Analysis of $MAP/PH/1$ Queueing model with Multiple Vacations, Optional Service, Close-down, Setup, Breakdown, Phase Type Repair and Impatient Customers

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Abstract

The purpose of this paper is to analyse a single server queueing model with multiple vacations, optional service, close-down, setup, balking, breakdown and repair under the assumption that the customers arrive according to a Markovian Arrival Process (MAP). The service and repair times follow the phase-type distributions. At the completion of service, in case there are no customers in the system, the server closes down the system and goes for vacation. After completion of the vacation, the server has to start the setup process if a minimum of one customer is present in the system or else the server goes for another vacation. The server provides optional service to the customers those who are in need of additional services. By employing the matrix analytic method, the stationary probability vector has been evaluated. The stability condition, busy period analysis, distribution function for waiting time and some of the system performance measures concerning this model are derived. The outcome arising out of numerical values and graphical representations are also presented for this model.

Keywords: Multiple vacations, Optional service, Close-down, Setup, Breakdown, Phase type repair, Balking.

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1. Introduction

During almost all the day-to-day activities of our life, we come across various queues in many places like shopping malls, traffic signals, billing section, railway counters, communication networks, telecommunication systems, etc in which the queues are either visible or invisible. Mostly people do not prefer to stand in a queue for a long time. Considering the attendant consequences of spending enormous time in queues, it becomes imperative to employ appropriate queueing models to offer remedial measures for these congestion situations.

In the analysis of a queue, Markovian Arrival Process (MAP) proves to be a very useful appliance in the point process which includes the Markov Modulated Poisson Process and phase type renewal process. Neuts [24] has introduced Versatile Markovian Point Processes (VMPP) through which the arrival process is formulated. Later, MAP and BMAP were introduced by Lucantoni et al. [17] . A method of analysis was provided by Chakravarthy [4] who considered the Markovian arrival process with various types of arrivals in the representation of parameter matrices (D_0, D_1) with dimension n where D_0 governs for no arrivals and D_1 governs for arrivals. Let D be the generator matrix defined by $D = D_0 + D_1$.

A queueing system with a single server and K waiting rooms with finite capacity was examined by Niu et al. [28]. They considered the arrival of a batch of customers with the server taking multiple vacations during the idle time. Jau-Chuan Ke et al. [10] have investigated a multi server retrial queue with single and multiple vacations. In this system, they have analysed the cost function to determine the optimum value of the server at minimum cost.

Artalejo et al. [1] have carried out a busy period analysis by considering the distribution function of the waiting time with multi-servers and finite retrial group. They have dealt with a system possessing finite buffer for retrial group and provided certain numerical illustrations in their model.

Chakravarthy [7] explored the working of a single server queueing model with multiple vacations and optional secondary services. Choudhury and Paul [8] have recognized a single server queueing system with Bernoulli schedule and multiple vacations policy. They have evaluated the distribution function of the waiting time with busy period analysis and gone into the performance measures of effectiveness.

Kulkarni and Choi [14] have investigated a single server retrial queue with breakdown and repair. Wang and Zhang [27] have analysed a queueing system with balking, reneging and motivating. They also have performed a busy period analysis.

Chakravarthy and Agarwal [5] have analysed a machine repair problem with an unreliable server and phase type repairs and services. Maragathasundari et al. [18] have examined a single server queueing model with compulsory short vacation and considered the reneging during long vacation as optional.

Wang and Zhang [26] studied a single server Discrete-time Retrial G-queue with breakdown and repairs due to negative arrivals. They considered the negative arrival of a customer which distracts the positive arrival of a customer. A single server queue with Bernoulli vacation, setup time, reneging, balking, Bernoulli feedback, breakdown and repair have been examined by Ayyappan and Gowthami [2]. With the help of matrix-geometric method, they have evaluated the probability vector and the rate matrix. They have derived the probability of the server to be idle, busy and the repair time.

MAP arrivals, impatient customers and a perishable inventory system with N-policy have been discussed by Suganya et al. [25]. They have analysed the cost function and presented numerical illustrations of their queueing model. Kumar and Sharma [13] studied a finite capacity single server queueing model with retention of reneging and balking. In addition, they derived the stability condition and the probability of server being idle, busy and on vacation.

A multi-server queueing system with Bernoulli feedback, impatient customers, single and multiple vacation policies have been examined by Kadi et al. [20]. Further, cost analysis and performance measure are also presented for their model. Ayyappan and Thilagavathy [3] have examined a single server queue with vacation, immediate feedback, breakdown, delayed repair, starting failures, stand-by server, and impatient customers.

