RT&A, No 2 (68)
Volume 17, June 2022

On the Use of Entropy as a Measure of Dependence of Two Events. Part 2

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Abstract

The joint experiment $\mathfrak{J}_{(A,B)}$ of two binary trials $A \cup A^c$ and $B \cup B^c$ in a probability space can be produced not only by the ordered pair (A, B) but by a set consisting, in general, of 24 ordered pairs of events (named Yule's pairs). The probabilities $\xi_1, \xi_2, \xi_3, \xi_4$ of the four results of $\mathfrak{J}_{(A,B)}$ are linear functions in three variables $\alpha = \Pr(A)$, $\beta = \Pr(B)$, $\theta = \Pr(A \cap B)$, and constitute a probability distribution. The symmetric group S_4 of degree four has an exact representation in the affine group $\mathrm{Aff}(3, \mathbb{R})$, which is constructed by using the types of the form $[\alpha, \beta, \theta]$ of those 24 Yule's pairs. The corresponding action of S_4 permutes the components of the probability distribution $(\xi_1, \xi_2, \xi_3, \xi_4)$, and, in particular, its entropy function is S_4-invariant. The function of degree of dependence of two events, defined in the first part of this paper via modifying the entropy function, turns out to be a relative invariant of the dihedral group of order 8.

Keywords: probability space; experiment in a sample space; probability distribution; entropy; degree of dependence; relative invariant.

1. Introduction

The initial idea of this work was to describe all symmetries of the sequence of Yule's pairs from (1) which produce one and the same experiment [3, 4.1,(1)]. If we consider the equivalence classes of the form $[(\alpha, \beta, \theta)]$ that contain the members of (1), then the naturally constructed in terms of coordinate functions α, β, θ affine automorphisms of the linear space $\mathbb{R}^3$ form a group which is isomorphic to the symmetric group S_4, see Section 2, Theorem 1. The components $\xi_1, \xi_2, \xi_3, \xi_4$ of the probability distribution [3, 4.1,(2)] are linear functions in α, β, θ. The group S_4 naturally acts via above isomorphism and permutes ξ_i's. As a consequence we obtain Theorem 2 which asserts that the entropy function $E_{\alpha,\beta}(\theta) = E(\alpha, \beta, \theta)$ of the probability distribution $(\xi_1, \xi_2, \xi_3, \xi_4)$ (see [3, 5.1]) is an absolute S_4-invariant.

In Section 3, Theorem 3, we show that the degree of dependence function $e_{\alpha,\beta}(\theta)$, defined in [3, 5.2] via "normalization" of the entropy function $E_{\alpha,\beta}(\theta)$, is a relative invariant of the dihedral group D_8, see [2, Ch.1,1.]. The proof uses the embedding of D_8 as one of the three Sylow 2-subgroups of S_4.

We use definitions and notation from [3, 2].

2. Methods

In this paper we are using fundamentals of:

- Affine geometry and Real algebraic geometry
- Invariant Theory.

3. The Group of Symmetry of an Experiment

3.1. Yule's Pairs and Experiments

Let $A, B \in \mathcal{A}$. We define $A \Diamond B = (A \triangle B)^c$, where $A \triangle B = (A^c \cap B) \cup (A \cap B^c)$ is the symmetric difference of A and B.

Any ordered pair $(A, B) \in \mathcal{A}^2$ produces the experiment $\mathfrak{J} = \mathfrak{J}_{(A,B)}$ from [3, 4.1,(1)], which is naturally identified with the partition $\{A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c\}$ of Ω (cf. [4, I,§5]). The proof of the next Lemma is straightforward.

