13, 25.17, 23.72, 21.28, 20.87, 19.00, 22.04, 20.12, 21.35, 28.57, 26.95, 28.13, 26.85, 25.27, 31.93, 16.75, 19.54, 20.42, 22.76, 20.12, 22.35, 19.16, 20.77, 19.37, 22.37, 17.54, 19.06, 20.30, 20.15, 25.36, 22.12, 21.25, 20.53, 17.06, 18.29, 18.37, 18.93, 17.79, 17.05, 20.31.
We obtained the maximum likelihood estimate (MLE) of the parameters by using the data sets with the help of the MATHCAD software. The numerical results obtained are presented in Table 1, which includes the estimated values of the parameters and the corresponding Kolmogorov-Smirnov Statistics (KSS) values of models: ESGND($\mu, \sigma; \lambda_1, \lambda_2, \alpha$) and EMSGND($\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta$). Also, its Akaike's Information Criterion(AIC), Bayesian Information Criterion(BIC) and corrected Akaike's Information Criterion(AICc) values are obtained.

Table 1: *Estimated values of the parameters for the model: ESGND($\mu, \sigma; \lambda_1, \lambda_2, \alpha$) and EMSGND($\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta$) with respective values of KSS, P-value, log-likelihood, AIC, BIC and AICc in case of Data Set 1, 2 and 3.*

Data set	Estimates of the parameters	ESGND($\mu, \sigma; \lambda_1, \lambda_2, \alpha$)	EMSGND ($\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta$)
1	$\hat{\mu}$	102.18167	106.654
	$\hat{\sigma}$	4.94535	8
	$\hat{\alpha}$	1.89352	2
	$\hat{\beta}$	-	8
	$\hat{\lambda}_1$	6.23351	0.809
	$\hat{\lambda}_2$	0.38068	0.349
	KSS	0.5	0.129961
	P-Value	3.91952×10^{-12}	0.315666
	Log-likelihood	-193.257	-183.43
	AIC	396.515	378.859
	BIC	406.271	390.567
	AICc	397.819	380.726
2	$\hat{\mu}$	31.43211	31.5934
	$\hat{\sigma}$	2.0174	4.464
	$\hat{\alpha}$	0.96229	0.5
	$\hat{\beta}$	-	20.006
	$\hat{\lambda}_1$	0.64746	6.002
	$\hat{\lambda}_2$	15.91043	8
	KSS	0.451625	0.0783872
	P-Value	9.83689×10^{-6}	0.98998
	Log-likelihood	-100.348	-81.1221
	AIC	210.697	174.244
	BIC	217.358	182.237
	AICc	213.424	178.244
3	$\hat{\mu}$	20.1321	21.865
	$\hat{\sigma}$	1.07639	3.33
	$\hat{\alpha}$	1.17325	0.6
	$\hat{\beta}$	-	8
	$\hat{\lambda}_1$	7.85813	7
	$\hat{\lambda}_2$	10.13911	5
	KSS	0.5	0.121453
	P-Value	1.07824×10^{-11}	0.418815
	Log-likelihood	-324.56	-130.59
	AIC	659.119	273.18
	BIC	668.68	284.652
	AICc	660.483	275.134

It is clear from Table 1, that the EMSGND($\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta$) is a more appropriate model to all the three data sets compared to the existing model ESGND($\mu, \sigma; \lambda_1, \lambda_2, \alpha$). We have plotted the histogram of the respective data sets along with the corresponding fitted values of the $ESGND$ and $EMSGND$ in Figures 2, 3 and 4 respectively. It shows that $EMSGND$ yields a better fit than $ESGND$ in all the cases. Thus, the model discussed in this paper provides more flexibility in modeling in case of all the three datasets due to the presence of the extra parameter.