An overview of the remaining part of this article is as follows: In section 2, we briefly explain the implementation of our model. The description of the model is provided in section 3. In section 4, we present a generation of the matrix and the notations of our model. In section 5, the stability condition, the stationary probability vector and the rate matrix R are evaluated. We have performed the analysis of busy period in section 6. In section 7, the features of some performance measures are examined. Particular case of our model have been dealt with in section 8. In section 9, we have evaluated the distribution of the waiting time in our model. In section 10, the numerical illustrations and graphical representation are provided. The conclusion part of this model has been presented in section 11.

2. Implementation of our model

As a measure of illustration of our model, let us consider a nationalized bank which has more than one serving counters. We choose any one of these counters for our model. In the counter, the server deals with many transaction processes listed below:

1. Deposit or withdrawal.
2. Foreign currency transaction.
3. Loan process, etc.

A customer may demand money transaction process in any of the ways as mentioned above. When the customer arrives, if he finds the availability of the server, then he receives the service immediately; otherwise, the customer has to wait in the line until he reaches the service point. After the customer gets service, he may exit the counter or he has an option to go for another service (optional service). After providing the service, the server goes for vacation (like receiving the telephone calls, arranging the money, checking the transactions, etc.). After the completion of the service, the server will put down the system and go for vacation (close-down). At the completion of vacation, if there is no customer in the system for receiving the service, then the server takes another vacation (multiple vacation). Once the vacation is completed, the server will do some settings in the system and refresh the system to give the service (setup). During the period of vacation, the incoming customer may balk in the particular counter due to the impatience (balking). In the time of busy period, the server can attain breakdown (like power problem, lack of network, hanging the system etc.). After carrying out the repair process, the server will be ready to provide the service to the waiting customers, those are in interruption of service in front of the queue. Our model has been formulated to hold in all these situations.

3. Model Description

A single server queueing system with infinite capacity has been dealt with in this model. The arrival of customers is according to Markovian arrival process with (D_0, D_1) as its parameter matrices of order m. The matrix D_0 is governed for the transition which deals for no arrival and the matrix D_1 deals for the arrival of customers.
The time duration of both normal and optional services follow PH-distributions with the notation (α_1, T_1) and (α_2, T_2) of order t_1 and t_2 where $T_1^0 + T_1 e = 0$ and $T_2^0 + T_2 e = 0$. The repair times of the server during both the normal and optional services are based on the PH-distributions with notation (β_1, S_1) and (β_2, S_2) of order s_1 and s_2, respectively where $S_1^0 + S_1 e = 0$ and $S_2^0 + S_2 e = 0$.
Upon the completion of the process of service, the customer may either leave the system with probability q, or he may need optional service by the server with probability p, with $p + q = 1$.
After the service is completed, the system will be close-down by the server only if the system becomes empty in which case the server moves on to vacation.
After the completion of vacation, if there is a customer waiting in the system for the service then the server starts the setup process. Otherwise, the server goes on to another vacation.
During the busy period (both normal and optional), the server may encounter failure of its service and it would start the repair process immediately. At that time, the customer who are getting the service from the server have to join the head of the queue.
After the completion of the repair process, the server begins the service to the customers those who are waiting in the queue.
During the vacation period, the arriving customers may balk the system with probability b or they may join the system with probability $(1 - b)$.
The close-down times, vacation times, setup times, breakdown times all follow exponential distribution with the parameters $\varphi, \eta, \psi, \tau$ respectively.

4. Generation of the matrix under QBD process

Let us list down some notations concerning this model and describe the construction of the generator matrix of the Quasi-Birth and Death process in this section:

Notations:

- $\otimes$ - Kronecker product of two matrices with various orders.
- $\oplus$ - Kronecker sum of two matrices with various orders.
- I_m - An Identity matrix of m-dimensional.
- $e = e_{(t_1+t_2+s_1+s_2+3)m}$.
- $e_2(g)$ - $2m \times 1$ vector with $m+1$ to $2m$ elements are 1 and the remaining elements are zero.
- $e(a)$ - $(t_1+t_2+s_1+s_2+3)m \times 1$ vector with first $t_1 m$ elements are 1 and the remaining elements are zero.
- $e(b)$ - $(t_1+t_2+s_1+s_2+3)m \times 1$ vector with $t_1 m+1$ to $t_1 m + t_2 m$ elements are 1 and the remaining elements are zero.
- $e(c)$ - $(t_1+t_2+s_1+s_2+3)m \times 1$ vector with $t_1 m + t_2 m + 1$ to $t_1 m + t_2 m + s_1 m$ elements are 1 and the remaining elements are zero.
- $e(d)$ - $(t_1+t_2+s_1+s_2+3)m \times 1$ vector with $t_1 m + t_2 m + s_1 m + 1$ to $t_1 m + t_2 m + s_1 m + s_2 m$ elements are 1 and the remaining elements are zero.
- $e(g)$ - $(t_1+t_2+s_1+s_2+3)m \times 1$ vector with $t_1 m + t_2 m + s_1 m + s_2 m + m + 1$ to $t_1 m + t_2 m + s_1 m + s_2 m + 2m$ elements are 1 and the remaining elements are zero.
- Let λ be the fundamental arrival rate defined by $\lambda = \pi D_1 e_m$ where π is the stationary probability vector.
- The normal and optional service rates of the server are indicated as $\mu_1 = [\alpha_1(-T_1^{-1})e_{t_1}]^{-1}$ and $\mu_2 = [\alpha_2(-T_2^{-1})e_{t_2}]^{-1}$.
- The repair rates (breakdown occurred during normal and optional services) for the server are indicated as
$\sigma_1 = [\beta_1(-S_1^{-1})e_{s_1}]^{-1}$ and $\sigma_2 = [\beta_2(-S_2^{-1})e_{s_2}]^{-1}$ respectively.
- $N(t)$ is the number of customers in the system at time t.
- $C(t)$ be the status of the server at time t, where

$$C(t) = \begin{cases} 0, & \text{if the server is busy with normal service} \\ 1, & \text{if the server is busy with optional service} \\ 2, & \text{if the server is under PH repair process (breakdown occurred during normal service)} \\ 3, & \text{if the server is under PH repair process (breakdown occurred during optional service)} \\ 4, & \text{if the server is on close-down process} \\ 5, & \text{if the server is on vacation} \\ 6, & \text{if the server is on setup process} \end{cases}$$

- $S(t)$ is the service phase.
- $K(t)$ is the repair phase.
- $A(t)$ is the Markovian arrival process phase.

Let $\{N(t), C(t), S(t), K(t), A(t), \quad t \geq 0\}$ be the Continuous Time Markov Chain ($CTMC$) with state level independent QBD structure for which the state space is provided by

$$\Omega = l(0) \cup l(i)$$

where

$l(0) = \{(0, j, a) : j = 4, 5 : \ 1 \leq a \leq m\}$

and for $\quad i \geq 1$,

$$
\begin{aligned}
l(i) = & \{(i,0,k_1,a):\ i\epsilon\mathbb{Z}^+,\ 1 \le k_1 \le t_1,\ 1 \le a \le m\} \cup \{(i,1,k_2,a):\ i\epsilon\mathbb{Z}^+,\ 1 \le k_2 \le t_2,\ 1 \le a \le m\} \\
& \cup\{(i,2,r_1,a):\ i\epsilon\mathbb{Z}^+,\ 1 \le r_1 \le s_1,\ 1 \le a \le m\} \cup \{(i,3,r_2,a):\ i\epsilon\mathbb{Z}^+,\ 1 \le r_2 \le s_2,\ 1 \le a \le m\} \\
& \cup\{(i,4,a):\ i\epsilon\mathbb{Z}^+,\ 1 \le a \le m\} \cup \{(i,5,a):\ i\epsilon\mathbb{Z}^+,\ 1 \le a \le m\} \cup \{(i,6,a):\ i\epsilon\mathbb{Z}^+,\ 1 \le a \le m\}
\end{aligned}
$$

4.1. The Infinitesimal Matrix Generation

The QBD process has infinitesimal generator matrix Q is given by

$$
Q = \begin{bmatrix}
B_{00} & B_{01} & 0 & 0 & 0 & 0 & \dots \\
B_{10} & A_1 & A_0 & 0 & 0 & 0 & \dots \\
0 & A_2 & A_1 & A_0 & 0 & 0 & \dots \\
0 & 0 & A_2 & A_1 & A_0 & 0 & \dots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots
\end{bmatrix}
$$

The entries of the Q matrix are defined by

$$
B_{00} = \begin{bmatrix} D_0 - \varphi I_m & \varphi I_m \\ 0 & D_0 + bD_1 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 & 0 & 0 & 0 & D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-b)D_1 & 0 \end{bmatrix},
$$

$$
B_{10} = \begin{bmatrix}
qT_1^0 \otimes I_m & 0 \\
T_2^0 \otimes I_m & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}, \quad
A_0 = \begin{bmatrix}
I_{t_1} \otimes D_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{t_2} \otimes D_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{s_1} \otimes D_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{s_2} \otimes D_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (1-b)D_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & D_1
\end{bmatrix},
$$

