CERTAIN CURVATURE CONDITIONS ON LORENTZIAN PARA-KENMOTSU MANIFOLDS

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Abstract

We classify Lorentzian para-Kenmotsu manifolds which satisfy the curvature conditions $W_2.C = 0$, $Z.C = L_C Q(g, C)$, $W_2.Z - Z.W_2 = 0$ and $W_2.Z + Z.W_2 = 0$, where W_2 is the Weyl- projective tensor, Z is the concircular tensor, and C is the Weyl conformal curvature tensor. We study and have shown that the manifold M is η-Einstein provided that the Weyl-projective curvature tensor W_2 meets the condition $W_2.Z - Z.W_2 = 0$, and it is an Einstein manifold if $W_2.Z + Z.W_2 = 0$. Finally, in this article, we derive the conditions in relation to conformally flatness of the manifold, whenever the LP-Kenmotsu manifold satisfies the condition $Z.C = L_C Q(g, C)$.

Keywords: Para-contact metric manifold, *LP*-Kenmotsu manifold, concircular curvature tensor, conformal curvature tensor, Weyl-projective tensor.
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I. Introduction

In 1989, K. Matsumoto [7] introduced the notion of Lorentzian paracontact and in particular, Lorentzian para-Sasakian (*LP*-Sasakian) manifolds. Later, these manifolds have been widely studied by many geometers Matsumoto and Mihai [8], Mihai and Rosca [6], Mihai, Shaikh and De [5], Venkatesha and Bagewadi [15], Venkatesha, Pradeep Kumar and Bagewadi [16, 17] and obtained several results of these manifolds.

In 1995, Sinha and Sai Prasad [2] defined a class of almost paracontact metric manifolds namely para-Kenmotsu (briefly *P*-Kenmotsu) and special para-Kenmotsu (briefly *SP*-Kenmotsu) manifolds in similar to *P*-Sasakian and *SP*-Sasakian manifolds. In 2018, Abdul Haseeb and Rajendra Prasad defined a class of Lorentzian almost paracontact metric manifolds namely Lorentzian para-Kenmotsu (briefly *LP*-Kenmotsu) manifolds [1] and they studied ϕ-semisymmetric *LP*-Kenmotsu manifolds with a quarter-symmetric non-metric connection admitting Ricci solitons [13].

On the other hand, In 1970 [4], Pokhariyal and Mishra introduced new tensor fields, called the Weyl-projective curvature tensor W_2 of type $(1, 3)$ and the tensor field E on a Riemannian manifold.The Weyl-projective curvature tensor W_2 with respect to Riemannian connection on a Riemannian manifold M is given by:

$$W_2(X, Y)W = R(X, Y)W + \frac{1}{n-1}[g(X, W)QY - g(Y, W)QX], \tag{1}$$

where $QX = (n-1)X$, which plays an important role in the theory of the projective transformations of connections.

Further, Pokhariyal [3] studied the properties of these tensor fields on a Sasakian manifold. Matsumoto, Ianus and Mihai extended these concepts to almost para-contact structures and studied para-Sasakian manifolds admitting these tensor fields [9] in 1986 and these results were generalised by De and Sarkar, in 2009 [14]. Sai Prasad and Satyanarayana studied the W_2-tensor field in an SP-Kenmotsu manifold [10]. In our earlier work, we consider LP-Kenmotsu manifolds admitting the Weyl-projective curvature tensor W_2 and shown that these manifolds admiting a Weyl-flat projective curvature tensor, an irrotational Weyl-projective curvature tensor and a conservative Weyl-projective curvature tensor are an Einstein manifolds of constant scalar curvature [11, 12].