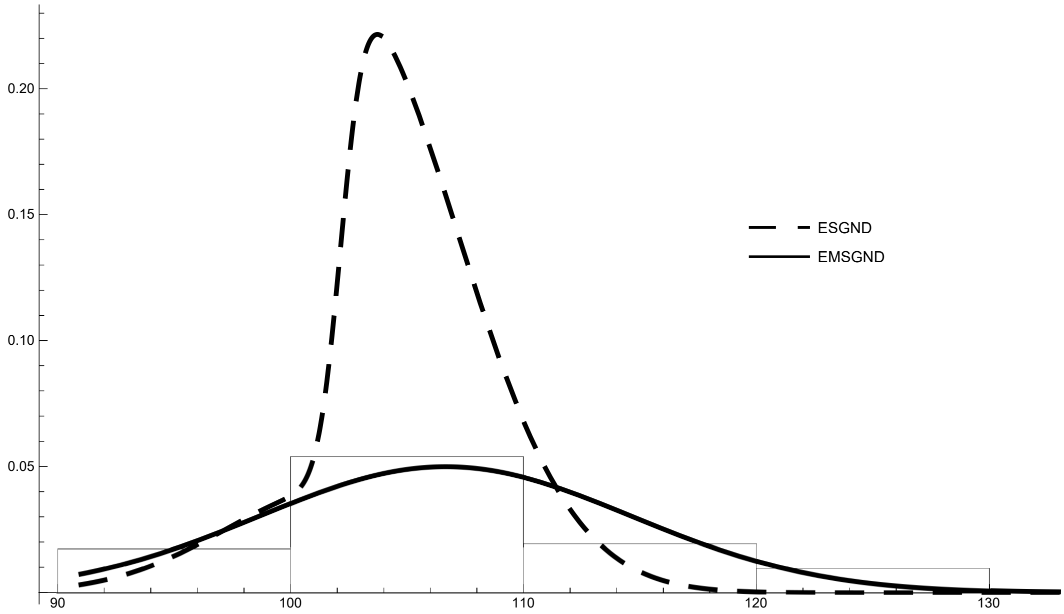

Figure 2: *Histogram of Data set 1 and fitted distributions*

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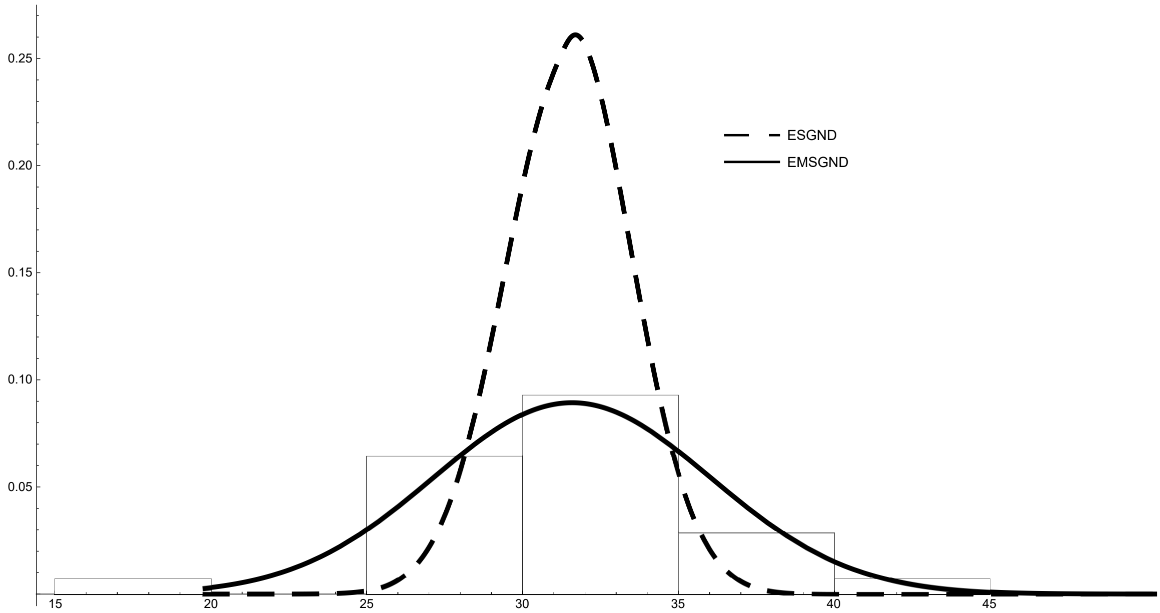