$$
A_1 = \begin{bmatrix}
T_1 \oplus D_0 - \tau_1 I_{mt_1} & \alpha_2 \otimes pT_1^0 \otimes I_m & e_2 \otimes \tau_1 \beta_1 \otimes I_m & 0 & 0 & 0 & 0 \\
0 & T_2 \oplus D_0 - \tau_2 I_{mt_2} & 0 & e_2 \otimes \tau_2 \beta_2 \otimes I_m & 0 & 0 & 0 \\
\alpha_1 \otimes S_1^0 \otimes I_m & 0 & S_1 \oplus D_0 & 0 & 0 & 0 & 0 \\
0 & \alpha_2 \otimes S_2^0 \otimes I_m & 0 & S_2 \oplus D_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D_0 - \varphi I_m & \varphi Im & 0 \\
0 & 0 & 0 & 0 & 0 & bD_1 + D_0 - \eta I_m & \eta I_m \\
\psi(\alpha_1 \otimes I_m) & 0 & 0 & 0 & 0 & 0 & D_0 - \psi I_m
\end{bmatrix},
$$

$$
A_2 = \begin{bmatrix}
qT_1^0 \otimes \alpha_1 \otimes I_m & 0 & 0 & 0 & 0 & 0 & 0 \\
T_2^0 \otimes \alpha_1 \otimes I_m & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

5. System Analysis

To ensure that the system is stable, we have to evaluate our model under certain conditions are described in the sequel:

5.1. Analysis of stability condition

Let A be the matrix defined by $A = A_0 + A_1 + A_2$ and ς be the stationary probability vector of A which satisfies the condition

$$\varsigma A = 0, \quad \varsigma e = 1.$$

where the vector ς represents the state given by $\varsigma = (\varsigma_0, \varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6)$.

The vector ς, partitioned as $\varsigma = (\varsigma_0, \varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6)$ is evaluated with the help of the equation:

$$\begin{aligned}
&\varsigma_0[(I_{t_1} \otimes D_1) + (T_1 + D_0 - \tau_1 I_{mt_1}) + (qT_1^0 \otimes \alpha_1 \otimes I_m)] + \varsigma_1[T_2^0 \otimes \alpha_2 \otimes I_m] \\
&\qquad\qquad + \varsigma_2[\alpha_1 \otimes S_1^0 \otimes I_m] + \varsigma_6[\psi\alpha_1 \otimes I_m] = 0, \\
&\varsigma_0[\alpha_2 \otimes pT_1^0 \otimes I_m] + \varsigma_1[(I_{t_2} \otimes D_1) + (T_2 + D_0 - \tau_2 I_{mt_2})] + \varsigma_3[\alpha_2 \otimes S_2^0 \otimes I_m] = 0, \\
&\varsigma_0[e_2 \otimes \tau_1\beta_1 \otimes I_m] + \varsigma_2[(I_{s_1} \otimes D_1) + (S_1 \otimes D_0)] = 0, \\
&\varsigma_1[e_2 \otimes \tau_2\beta_2 \otimes I_m] + \varsigma_3[(I_{s_2} \otimes D_1) + (S_2 \otimes D_0)] = 0, \\
&\varsigma_4[D - \varphi I_m] = 0, \\
&\varsigma_4[\varphi I_m] + \varsigma_5[D - \eta I_m] = 0, \\
&\varsigma_5[\eta I_m] + \varsigma_6[D - \psi I_m] = 0
\end{aligned}$$

subject to

$$\varsigma_0 e_{mt_1} + \varsigma_1 e_{mt_2} + \varsigma_2 e_{ms_1} + \varsigma_3 e_{ms_2} + \varsigma_4 e_m + \varsigma_5 e_m + \varsigma_6 e_m = 1.$$

The necessary and sufficient condition for the stability of the system is that the QBD process satisfies the condition $\varsigma A_0 e < \varsigma A_2 e$.
i.e.,

$$\begin{aligned}
&\varsigma_0[I_{t_1} \otimes D_1] + \varsigma_1[I_{t_2} \otimes D_1] + \varsigma_2[I_{s_1} \otimes D_1] + \varsigma_3[I_{s_2} \otimes D_1] + (\varsigma_4 + \varsigma_6)[D_1] + \varsigma_5[(1-b)D_1] \\
&\qquad\qquad < \varsigma_0[qT_1^0 \otimes \alpha_1 \otimes I_m] + \varsigma_1[T_2^0 \otimes \alpha_1 \otimes I_m].
\end{aligned}$$

5.2. Analysis of stationary probability vector

Let X be the stationary probability vector. It is subdivided as X=$(X_0, X_1, X_2, ...)$ which is the steady state probability vector of Q.