Inspired by these studies, in the present work, we explore a class of Lorentzian para-Kenmotsu manifolds that admits certain curvature conditions. The current study is arranged as follows: Section 2 has certain prerequisites. In section 3, it is illustrated that the manifold M is η-Einstein provided that the Weyl-projective curvature tensor W_2 meets the condition $W_2.Z - Z.W_2 = 0$, and it is an Einstein manifold if $W_2.Z + Z.W_2 = 0$. Finally, we derive the conditions in relation to conformally flatness of the manifold, whenever the LP-Kenmotsu manifold satisfying the condition $Z.C = L_C Q(g, C)$, where the concircular curvature tensor $Z(X,Y)$ is given by:

$$Z(X,Y)W = R(X,Y)W - \frac{r}{n(n-1)}[g(Y,W)X - g(X,W)Y]. \tag{2}$$

II. Preliminaries

An n-dimensional differentiable manifold M admitting a (1, 1) tensor field ϕ, contravariant vector field ξ, a 1-form η and the Lorentzian metric $g(X,Y)$ satisfying

$$\phi^2 X = X + \eta(X)\xi, \;\; g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{3}$$

and

$$\eta(\xi) = -1, \;\; \phi\xi = 0, \;\; \eta(\phi X) = 0, \;\; g(X,\xi) = \eta(X), \; rank\phi = n-1. \tag{4}$$

is called Lorentzian almost paracontact manifold [7].

In a Lorentzian almost paracontact manifold, we have

$$\Phi(X,Y) = \Phi(Y,X), \tag{5}$$

where $\Phi(X,Y) = g(X, \phi Y)$.

A Lorentzian almost paracontact manifold M is called Lorentzian para-Kenmotsu manifold if [1]

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{6}$$

for any vector fields X and Y on M and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric g.

In the Lorentzian para-Kenmotsu manifold, the following relations hold good:

$$\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi \tag{7}$$

and

$$(\nabla_X \eta)Y = -g(X,Y)\xi - \eta(X)\eta(Y). \tag{8}$$

Further, on a Lorentzian para-Kenmotsu manifold M, the following relations hold [1]:

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{9}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{10}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y; \text{ when } X \text{ is orthogonal to } \xi, \tag{11}$$

$$R(\xi,X)\xi = X + \eta(X)\xi, \tag{12}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{13}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y). \tag{14}$$

A Lorentzian para-Kenmotsu manifold M is said to be an η-Einstein manifold if its Ricci tensor $S(X,Y)$ is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{15}$$

where a and b are scalar functions on M.

Next we define endomorphisms $R(X,Y)$ and $X \wedge_A Y$ by

$$R(X,Y)W = \nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_[X,Y]W, \tag{16}$$

$$(X \wedge_A Y)W = A(Y,W)X - A(X,W)Y, \tag{17}$$

A is the symmetric $(0,2)$- tensor.

For a $(0,k)$-tensor field T, $K \geq 1$, on (M_n, g) we define $W_2.T$, $Z.T$ and $Q(g,T)$ by

$$\begin{aligned}(W_2(X,Y).T(X_1,X_2,...,X_k) = &- T(W_2(X,Y)X_1,X_2,...,X_k)\\ &- T(X_1,W_2(X,Y)X_2,...,X_k)\\ &- ... - T(X_1,X_2,...,W_2(X,Y)X_k),\end{aligned} \tag{18}$$

$$\begin{aligned}(Z(X,Y).T(X_1,X_2,...,X_k) = &- T(Z(X,Y)X_1,X_2,...,X_k)\\ &- T(X_1,Z(X,Y)X_2,...,X_k)\\ &- ... - T(X_1,X_2,...,Z(X,Y)X_k),\end{aligned} \tag{19}$$

$$\begin{aligned}Q(g,T)(X_1,X_2,...,X_k;X,Y) = &- T((X\wedge Y)X_1,X_2,...,X_k)\\ &- T(X_1,(X\wedge Y)X_2,...,X_k)\\ &- ... - T(X_1,X_2,...,(X\wedge Y)X_k),\end{aligned} \tag{20}$$

respectively.