Lemma 1. Yule's pairs from the sequence with members

$$
\begin{gathered}
(A,B) \text{ of type } (\alpha,\beta,\theta),\\
(A,B^c) \text{ of type } (\alpha,1-\beta,\alpha-\theta),\\
(A^c,B) \text{ of type } (1-\alpha,\beta,\beta-\theta),\\
(A^c,B^c) \text{ of type } (1-\alpha,1-\beta,1-\alpha-\beta+\theta),\\
(B,A) \text{ of type } (\beta,\alpha,\theta),\\
(B,A^c) \text{ of type } (\beta,1-\alpha,\beta-\theta),\\
(B^c,A) \text{ of type } (1-\beta,\alpha,\alpha-\theta),\\
(B^c,A^c) \text{ of type } (1-\beta,1-\alpha,1-\alpha-\beta+\theta),\\
(A,A\Diamond B) \text{ of type } (\alpha,1-\alpha-\beta+2\theta,\theta),\\
(A\Diamond B,A) \text{ of type } (1-\alpha-\beta+2\theta,\alpha,\theta),\\
(B,A\Diamond B) \text{ of type } (\beta,1-\alpha-\beta+2\theta,\theta),\\
(A\Diamond B,B) \text{ of type } (1-\alpha-\beta+2\theta,\beta,\theta),\\
(A^c,A\Diamond B) \text{ of type } (1-\alpha,1-\alpha-\beta+2\theta,1-\alpha-\beta+\theta),\\
(A\Diamond B,A^c) \text{ of type } (1-\alpha-\beta+2\theta,1-\alpha,1-\alpha-\beta+\theta),\\
(B^c,A\Diamond B) \text{ of type } (1-\beta,1-\alpha-\beta+2\theta,1-\alpha-\beta+\theta),\\
(A\Diamond B,B^c) \text{ of type } (1-\alpha-\beta+2\theta,1-\beta,1-\alpha-\beta+\theta),\\
(A,A\triangle B) \text{ of type } (\alpha,\alpha+\beta-2\theta,\alpha-\theta),\\
(A\triangle B,A) \text{ of type } (\alpha+\beta-2\theta,\alpha,\alpha-\theta),\\
(B,A\triangle B) \text{ of type } (\beta,\alpha+\beta-2\theta,\beta-\theta),\\
(A\triangle B,B) \text{ of type } (\alpha+\beta-2\theta,\beta,\beta-\theta),\\
(A^c,A\triangle B) \text{ of type } (1-\alpha,\alpha+\beta-2\theta,\beta-\theta),\\
(A\triangle B,A^c) \text{ of type } (\alpha+\beta-2\theta,1-\alpha,\beta-\theta),\\
(B^c,A\triangle B) \text{ of type } (1-\beta,\alpha+\beta-2\theta,\alpha-\theta),\\
(A\triangle B,B^c) \text{ of type } (\alpha+\beta-2\theta,1-\beta,\alpha-\theta),
\end{gathered}
\tag{1}
$$

are exactly the pairs that produce the experiment $\mathfrak{J}_{(A,B)}$.

Remark 1. (i) According to [1, 2.1, 2.7.1, 2.8.4], the set of points (α, β, θ) in $\mathbb{R}^3$ where the types from Lemma 1 are pair-wise different is semi-algebraic, open, and three-dimensional. Its trace U_3 on the interior $\mathring{T}_3$ of the classification tetrahedron T_3 from [3, 4.1] is not empty because otherwise $\mathring{T}_3$ would be subset of a finite union of planes. Theorem 2.2.1 from [1, 2.1] guaranties that the open two dimensional projection U_2 of U_3 onto $\alpha\beta$-plane is semi-algebraic. Note that "openness" is with respect to the standard topology in $\mathbb{R}^3$.

(ii) Under some "plentifulness" condition on Boolean algebra $\mathcal{A}$ (for example, if it is non-atomic), there exist plenty of Yule's pairs (A, B) of type $(\alpha, \beta) \in U_2$. In this case (we call it "general") the sequence from Lemma 1 consists of 24 Yule's pairs.