The dimensions of X_0 and X_i , $i \geq 1$ are $2m$ and $(t_1 + t_2 + s_1 + s_2 + 3)m$ respectively. The vector X of Q satisfies the condition

$$XQ = 0 \text{ and } Xe = 1.$$

However, once the stability condition is satisfied, we find the vector X as invariant probability with the help of the equation,

$$X_i = X_1 R^{i-1}, \quad i \geq 2.$$

where the matrix R is the minimal non-negative solution to

$$R^2 A_2 + RA_1 + A_0 = 0.$$

By means of the equations

$$X_0 B_{00} + X_1 B_{10} = 0,$$

$$X_0 B_{01} + X_1[A_1 + RA_2] = 0$$

we can find the vectors namely X_0 and X_1 subject to normalizing condition

$$X_0 e_{2m} + X_1[I - R]^{-1} e_{(t_1+t_2+s_1+s_2+3)m} = 1.$$

The "Logarithmic Reduction Algorithm" can be used to quickly calculate the rate matrix R as specified by Latouche and Ramaswami [16].

6. Busy Period Analysis

A busy period can be measured as the interval between the customers entering into an empty system and when the system size reduces to empty for the first time. As a result, this is the first passage time between the level i and level $i-1$, $i \geq 2$ under the consideration of QBD process. The first return time to level 0 with minimum one visit to a state in any other level is known as busy cycle. It is necessary to deal with $i = 0, 1$ independently for the boundary states. For each and every level i, where $i \geq 1$, we can observe that there are $m(t_1 + t_2 + s_1 + s_2 + 3)$ states.

Notations:

1. $G_{j,j'}(k, x)$ - The probability that the QBD moves by making k left transitions to the level $(i-1)$ and entering the state (i, j'), with the condition of beginning from the state (i, j) at time $t = 0$.
2. Let the joint transform matrix be
$$\tilde{G}_{j,j'}(z, s) = \sum_{k=1}^{\infty} z^k \int_0^{\infty} e^{-sx} dG_{j,j'}(k, x); \quad |z| \leq 1, \quad Re(s) \geq 0$$
3. The matrix $\tilde{G}(z, s) = (\tilde{G}_{j,j'}(z, s))$. (Neuts [21])
4. The matrix $G = (G_{j,j'}) = \tilde{G}(1, 0)$ concerns the first passage times except for the boundary states.
5. $G^{(1,0)}_{j,j'}(k, x)$ - The probability that the system moves from level 0 to level 1 at time $t = 0$.
6. $G^{(0,0)}_{j,j'}(k, x)$ - The probability that the system returns to level 0 at time $t = 0$.
7. S_{1j} - The average first passage time among the levels i and $i-1$, the process in the state (i, j) at time $t = 0$.
8. $\tilde{S}_1$ - The column vector with S_{1j} as its entries.
9. S_{2j} - The average number of customers who receive the service at the first passage time among the levels i and $i-1$, beginning in the state (i, j) at time $t = 0$.
10. $\tilde{S}_2$ - The column vector with S_{2j} as its entries.
11. $\tilde{S}^{(1,0)}_1$ - The average first return time from level 1 to 0.
12. $\tilde{S}^{(1,0)}_2$ - The average number of services completed during the first return time from level 1 to 0.
13. $\tilde{S}^{(0,0)}_1$ - The average first return time to level 0.
14. $\tilde{S}^{(0,0)}_2$ - The average number of services completed during the first return time to level 0.

We can compute the matrix $\tilde{G}(z,s)$ with the equation

$$\tilde{G}(z,s) = z(sI - A_1)^{-1}A_2 + (sI - A_1)^{-1}A_0\tilde{G}^2(z,s).$$

After evaluating the rate matrix R, the G matrix can be computed with the help of logarithmic reduction algorithm (Lautouche and Ramaswami, [16]) provided by

$$G = -(A_1 + RA_2)^{-1}A_2.$$

The equations

$$\begin{aligned}
\tilde{G}^{(1,0)}(z,s) &= z(sI - A_1)^{-1}B_{10} + (sI - A_1)^{-1}A_0\tilde{G}(z,s)\tilde{G}^{(1,0)}(z,s)\\
\tilde{G}^{(0,0)}(z,s) &= (sI - B_{00})^{-1}B_{01}\tilde{G}^{(1,0)}(z,s).
\end{aligned}$$

which are satisfied by $\tilde{G}^{(1,0)}(z,s)$ and $\tilde{G}^{(0,0)}(z,s)$ lead to the boundary states namely 1 and 0, respectively.