By definition the Weyl Conformal curvature tensor C is given by

$$\begin{aligned}C(X,Y)Z =& R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]\\ &\frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{21}$$

where Q denotes the Ricci operator, i.e., $S(X,Y) = g(QX,Y)$ and r is scalar curvature. The Weyl conformal curvature tensor C is defined by $C(X,Y,Z,W) = g(C(X,Y)Z,W)$. If $C = 0, n \geq 4$, then M is conformally flat.

III. MAIN RESULTS

In the present section we consider the LP-Kenmotsu manifold satisfying the curvature conditions $W_2.C = 0$, $Z.C = L_C Q(g, C)$, $W_2.Z - Z.W_2 = 0$, and $W_2.Z + Z.W_2 = 0$.

First we give the following proposition.

Proposition 1. Let M be an n-dimensional $(n > 3)$ LP-Kenmotsu manifold. If the condition $W_2.C = 0$ holds on M, then

$$S^2(X, U) = (n-1)(r-2)\eta(X)\eta(U) + (r+n-2)S(U, X) - (n-1)g(X, U)$$

is satisfied on M, where $S^2(X, U) = S(QX, U)$.

Proof: Assume that M is an n-dimensional, $n > 3$, LP-Kenmotsu manifold satisfying the condition $W_2.C = 0$. From (18) we have

$$\begin{aligned}(W_2(V, X).C)(Y, U)W = &- W_2(V, X)C(Y, U)W \\ &- C(W_2(V, X)Y, U)W - C(Y, W_2(V, X)U)W \\ &- C(Y, U)W_2(V, X)W = 0,\end{aligned} \tag{22}$$

where $X, Y, U, V, W \in \chi(M)$. Taking $V = \xi$ in (22), we have

$$\begin{aligned}(W_2(\xi, X).C)(Y, U)W = &- W_2(\xi, X)C(Y, U)W \\ &- C(W_2(\xi, X)Y, U)W - C(Y, W_2(\xi, X)U)W \\ &- C(Y, U)W_2(\xi, X)W = 0,\end{aligned} \tag{23}$$

Furthermore, substituting (1), (9), (13), (21) into (23) and multiplying with ξ, we get.

$$\begin{aligned}&- g(X, C(Y, U)W) - g(X, Y)\eta(C(\xi, U)W + \eta(Y)\eta(C(X, U)W) \\ &- g(X, U)\eta(C(Y, \xi)W) + \eta(U)\eta(C(Y, X)W) - g(X, W)\eta(C(Y, U)\xi) \\ &+ \eta(W)\eta(C(Y, U)X) + \frac{1}{n-1}[\eta(C(Y, U)W) - \eta(Y)\eta(C(QX, U)W) \\ &+ g(X, Y)\eta(C(Q\xi, U)W) + g(X, U)\eta(C(Y, Q\xi)W) - \eta(U)\eta(C(Y, QX)W) \\ &- \eta(W)\eta(C(Y, U)QX) + g(X, W)\eta(C(Y, U)Q\xi)] = 0.\end{aligned} \tag{24}$$

Thus replacing W with ξin (24), we have

$$- g(X, C(Y, U)\xi) - \eta(C(Y, U)X) + \frac{1}{n-1}[\eta(C(Y, U)QX)] = 0. \tag{25}$$

Again taking $Y = \xi$ in (25)and after some calculations, since $n > 3$, we get

$$S^2(U, X) = (n-1)(r-2)\eta(X)\eta(U) + (r+n-2)S(U, X) - (n-1)g(X, U).$$

Theorem 2. Let M be an n-dimensional $(n > 3)$ LP-Kenmotsu manifold. If the condition $Z.C = L_C Q(g, C)$ holds on M, then either M is conformally flat or $L_C = \frac{r}{n(n-1)} - 1$.