3.2. The Group of Symmetry

Let $\mathcal{E}$ be the set of all experiments in the probability space $(\Omega, \mathcal{A}, \Pr)$, that is, the set of all finite partitions of Ω with members from $\mathcal{A}$. The rule $(A, B) \mapsto \mathfrak{J}_{(A,B)}$ defines a map $\mathfrak{J}\colon \mathcal{A}^2 \to \mathcal{E}$ and Lemma 1 implies that the inverse image $\mathfrak{J}^{-1}(\mathfrak{J}_{(A,B)})$ coincides with the associated set of the sequence (1). Let us denote by $\mathcal{I}_{(A,B)}$ the set of equivalence classes in $\mathcal{A}^2$ of the form $[(\alpha, \beta, \theta)]$, which contain the members of $\mathfrak{J}^{-1}(\mathfrak{J}_{(A,B)})$. If $\alpha = \Pr(A)$, $\beta = \Pr(B)$, $\theta = \Pr(A \cap B)$, then (A, B) is a Yule's pair of type (α, β, θ), (A, B^c) is a Yule's pair of type $(\alpha, 1 - \beta, \alpha - \theta)$, (A^c, B) is a Yule's pair of type $(1 - \alpha, \beta, \beta - \theta)$, etc. Considering α, β, θ as coordinate functions in $\mathbb{R}^3$, the members of $\mathcal{I}_{(A,B)}$ produce the set $\mathfrak{S}_4$ consisting of 24 affine automorphisms of $\mathbb{R}^3$ from the following list:

$$
\begin{aligned}
\varphi_{(1)}(\alpha, \beta, \theta) &= (\alpha, \beta, \theta),\\
\varphi_{(12)(34)}(\alpha, \beta, \theta) &= (\alpha, 1 - \beta, \alpha - \theta),\\
\varphi_{(13)(24)}(\alpha, \beta, \theta) &= (1 - \alpha, \beta, \beta - \theta),\\
\varphi_{(14)(23)}(\alpha, \beta, \theta) &= (1 - \alpha, 1 - \beta, 1 - \alpha - \beta + \theta),\\
\varphi_{(23)}(\alpha, \beta, \theta) &= (\beta, \alpha, \theta),\\
\varphi_{(1342)}(\alpha, \beta, \theta) &= (\beta, 1 - \alpha, \beta - \theta),\\
\varphi_{(1243)}(\alpha, \beta, \theta) &= (1 - \beta, \alpha, \alpha - \theta),\\
\varphi_{(14)}(\alpha, \beta, \theta) &= (1 - \beta, 1 - \alpha, 1 - \alpha - \beta + \theta),\\
\varphi_{(34)}(\alpha, \beta, \theta) &= (\alpha, 1 - \alpha - \beta + 2\theta, \theta),\\
\varphi_{(243)}(\alpha, \beta, \theta) &= (1 - \alpha - \beta + 2\theta, \alpha, \theta),\\
\varphi_{(234)}(\alpha, \beta, \theta) &= (\beta, 1 - \alpha - \beta + 2\theta, \theta),\\
\varphi_{(24)}(\alpha, \beta, \theta) &= (1 - \alpha - \beta + 2\theta, \beta, \theta),\\
\varphi_{(142)}(\alpha, \beta, \theta) &= (1 - \alpha, 1 - \alpha - \beta + 2\theta, 1 - \alpha - \beta + \theta),\\
\varphi_{(1423)}(\alpha, \beta, \theta) &= (1 - \alpha - \beta + 2\theta, 1 - \alpha, 1 - \alpha - \beta + \theta),\\
\varphi_{(143)}(\alpha, \beta, \theta) &= (1 - \beta, 1 - \alpha - \beta + 2\theta, 1 - \alpha - \beta + \theta),\\
\varphi_{(1432)}(\alpha, \beta, \theta) &= (1 - \alpha - \beta + 2\theta, 1 - \beta, 1 - \alpha - \beta + \theta),\\
\varphi_{(12)}(\alpha, \beta, \theta) &= (\alpha, \alpha + \beta - 2\theta, \alpha - \theta),\\
\varphi_{(123)}(\alpha, \beta, \theta) &= (\alpha + \beta - 2\theta, \alpha, \alpha - \theta),\\
\varphi_{(132)}(\alpha, \beta, \theta) &= (\beta, \alpha + \beta - 2\theta, \beta - \theta),\\
\varphi_{(13)}(\alpha, \beta, \theta) &= (\alpha + \beta - 2\theta, \beta, \beta - \theta),\\
\varphi_{(1324)}(\alpha, \beta, \theta) &= (1 - \alpha, \alpha + \beta - 2\theta, \beta - \theta),\\
\varphi_{(134)}(\alpha, \beta, \theta) &= (\alpha + \beta - 2\theta, 1 - \alpha, \beta - \theta),\\
\varphi_{(124)}(\alpha, \beta, \theta) &= (1 - \beta, \alpha + \beta - 2\theta, \alpha - \theta),\\
\varphi_{(1234)}(\alpha, \beta, \theta) &= (\alpha + \beta - 2\theta, 1 - \beta, \alpha - \theta).
\end{aligned}
$$