Since $G, \tilde{G}^{(1,0)}(1,0)$ and $\tilde{G}^{(0,0)}(1,0)$ are all stochastic matrices, we can calculate the moments as follows:

$$\begin{aligned}
\tilde{S}_1 &= -\frac{\partial\tilde{G}(z,s)}{\partial s}\Big|_{s=0,z=1} = -[A_0(G+I) + A_1]^{-1}e,\\
\tilde{S}_2 &= \frac{\partial\tilde{G}(z,s)}{\partial z}\Big|_{s=0,z=1} = -[A_0(G+I) + A_1]^{-1}]A_2e,\\
\tilde{S}_1^{(1,0)} &= -\frac{\partial\tilde{G}^{(1,0)}(z,s)}{\partial s}\Big|_{s=0,z=1} = -[A_1 + A_0G]^{-1}[e + A_0\tilde{S}_1],\\
\tilde{S}_2^{(1,0)} &= \frac{\partial\tilde{G}^{(1,0)}(z,s)}{\partial z}\Big|_{s=0,z=1} = -[A_1 + A_0G]^{-1}[B_{10}e + A_0\tilde{S}_2],\\
\tilde{S}_1^{(0,0)} &= -\frac{\partial\tilde{G}^{(0,0)}(z,s)}{\partial s}\Big|_{s=0,z=1} = -B_{00}^{-1}[e + B_{01}\tilde{S}_1^{(1,0)}],\\
\tilde{S}_2^{(0,0)} &= \frac{\partial\tilde{G}^{(0,0)}(z,s)}{\partial z}\Big|_{s=0,z=1} = -B_{00}^{-1}[B_{01}\tilde{S}_2^{(1,0)}].
\end{aligned}$$

7. Performance Measure

- Probability that the server is busy with the normal service

$$P_{BNS} = X_1(I-R)^{-1}e(a)$$

- Probability that the server is busy with the optional service

$$P_{BOS} = X_1(I-R)^{-1}e(b)$$

- Probability that the server is in breakdown during the normal service

$$P_{BDNS} = X_1(I-R)^{-1}e(c)$$

- Probability that the server is in breakdown during the optional service

$$P_{BDOS} = X_1(I-R)^{-1}e(d)$$

- Probability of the server being in vacation

$$P_{VAC} = X_1(I-R)^{-1}e(g) + X_0e_2(g)$$

- Expected size of the system

$$E_S = \sum_{z=1}^{\infty} zX_z e = X_1(I-R)^{-2}e$$

- Expected system size that the server is busy with the normal service

$$E_{BNS} = X_1(I-R)^{-2}e(a)$$

- Expected system size that the server is busy with the optional service

$$E_{BOS} = X_1(I-R)^{-2}e(b)$$

- Expected system size that the server will be in breakdown during the normal service

$$E_{BDNS} = X_1(I-R)^{-2}e(c)$$

- Expected system size that the server will be in breakdown during the optional service

$$E_{BDOS} = X_1(I-R)^{-2}e(d)$$

- Expected system size that the server being in vacation

$$E_{VAC} = X_1(I-R)^{-2}e(g).$$

8. Particular case

We consider an exponential distribution for the arrival, service and repair times. Let us denote:

D_0=$[-\lambda]$, D_1=$[\lambda]$, α_1=[1], T_1=$[\mu_1]$, α_2=[1], T_2=$[\mu_2]$, β_1=[1], S_1=$[\sigma_1]$, β_2=[1], S_2=$[\sigma_2]$

With our assumption, the infinitesimal generator matrix becomes

$$Q = \begin{bmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \dots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & \dots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \dots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots \end{bmatrix}.$$

The entries of the Q matrix are defined by

$$B_{00} = \begin{bmatrix} -\lambda-\varphi & \varphi \\ 0 & -\lambda+b\lambda \end{bmatrix}, B_{01} = \begin{bmatrix} 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-b)\lambda & 0 \end{bmatrix}, B_{10} = \begin{bmatrix} q\mu_1 & 0 \\ \mu_2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -\lambda-\mu_1-\tau_1 & p\mu_1 & \tau_1 & 0 & 0 & 0 & 0 \\ 0 & -\lambda-\mu_2-\tau_2 & 0 & \tau_2 & 0 & 0 & 0 \\ \sigma_1 & 0 & -\sigma_1-\lambda & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & -\sigma_2-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda-\varphi & \varphi & 0 \\ 0 & 0 & 0 & 0 & 0 & b\lambda-\lambda-\eta & \eta \\ \psi & 0 & 0 & 0 & 0 & 0 & -\lambda-\psi \end{bmatrix},$$

$$A_2 = \begin{bmatrix} q\mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-b)\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix},$$