Proof. Let M be an LP-Kenmotsu manifold. So we have

$$(Z(V, X).C)(Y, U)W = L_C Q(g, C)(Y, U, W; V, X).$$

Then from (19) and (20) we can write,

$$\begin{aligned}Z(V, X)C(Y, U)W &- C(Z(V, X)Y, U)W - C(Y, Z(V, X)U)W \\ &- C(Y, U)Z(V, X)W \\ &= L_C[(V \wedge X)C(Y, U)W - C((V \wedge X)Y, U)W \\ &- C(Y, (V \wedge X)U)W - C(Y, U)(V \wedge X)W].\end{aligned} \tag{26}$$

Therefore, replacing v with ξ in (26), we have

$$\begin{aligned} Z(\xi,X)C(Y,U)W &- C(Z(\xi,X)Y,U)W - C(Y,Z(\xi,X)U)W \\ &- C(Y,U)Z(\xi,X)W \\ &= L_C[(\xi\wedge X)C(Y,U)W - C((\xi\wedge X)Y,U)W \\ &- C(Y,(\xi\wedge X)U)W - C(Y,U)(\xi\wedge X)W]. \end{aligned} \tag{27}$$

Using (20), (9) and taking the inner product of (27) with ξ, we get

$$\begin{aligned} &\big[1-\frac{r}{n(n-1)}-L_C\big][-g(X,C(Y,U)W) - \eta(X)\eta(C(Y,U)W) \\ &- g(X,Y)\eta(C(\xi,U)W) + \eta(Y)\eta(C(X,U)W) \\ &- g(X,U)\eta(C(Y,\xi)W) + \eta(U)\eta(C(Y,X)W) + \eta(W)\eta(C(Y,U)X)\big] = 0. \end{aligned} \tag{28}$$

Putting $X = Y$ in (28), we have

$$\begin{aligned} &\big[1-\frac{r}{n(n-1)}-L_C\big][-g(Y,C(Y,U)W) + \eta(W)\eta(C(Y,U)Y) \\ &- g(Y,Y)\eta(C(\xi,U)W) - g(Y,U)\eta(C(Y,\xi)W)\big] = 0. \end{aligned} \tag{29}$$

A contraction of (29) with respect to Y gives us

$$\big[1-\frac{r}{n(n-1)}-L_C\big]\eta(C(\xi,U)W) = 0. \tag{30}$$

If $L_C \neq 1-\frac{r}{n(n-1)}$, then eq.(30) is reduced to

$$\eta(C(\xi,U)W) = 0, \tag{31}$$

which gives

$$S(U,W) = \big(\frac{r}{(n-1)}-1\big)g(U,W) + \big(\frac{r}{(n-1)}-n\big)\eta(U)\eta(W). \tag{32}$$

Therefore, M is a η-Einstein manifold. So, using (31) and (32), we have eq. (28) in the form

$$C(Y,U,W,X) = 0,$$

which means that M is conformally flat.

If $L_C \neq 0$ and $\eta(C(\xi,U)W) \neq 0$, then $1-\frac{r}{n(n-1)}-L_C = 0$, which gives $L_C = 1-\frac{r}{n(n-1)}$. This completes the proof of the theorem.

Corollary 3. Every n-dimensional $(n > 3)$ nonconformally flat *LP*-Kenmotsu manifold satisfies $Z.C = \big(1-\frac{r}{n(n-1)}\big)Q(g,C)$.

Theorem 4. Let M be an n-dimensional $(n > 3)$ *LP*-Kenmotsu manifold. M satisfies the condition

$$W_2.Z - Z.W_2 = 0$$

if and only if M is a η-Einstein manifold.