The above affine automorphisms of $\mathbb{R}^3$ are indexed by the permutations σ from the symmetric group S_4 because of the theorem below.

The operator of symmetry

$$\sigma\colon H \to H, (\xi_1, \xi_2, \xi_3, \xi_4) \mapsto (\xi_{\sigma^{-1}(1)}, \xi_{\sigma^{-1}(2)}, \xi_{\sigma^{-1}(3)}, \xi_{\sigma^{-1}(4)}),$$

permutes the components of the probability distribution [3, 4.1,(2)] produced by the experiment $\mathfrak{J}_{(A,B)}$ and we have

Theorem 1. (i) One has $\iota \circ \varphi_{\sigma^{-1}} = \sigma \circ \iota$.

(ii) The map

$$S_4 \to \mathrm{Aff}(3, \mathbb{R}), \sigma \mapsto \varphi_{\sigma^{-1}}, \tag{2}$$

is a group anti-monomorphism with image $\mathfrak{S}_4$.

(iii) The group $\mathfrak{S}_4$ is the affine symmetry group of the classification tetrahedron T_3.

Proof. (i) It is enough to check the equality $\varphi_{\sigma^{-1}} = \iota^{-1} \circ \sigma \circ \iota$ for all $\sigma \in S_4$. For example, let $\sigma = (1243)$, so $\sigma^{-1} = (1342)$. We have

$$(\sigma \circ \iota)(\alpha, \beta, \theta) = (\xi_{\sigma^{-1}(1)}, \xi_{\sigma^{-1}(2)}, \xi_{\sigma^{-1}(3)}, \xi_{\sigma^{-1}(4)}) = (\xi_3, \xi_1, \xi_4, \xi_2),$$

$$(\iota^{-1} \circ \sigma \circ \iota)(\alpha, \beta, \theta) = \iota^{-1}(\xi_3, \xi_1, \xi_4, \xi_2) = (\beta, 1 - \alpha, \beta - \theta) =$$

$$\varphi_{(1342)}(\alpha, \beta, \theta) = \varphi_{\sigma^{-1}}(\alpha, \beta, \theta).$$

(ii) The map (2) is injective; moreover, it is a group anti-homomorphism because $\varphi_{(1)} = \iota^{-1} \circ (1) \circ \iota = (1)$ and $\varphi_{\tau^{-1}\sigma^{-1}} = \varphi_{(\sigma\tau)^{-1}} = \iota^{-1} \circ (\sigma\tau) \circ \iota = \iota^{-1} \circ \sigma \circ \tau \circ \iota = \iota^{-1} \circ \sigma \circ \iota \circ \iota^{-1} \circ \tau \circ \iota = \varphi_{\sigma^{-1}} \circ \varphi_{\tau^{-1}}$.