Figure 3: *Histogram of Data set 2 and fitted distributions*

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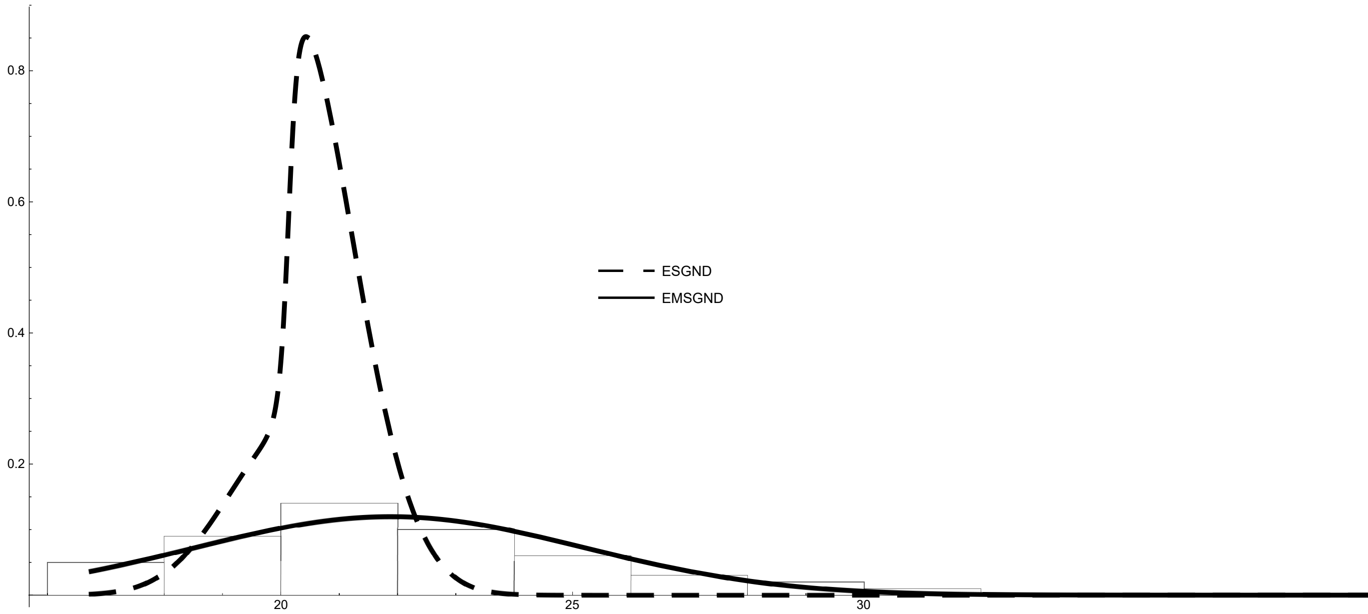

Figure 4: *Histogram of Data set 3 and fitted distributions*

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Also, we conduct a generalized likelihood ratio test for illustrating the usefulness of the model, which is described as follows.
Let us consider the problem of testing a hypothesis $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$ in the case of Data set 1. The MLEs and values of the likelihood for $ESGND$ and $EMSGND$ are

$$\hat{\mu} = 102.18167,\ \hat{\sigma} = 4.94535,\ \hat{\lambda_1} = 6.23351,\ \hat{\lambda_2} = 0.38068,\ \hat{\alpha} = 1.89352,$$

$L(\hat{\Theta}^*; x) = 1.17315 \times 10^{-84}$ and

$$\hat{\mu} = 106.654,\ \hat{\sigma} = 8,\ \hat{\lambda_1} = 0.809,\ \hat{\lambda_2} = 0.349,\ \hat{\alpha} = 2,\ \hat{\beta} = 8,$$

$L(\hat{\Theta}; x) = 2.17542 \times 10^{-80}$, respectively. The value of likelihood ratio (LR) test statistic is 19.6557. Since the critical value for the test with significance level 0.05 at one degrees of freedom is 3.84, the null hypothesis is rejected.

Similarly we consider the problem of testing $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$ using the Data set 2. The MLEs and values of the likelihood for $ESGND$ and $EMSGND$ are

$$\hat{\mu} = 31.43211,\ \hat{\sigma} = 2.0174,\ \hat{\lambda_1} = 0.64746,\ \hat{\lambda_2} = 15.91043,\ \hat{\alpha} = 0.96229,$$

$L(\hat{\Theta}^*; x) = 2.62584 \times 10^{-44}$ and

$$\hat{\mu} = 31.5934,\ \hat{\sigma} = 4.464,\ \hat{\lambda_1} = 6.002,\ \hat{\lambda_2} = 8,\ \hat{\alpha} = 0.5,\ \hat{\beta} = 20.006,$$

$L(\hat{\Theta}; x) = 5.87655 \times 10^{-36}$, respectively. The value of likelihood ratio (LR) test statistic is 38.4525. Since the critical value for the test with significance level 0.05 at one degrees of freedom is 3.84, the null hypothesis is rejected.