Consequently, the matrix A becomes

$$A = \begin{bmatrix} -\mu_1 - \tau_1 + q\mu_1 & p\mu_1 & \tau_1 & 0 & 0 & 0 & 0 \\ \mu_2 & -\mu_2 - \tau_2 & 0 & \tau_2 & 0 & 0 & 0 \\ \sigma_1 & 0 & -\sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & -\sigma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\varphi & \varphi & 0 \\ 0 & 0 & 0 & 0 & 0 & -\eta & \eta \\ \psi & 0 & 0 & 0 & 0 & 0 & -\psi \end{bmatrix}.$$

The stationary probability vector ξ of A which satisfies $\xi A = 0$ and $\xi e = 1$ is given by $\xi = (\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$, where

$$\xi_0 = \frac{\mu_2\sigma_1\sigma 2}{\mu_2\sigma_1\sigma_2+\mu_2\sigma_2\tau+\mu_1 p\sigma_1\sigma_2+\mu_1 p\sigma_1\tau},\ \xi_1 = \frac{\mu_1 p\sigma_1\sigma 2}{\mu_2\sigma_1\sigma_2+\mu_2\sigma_2\tau+\mu_1 p\sigma_1\sigma_2+\mu_1 p\sigma_1\tau},\ \xi_2 = \frac{\mu_2\sigma_2\tau}{\mu_2\sigma_1\sigma_2+\mu_2\sigma_2\tau+\mu_1 p\sigma_1\sigma_2+\mu_1 p\sigma_1\tau},$$

$$\xi_3 = \frac{\mu_1 p\sigma_1\tau}{\mu_2\sigma_1\sigma_2+\mu_2\sigma_2\tau+\mu_1 p\sigma_1\sigma_2+\mu_1 p\sigma_1\tau},\ \xi_4 = 0,\ \xi_5 = 0,\ \xi_6 = 0.$$

The necessary and sufficient condition required by the system to remain stable is $\xi A_0 e < \xi A_2 e$. Hence

$$\lambda < \frac{\mu_1\mu_2\sigma_1\sigma_2}{\mu_2\sigma_1\sigma_2 + \mu_2\sigma_2\tau + \mu_1 p\sigma_1\sigma_2 + \mu_1 p\sigma_1\tau}.$$

9. Waiting Time Distribution

Using the first passage time, we perform an analysis of waiting time distribution of such of those customers who arrive in the queueing line. Let $W(t)$, $t \geq 0$ denote the distribution function of the waiting time of the incoming tagged customer in the queue. When the server is in busy, repair process or in vacation, the customer has to wait in the queueing line to get service from the server.

The absorbing state $(*)$ corresponds to the upcoming tagged customer to receive the service without waiting in the queue. Let us introduce the absorption time in a continuous time Markov chain with the state space as follows:

$$\tilde{\Omega} = (*) \cup \{0, 1, 2, 3, ...\}$$

where

$$l(0) = \{(0, j) : j = 4, 5\}$$

and for $i \geq 1$,

$$\begin{aligned} l(i) = &\{(i, 0, k_1) : i \epsilon \mathbb{Z}^+,\ 1 \leq k_1 \leq t_1\} \cup \{(i, 1, k_2) : i \epsilon \mathbb{Z}^+,\ 1 \leq k_2 \leq t_2\} \\ &\cup \{(i, 2, r_1) : i \epsilon \mathbb{Z}^+,\ 1 \leq r_1 \leq s_1\} \cup \{(i, 3, r_2) : i \epsilon \mathbb{Z}^+,\ 1 \leq r_2 \leq s_2\} \\ &\cup \{(i, 4) : i \epsilon \mathbb{Z}^+,\ 1 \leq a \leq m\} \cup \{(i, 5) : i \epsilon \mathbb{Z}^+\} \cup \{(i, 6) : i \epsilon \mathbb{Z}^+\} \end{aligned}$$

Let the transition matrix $\tilde{Q}$ of this Markov process be

$$\tilde{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ M_0 & P_{00} & 0 & 0 & 0 & 0 & \dots \\ M_1 & P_1 & P & 0 & 0 & 0 & \dots \\ 0 & 0 & P_2 & P & 0 & 0 & \dots \\ 0 & 0 & 0 & P_2 & P & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots \end{bmatrix}$$

where the entries of the $\tilde{Q}$ matrix are defined by

$$M_0 = \begin{bmatrix} 0 \\ \eta \end{bmatrix}, P_{00} = \begin{bmatrix} -\varphi & \varphi \\ 0 & -\eta \end{bmatrix}, M_1 = \begin{bmatrix} qT_1^0 \\ T_2^0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} qT_1^0 \otimes \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ T_2^0 \otimes \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} T_1 - \tau_1 I_{ns} & pT_1^0 \otimes \alpha_2 & e_2 \otimes \tau_1 \beta_1 & 0 & 0 & 0 & 0 \\ 0 & T_2 - \tau_2 I_{os} & 0 & e_2 \otimes \tau_2 \beta_2 & 0 & 0 & 0 \\ \alpha_1 \otimes S_1^0 & 0 & S_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 \otimes S_2^0 & 0 & S_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\varphi & \varphi & 0 \\ 0 & 0 & 0 & 0 & 0 & -\eta & \eta \\ \psi\alpha_1 & 0 & 0 & 0 & 0 & 0 & -\psi \end{bmatrix}.$$