(iii) In accord with part (i), for any $\sigma \in S_4$ we have $\iota(\varphi_\sigma(T_3)) = \sigma^{-1}(\iota(T_3)) = \sigma^{-1}(\Delta_3) = \Delta_3$, hence $\varphi_\sigma(T_3) = \iota^{-1}(\Delta_3) = T_3$. On the other hand, S_4 is the symmetry group of the regular tetrahedron (see, for example, [5, 8.4]). Since both tetrahedrons are isomorphic as affine spans, the proof is done.

■

For any $\sigma \in S_4$ we write down the affine automorphism φ_σ in terms of coordinates in $\mathbb{R}^3$: $\varphi_\sigma(\alpha, \beta, \theta) = (\alpha^{(\sigma)}, \beta^{(\sigma)}, \theta^{(\sigma)})$ and obtain that φ_σ maps the components of the partition $T_3 = \cup_{(\alpha,\beta)\in[0,1]^2}\{\alpha\} \times \{\beta\} \times I(\alpha, \beta)$ onto the corresponding components of the partition $T_3 = \cup_{(\alpha,\beta)\in[0,1]^2}\{\alpha^{(\sigma)}\} \times \{\beta^{(\sigma)}\} \times I(\alpha^{(\sigma)}, \beta^{(\sigma)})$. Moreover, φ_σ maps the components of the partition $\mathring{T}_3 = \cup_{(\alpha,\beta)\in(0,1)^2}\{\alpha\} \times \{\beta\} \times \mathring{I}(\alpha, \beta)$ onto the corresponding components of the partition $\mathring{T}_3 = \cup_{(\alpha,\beta)\in(0,1)^2}\{\alpha^{(\sigma)}\} \times \{\beta^{(\sigma)}\} \times \mathring{I}(\alpha^{(\sigma)}, \beta^{(\sigma)})$.

Let us set $\hat{T}_3 = \cup_{(\alpha,\beta)\in(0,1)^2}\{\alpha\} \times \{\beta\} \times I(\alpha, \beta)$. In particular, we obtain the following

Lemma 2. Let $(\alpha, \beta) \in (0, 1)^2$, $\sigma \in S_4$. (i) The automorphism φ_σ maps the set

$$\{(\alpha, \beta, \ell(\alpha, \beta)), (\alpha, \beta, r(\alpha, \beta))\}$$

of endpoints of the segments $\{\alpha\} \times \{\beta\} \times I(\alpha, \beta)$ onto the set

$$\{(\alpha^{(\sigma)}, \beta^{(\sigma)}, \ell(\alpha^{(\sigma)}, \beta^{(\sigma)})), (\alpha^{(\sigma)}, \beta^{(\sigma)}, r(\alpha^{(\sigma)}, \beta^{(\sigma)}))\}$$

of endpoints of their images $\{\alpha^{(\sigma)}\} \times \{\beta^{(\sigma)}\} \times I(\alpha^{(\sigma)}, \beta^{(\sigma)})$.

(ii) One has $\varphi_\sigma(\hat{T}_3) = \hat{T}_3$.

In accord with Theorem 1, (ii), the group S_4 acts on the real functions $F\colon \mathbb{R}^3 \to \mathbb{R}$ via the rule $\sigma \cdot F = F \circ \varphi_{\sigma^{-1}}$. Let

$$G\colon \mathring{\Delta}_3 \to \mathbb{R}, G(\xi_1, \xi_2, \xi_3, \xi_4) = -\xi_1 \ln \xi_1 - \xi_2 \ln \xi_2 - \xi_3 \ln \xi_3 - \xi_4 \ln \xi_4,$$