Similarly we consider the problem of testing $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$ using the Data set 3. The MLEs and values of the likelihood for $ESGND$ and $EMSGND$ are

$$\hat{\mu} = 20.1321,\ \hat{\sigma} = 1.07639,\ \hat{\lambda_1} = 7.85813,\ \hat{\lambda_2} = 10.13911,\ \hat{\alpha} = 1.17325,$$

$L(\hat{\Theta}^*; x) = 1.11045 \times 10^{-141}$ and

$$\hat{\mu} = 21.865,\ \hat{\sigma} = 3.33,\ \hat{\lambda_1} = 7,\ \hat{\lambda_2} = 5,\ \hat{\alpha} = 0.6,\ \hat{\beta} = 8,$$

$L(\hat{\Theta}; x) = 1.92965 \times 10^{-}57$, respectively. The value of likelihood ratio (LR) test statistic is 387.939. Since the critical value for the test with significance level 0.05 at one degrees of freedom is 3.84, the null hypothesis is rejected.

10. Simulation Study

In order to assess the performance of the maximum likelihood estimators of the parameters of the EMSGND($\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta$), we have conducted a brief simulation study as follows. We have simulated data sets of sizes 30, 50 and 100 from the EMSGND for the parameter values $\mu = 2, \sigma = 0.5, \lambda_1 = 0.8, \lambda_2 = 0.3, \alpha = 1$ and $\beta = 8$. We obtain likelihood estimates of these parameters and computed bias and mean square errors (MSE). The results obtained are presented in Table 2.

Table 2: *Estimate of the parameters and corresponding bias and mean square error(MSE).*

Sample size	Parameters	Estimate	Bias	MSE
30	$\hat{\mu}$	1.988331	0.1883313	0.03546867
	$\hat{\sigma}$	0.4833721	-0.0166279	0.0002764871
	$\hat{\lambda_1}$	0.83	0.54	0.2916
	$\hat{\lambda_2}$	0.29	-1.71	2.9241
	$\hat{\beta}$	7.98	1.98	3.9204
	$\hat{\alpha}$	1.37	1.07	1.1449
50	$\hat{\mu}$	1.99147	-0.008530334	7.27666×10^{-05}
	$\hat{\sigma}$	0.4910136	-0.00898637	8.075485×10^{-05}
	$\hat{\lambda_1}$	0.78	-0.02	4×10^{-04}
	$\hat{\lambda_2}$	0.285	-0.015	0.000225
	$\hat{\beta}$	7.87	-0.13	0.0169
	$\hat{\alpha}$	1.36	0.36	0.1296
100	$\hat{\mu}$	1.994749	-0.005251242	2.757554×10^{-05}
	$\hat{\sigma}$	0.5006707	0.0006706597	4.497845×10^{-07}
	$\hat{\lambda_1}$	0.795	-0.005	2.5×10^{-05}
	$\hat{\lambda_2}$	0.29	-0.01	1×10^{-04}
	$\hat{\beta}$	7.9	-0.1	0.01
	$\hat{\alpha}$	1	-3.248735×10^{-12}	1.055428×10^{-23}

From Table 2, it can be observed that both the bias and MSE are in decreasing order as sample size increases.

11. Summary and Conclusion

Through this paper we proposed a wide class of distributions which are suitable for asymmetric as well as plurimodal situations. Certain structural properties of the distribution are derived and discussed its reliability properties as well as unimodal and plurimodal properties. A location scale extension of this class of distribution is also considered and obtained its analogous properties. Further we discussed the maximum likelihood estimation of the parameters of the model and thereby illustrated the procedures through certain real life applications using three real data sets and shown that the model is suitable for all the data sets compared to the existing model. Also, we constructed a test procedure for establishing the significance of the additional parameter β. In order to assess the performance of the maximum likelihood estimation procedure, we carried out a brief simulation study. The proposed model is shown to be more appropriate for asymmetric as well as plurimodal data sets. Certain characteristic properties as well as inferential aspects of the model are yet to study, which we hope to publish in another article. Even though there is flexibility in the proposed model compared to the existing model from the practical point of view, there is scope for developing a further generalized version of the proposed model so as to model more complicated data sets. Such possibilities are under investigation and hope to publish through another article shortly.

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