Proof. Let M satisfy the condition $W_2.Z - Z.W_2 = 0$. Then we can write

$$\begin{aligned}
W_2.Z - Z.W_2 =& R(V,X)R(Y,U)W + \frac{1}{n-1}\left[g(V,R(Y,U)W)QX - g(X,R(Y,U)W)QV\right] \\
& - \frac{r}{n(n-1)}g(U,W)\left[R(V,X)Y + \frac{1}{n-1}\left(g(V,Y)QX - g(X,Y)QV\right)\right] \\
& + \frac{r}{n(n-1)}g(Y,W)\left[R(V,X)U + \frac{1}{n-1}\left(g(V,U)QX - g(X,U)QV\right)\right] \\
& - R(V,X)R(Y,U)W + \frac{r}{n(n-1)}\left[g(X,R(Y,U)W)V - g(V,R(Y,U)W)X\right] \\
& - \frac{1}{n-1}g(Y,W)\left[R(V,X)QU - \frac{r}{n(n-1)}\left(g(X,QU)V - g(V,QU)X\right)\right] \\
& + \frac{1}{n-1}g(U,W)\left[R(V,X)QY - \frac{r}{n(n-1)}\left(g(X,QY)V - g(V,QY)X\right)\right] = 0.
\end{aligned} \tag{33}$$

Therefore, replacing V with ξ in (33), we have

$$\begin{aligned}
W_2.Z - Z.W_2 =& \frac{1}{n-1}\left[g(\xi,R(Y,U)W)QX - g(X,R(Y,U)W)Q\xi\right] \\
& - \frac{r}{n(n-1)}g(U,W)\left[R(\xi,X)Y + \frac{1}{n-1}\left(g(\xi,Y)QX - g(X,Y)Q\xi\right)\right] \\
& + \frac{r}{n(n-1)}g(Y,W)\left[R(\xi,X)U + \frac{1}{n-1}\left(g(\xi,U)QX - g(X,U)Q\xi\right)\right] \\
& - R(\xi,X)R(Y,U)W + \frac{r}{n(n-1)}\left[g(X,R(Y,U)W)\xi - g(\xi,R(Y,U)W)X\right] \\
& - \frac{1}{n-1}g(Y,W)\left[R(\xi,X)QU - \frac{r}{n(n-1)}\left(g(X,QU)\xi - g(\xi,QU)X\right)\right] \\
& + \frac{1}{n-1}g(U,W)\left[R(\xi,X)QY - \frac{r}{n(n-1)}\left(g(X,QY)\xi - g(\xi,QY)X\right)\right] = 0.
\end{aligned} \tag{34}$$

Using (10), (13), we get

$$\begin{aligned}
W_2.Z - Z.W_2 =& \frac{1}{n-1}\left[g(\xi,R(Y,U)W)QX - g(X,R(Y,U)W)Q\xi\right] \\
& - \frac{r}{n(n-1)}g(U,W)\left[g(X,Y)\xi - \eta(Y)X\right] - \frac{r}{n(n-1)}g(U,W)\eta(Y)X \\
& + \frac{r}{n(n-1)}g(U,W)g(X,Y)\xi + \frac{r}{n(n-1)}g(Y,W)\left[g(X,U)\xi - \eta(U)X\right] \\
& - \frac{r}{n(n-1)}g(Y,W)\eta(U)X - \frac{r}{n(n-1)}g(Y,W)g(X,U)\xi \\
& + \frac{r}{n(n-1)}\left[g(X,R(Y,U)W)\xi - g(\xi,R(Y,U)W)X\right. \\
& - \frac{1}{(n-1)}g(Y,W)\left[g(X,QU)\xi - \eta(QU)X\right] + \frac{r}{n(n-1)^2}g(Y,W)g(X,QU)\xi \\
& - \frac{r}{n(n-1)^2}g(Y,W)\eta(QU)X + \frac{1}{(n-1)}g(U,W)\left[g(X,QY)\xi - \eta(QY)X\right] \\
& - \frac{r}{n(n-1)^2}g(U,W)g(X,QY)\xi - \frac{r}{n(n-1)^2}g(U,W)\eta(QY)X = 0.
\end{aligned} \tag{35}$$