$$E\colon \mathring{T}_3 \to \mathbb{R}, E = G \circ \iota.$$

The function G is continuously differentiable on the interior $\mathring{\Delta}_3$ and can be extended under the name $\hat{G}$ as continuous on $\hat{\Delta}_3 = \iota(\hat{T}_3)$. The function E is continuously differentiable on the interior $\mathring{T}_3$ and can be extended under the name $\hat{E}$ as continuous on $\hat{T}_3$ (cf. [3, 5.1,Theorem 2, (iii)]). Moreover, $\hat{G} = \hat{G} \circ \sigma$ (that is, $\hat{G}$ is an absolute S_4-invariant) and $\hat{E} = \hat{G} \circ \iota$. Lemma 2, (ii), allows us to extend the action of the symmetric group S_4 on $\hat{T}_3$ via the rule $\sigma \cdot \hat{E} = \hat{E} \circ \varphi_{\sigma^{-1}}$.

Throughout the end of the paper, with an abuse of the language, we designate $\hat{G}$ via G and $\hat{E}$ via E.

Theorem 2. The function $E\colon \hat{T}_3 \to \mathbb{R}$ is an (absolute) invariant of the symmetric group S_4.

Proof. Theorem 1, (i), yields $E = G \circ \iota = G \circ \sigma \circ \iota = G \circ \iota \circ \varphi_{\sigma^{-1}} = E \circ \varphi_{\sigma^{-1}} = \sigma \cdot E$ for all $\sigma \in S_4$.

■

4. Degree of Dependance: Further Properties

4.1. The Groups of Symmetry

Let us suppose $(\alpha, \beta) \in (0,1)^2$ and set

$$e(\alpha,\beta,\theta) = \begin{cases} -\frac{E(\alpha,\beta,\alpha\beta)-E(\alpha,\beta,\theta)}{E(\alpha,\beta,\alpha\beta)-E(\alpha,\beta,\ell(\alpha,\beta))} & \text{if } \ell(\alpha,\beta) \leq \theta \leq \alpha\beta \\ \frac{E(\alpha,\beta,\alpha\beta)-E(\alpha,\beta,\theta)}{E(\alpha,\beta,\alpha\beta)-E(\alpha,\beta,r(\alpha,\beta))} & \text{if } \alpha\beta \leq \theta \leq r(\alpha,\beta), \end{cases} \tag{3}$$

where $I(\alpha,\beta) = [\ell(\alpha,\beta), r(\alpha,\beta)]$. Note that in [3, 5.2] the function $e_{\alpha,\beta}(\theta) = e(\alpha,\beta,\theta)$ is said to be the degree of dependence of events A and B with $\alpha = \Pr(A)$, $\beta = \Pr(B)$, and $\theta = \Pr(A \cap B)$.

Let us consider the dihedral subgroup $D_8 = \langle (1342), (14) \rangle$ of S_4 and let $\chi\colon D_8 \to \mathbb{R}^*$ be its Abelian character with kernel $K = \langle (14), (23) \rangle$ and image $\{1, -1\}$.

Theorem 3. The function e from (3) is a relative invariant of weight χ of the dihedral group D_8.

Proof. Given $\sigma \in S_4$ we have

$$(\sigma^{-1} \cdot e)(\alpha,\beta,\theta) = e(\varphi_\sigma(\alpha,\beta,\theta)) = e(\alpha^{(\sigma)},\beta^{(\sigma)},\theta^{(\sigma)}) =$$

$$\begin{cases} -\frac{E(\alpha^{(\sigma)},\beta^{(\sigma)},\alpha^{(\sigma)}\beta^{(\sigma)})-E(\alpha^{(\sigma)},\beta^{(\sigma)},\theta^{(\sigma)})}{E(\alpha^{(\sigma)},\beta^{(\sigma)},\alpha^{(\sigma)}\beta^{(\sigma)})-E(\alpha^{(\sigma)},\beta^{(\sigma)},\ell(\alpha^{(\sigma)},\beta^{(\sigma)}))} & \text{if } \ell(\alpha^{(\sigma)},\beta^{(\sigma)}) \leq \theta^{(\sigma)} \leq \alpha^{(\sigma)}\beta^{(\sigma)} \\ \frac{E(\alpha^{(\sigma)},\beta^{(\sigma)},\alpha^{(\sigma)}\beta^{(\sigma)})-E(\alpha^{(\sigma)},\beta^{(\sigma)},\theta^{(\sigma)})}{E(\alpha^{(\sigma)},\beta^{(\sigma)},\alpha^{(\sigma)}\beta^{(\sigma)})-E(\alpha^{(\sigma)},\beta^{(\sigma)},\ell(\alpha^{(\sigma)},\beta^{(\sigma)}))} & \text{if } \alpha^{(\sigma)}\beta^{(\sigma)} \leq \theta^{(\sigma)} \leq r(\alpha^{(\sigma)},\beta^{(\sigma)}), \end{cases}$$

where $I(\alpha^{(\sigma)},\beta^{(\sigma)}) = [\ell(\alpha^{(\sigma)},\beta^{(\sigma)}), r(\alpha^{(\sigma)},\beta^{(\sigma)})]$. For any $\sigma \in D_8$ we have $\varphi_\sigma(\alpha,\beta,\alpha\beta) = (\alpha^{(\sigma)},\beta^{(\sigma)},\alpha^{(\sigma)}\beta^{(\sigma)})$. On the other hand, given $\sigma \in K$, the inequalities $\ell(\alpha,\beta) \leq \theta \leq \alpha\beta$ are equivalent to the inequalities $\ell(\alpha^{(\sigma)},\beta^{(\sigma)}) \leq \theta^{(\sigma)} \leq \alpha^{(\sigma)}\beta^{(\sigma)}$ and the inequalities $\alpha\beta \leq \theta \leq r(\alpha,\beta)$ are equivalent to the inequalities $\alpha^{(\sigma)}\beta^{(\sigma)} \leq \theta^{(\sigma)} \leq r(\alpha^{(\sigma)},\beta^{(\sigma)})$. Given $\sigma \in D_8 \backslash K$, the inequalities $\ell(\alpha,\beta) \leq \theta \leq \alpha\beta$ are equivalent to the inequalities $\alpha^{(\sigma)}\beta^{(\sigma)} \leq \theta^{(\sigma)} \leq r(\alpha^{(\sigma)},\beta^{(\sigma)})$ and the inequalities $\alpha\beta \leq \theta \leq r(\alpha,\beta)$ are equivalent to the inequalities $\ell(\alpha^{(\sigma)},\beta^{(\sigma)}) \leq \theta^{(\sigma)} \leq \alpha^{(\sigma)}\beta^{(\sigma)}$. The corresponding equalities hold simultaneously because of Lemma 2, (i). Now, Theorem 2 yields that $\sigma \cdot e = \chi(\sigma)e$ for all permutations $\sigma \in D_8$.

■

We obtain immediately the following

Corollary 1. For any $(\alpha,\beta) \in (0,1)^2$ and for any $\theta \in I(\alpha,\beta)$ one has

$$e_{\alpha,\beta}(\theta) = e_{\beta,\alpha}(\theta) = e_{1-\alpha,1-\beta}(1-\alpha-\beta+\theta) = e_{1-\beta,1-\alpha}(1-\alpha-\beta+\theta),$$

$$-e_{\alpha,\beta}(\theta) = e_{\alpha,1-\beta}(\alpha-\theta) = e_{1-\alpha,\beta}(\beta-\theta) = e_{\beta,1-\alpha}(\beta-\theta) = e_{1-\beta,\alpha}(\alpha-\theta).$$

Acknowledgements

I would like to thank the referees for their very useful remarks.
This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Declaration of Conflicting Interests

The Author declares that there is no conflict of interest.

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