Again, taking $U = \xi$ in (35), we get

$$\begin{aligned}
&\frac{1}{n-1}\big[g(\xi, g(Y,W)\xi - \eta(W)Y)(n-1)X - g(X, g(Y,W)\xi - \eta(W)Y)(n-1)\xi\big] \\
&- \frac{r}{n(n-1)}\eta(W)\big[g(X,Y)\xi - \eta(Y)X\big] - \frac{r}{n(n-1)}\eta(Y)\eta(W)X \\
&+ \frac{r}{n(n-1)}g(X,Y)\eta(W)\xi + \frac{r}{n(n-1)}g(Y,W)\big[\eta(X)\xi + X\big] \\
&- \frac{r}{n(n-1)}g(Y,W)\eta(U)X - \frac{r}{n(n-1)}g(Y,W)g(X,U)\xi \\
&+ \frac{r}{n(n-1)}g(Y,W)X - \frac{r}{n(n-1)}g(Y,W)\eta(X)\xi \\
&+ \frac{r}{n(n-1)}\big[g(X, g(Y,W)\xi - \eta(W)Y)\xi - g(\xi, g(Y,W)\xi - \eta(W)Y)x \\
&- \frac{1}{(n-1)}g(Y,W)\big[(n-1)\eta(X)\xi - (n-1)X\big] + \frac{r}{n(n-1)}g(Y,W)\eta(X)\xi \\
&- \frac{r}{n(n-1)^2}g(Y,W)X + \frac{1}{(n-1)}\eta(W)\big[(n-1)g(X,Y)\xi - (n-1)\eta(Y)X\big] \\
&- \frac{r}{n(n-1)^2}\eta(W)S(X,Y)\xi - \frac{r}{n(n-1)}\eta(W)\eta(Y)X = 0.
\end{aligned} \tag{36}$$

Taking the inner product of (36) with ξ, we find

$$\begin{aligned}
-2\eta(W)\eta(Y)\eta(X) - 2\eta(W)g(X,Y) + \frac{r}{n(n-1)}\eta(W)g(X,Y) + \frac{2r}{n(n-1)}\eta(W)\eta(Y)\eta(X) \\
+ \frac{2r}{n(n-1)}\eta(X)g(Y,W) + \frac{r}{n(n-1)^2}\eta(W)S(X,Y) = 0.
\end{aligned} \tag{37}$$

Again, taking $W = \xi$ and using (4) in (37), we get

$$S(X,Y) = \big[\frac{2(n-1)}{r}\eta(X)\eta(Y) + \frac{(n-r)(n-1)}{r}g(X,Y) \tag{38}$$

So, M is a η-Einstein manifold.

Conversely, if M is a η-Einstein manifold, then it is easy to show that $W_2.Z - Z.W_2 = 0$. Our theorem is thus proved.

Theorem 5. Let M be an n-dimensional $(n > 3)$ *LP*-Kenmotsu manifold. M satisfies the condition

$$W_2.Z + Z.W_2 = 0$$

if and only if M is an Einstein manifold.

Proof. Let M satisfy the condition $W_2.Z + Z.W_2 = 0$. Then from (33) and (34) we can write

$$\begin{aligned}
&2R(V,X)R(Y,U)W + \frac{1}{n-1}\big[g(V, R(Y,U)W)QX - g(X, R(Y,U)W)QV\big] \\
&- \frac{r}{n(n-1)}g(U,W)\big[R(V,X)Y + \frac{1}{n-1}\big(g(V,Y)QX - g(X,Y)QV\big)\big] \\
&+ \frac{r}{n(n-1)}g(Y,W)\big[R(V,X)U + \frac{1}{n-1}\big(g(V,U)QX - g(X,U)QV\big)\big] \\
&- \frac{r}{n(n-1)}\big[g(X, R(Y,U)W)V - g(V, R(Y,U)W)X\big] \\
&+ \frac{1}{n-1}g(Y,W)\big[R(V,X)QU - \frac{r}{n(n-1)}\big(g(X,QU)V - g(V,QU)X\big)\big] \\
&- \frac{1}{n-1}g(U,W)\big[R(V,X)QY - \frac{r}{n(n-1)}\big(g(X,QY)V - g(V,QY)X\big)\big] = 0.
\end{aligned} \tag{39}$$

Therefore, replacing V with ξ in (39), we have

$$\begin{aligned}
&2R(\xi,X)R(Y,U)W+\frac{1}{n-1}\left[g(\xi,R(Y,U)W)QX-g(X,R(Y,U)W)Q\xi\right]\\
&-\frac{r}{n(n-1)}g(U,W)\left[R(\xi,X)Y+\frac{1}{n-1}\left(g(\xi,Y)QX-g(X,Y)Q\xi\right)\right]\\
&+\frac{r}{n(n-1)}g(Y,W)\left[R(\xi,X)U+\frac{1}{n-1}\left(g(\xi,U)QX-g(X,U)Q\xi\right)\right]\\
&-\frac{r}{n(n-1)}\left[g(X,R(Y,U)W)\xi-g(\xi,R(Y,U)W)X\right]\\
&+\frac{1}{n-1}g(Y,W)\left[R(\xi,X)QU-\frac{r}{n(n-1)}\left(g(X,QU)\xi-g(\xi,QU)X\right)\right]\\
&-\frac{1}{n-1}g(U,W)\left[R(\xi,X)QY-\frac{r}{n(n-1)}\left(g(X,QY)\xi-g(\xi,QY)X\right)\right]=0.
\end{aligned} \tag{40}$$

Again, taking $Y=\xi$ in (40), we get

$$\begin{aligned}
&2R(\xi,X)R(\xi,U)W+\frac{1}{n-1}\left[g(\xi,R(\xi,U)W)QX-g(X,R(\xi,U)W)Q\xi\right]\\
&-\frac{r}{n(n-1)}g(U,W)\left[R(\xi,X)\xi-\frac{1}{n-1}QX-\frac{1}{n-1}\eta(X)Q\xi\right]\\
&+\frac{r}{n(n-1)}\eta(W)\left[R(\xi,X)U+\frac{1}{n-1}\eta(U)QX-\frac{1}{n-1}g(X,U)Q\xi\right]\\
&-\frac{r}{n(n-1)}\left[g(X,R(\xi,U)W)\xi-g(\xi,R(\xi,U)W)X\right]\\
&+\frac{1}{n-1}\eta(W)\left[R(\xi,X)QU-\frac{r}{n(n-1)}S(X,U)\xi+\frac{r}{n(n-1)}(n-1)\eta(U)\eta(X)\right]\\
&-\frac{1}{n-1}g(U,W)\left[(n-1)R(\xi,X)\xi-\frac{r}{n(n-1)}(n-1)\eta(X)\xi-\frac{r}{n(n-1)}(n-1)X\right]=0.
\end{aligned} \tag{41}$$

Taking the inner product of (41) with ξ and using (7), (10), we get

$$eta(W)g(X,U)-\frac{r}{n(n-1)}\eta(W)g(X,U)-\frac{1}{n-1}\eta(W)S(X,U)+\frac{r}{n(n-1)^2}S(X,U)=0. \tag{42}$$

Again, taking $W=\xi$ and using (4) in (42), we get

$$-g(X,U)-\frac{r}{n(n-1)}g(X,U)+\frac{1}{n-1}S(X,U)+\frac{r}{n(n-1)^2}S(X,U)=0. \tag{43}$$

Thus, from (43), we have

$$S(X,U)=(n-1)g(X,U) \tag{44}$$

So, M is an Einstein manifold.

Conversely, if M is an Einstein manifold, then it is easy to show that $W_2.Z+Z.W_2=0$. Our theorem is thus proved.

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