Let us define $z(0)$=$(z_0(0), z_1(0), z_2(0), ...)$ which is the conditional probability distribution of the system state defined on the arrival of the tagged customers. The probability vectors of $z_0(0)$ and $z_i(0)$ are respectively given by

$$z_0(0) = x_0 \big[I_2 \otimes \frac{D_1 e_m}{\lambda} \big]$$
$$z_i(0) = x_i [I_{t_1+t_2+s_1+s_2+3} \otimes \frac{D_1 e_m}{\lambda}], \quad \text{for } i \geq 1$$

where the fundamental arrival rate of the Markovian arrival process is denoted by λ. Let

$$z(t) = (z_*(t), z_0(t), z_1(t), ...),$$

where

$$\begin{aligned} z_0(t) \quad &: \quad \text{vector of order } (1 \times 2), \\ z_i(t),\ i \geq 1 \quad &: \quad \text{vector of order } 1 \times (t_1 + t_2 + s_1 + s_2 + 3). \end{aligned}$$

The elements of $z_i(t),\ i \geq 1$ are the probabilities of the $CTMC$ wherein the respective states of level i with the generator matrix $\tilde{Q}$ are at epoch t. The probability of the process being in the absorbing state at time t is given by $z_*(t)$.

We have $W(t)$ = $z_*(t)$, for $t \geq 0$.

The differential equation $z'(t)$ = $z(t)\tilde{Q}$, where $t \geq 0$ becomes

$$\begin{aligned} z'_*(t) &= z_0(t) M_0 + z_1(t) M_1, \\ z'_0(t) &= z_0(t) P_{00} + z_1(t) P_1, \\ z'_i(t) &= z_i(t) P + z_{i+1}(t) P_2, \quad i \geq 1. \end{aligned}$$

where $'$ denotes the derivative with respect to t.

Let us compute the Laplace Stieltjes Transform (LST) of $W(t)$ with the aid of the technique mentioned by Neuts [23].

The process is commenced at the state i with initial probability vector $z_i(0)$, $i \geq 1$.

Let $w(s)$ be the row vector which specifies the Laplace-Stieltjes transform of the initial transit time to level 1.

As per the scheme specified by Neuts [22], we have

$$w(s) = \sum_{i=1}^{\infty} [(sI - P)^{-1} P_2]^{i-1}. \tag{1}$$

Let the Laplace-Stieltjes transform of absorbing time to the state $(*)$ correspond to the process starting at state level $i = 0, 1$ be indicated by $\phi(i, s)$. Applying a result in Neuts [22], we have

$$\phi(0, s) = [sI - P_{00}]^{-1} M_0, \tag{2}$$

$$\phi(1, s) = [sI - P]^{-1} P_1 \phi(0, s) + [sI - P]^{-1} M_1. \tag{3}$$

Thus, the LST of the waiting time distribution $\tilde{W}(s)$ is evaluated as

$$\tilde{W}(s) = z_0(0)\phi(0, s) + w(s)\phi(1, s). \tag{4}$$

9.1. Expected Waiting Time

The expected waiting time is denoted by

$$E(W) = -z_0(0)\phi'(0, 0) - w'(0)e_{t_1+t_2+s_1+s_2+3} - w(0)\phi'(1, 0)e_n. \tag{5}$$

When the system has the state level $i = 0$, the average time to enter into the absorbing state $(*)$ is denoted by the foremost terms of the preceding equation.

Likewise, if the system has the state level $i \geq 1$, the average amount of time to enter into the absorbing state $(*)$ is denoted by the last two terms of the above equation.

On differentiating (2) and (3) with respect to t and substituting $s = 0$, we get

$$\phi'(0, 0) = (-1)[-P_{00}]^{-2} M_0, \tag{6}$$

$$\phi'(1, 0) = (-1)[-P]^{-2} P_1 \phi(0, 0) + [-P]^{-1} P_1 \phi'(0, 0) - [-P]^{-2} M_1. \tag{7}$$

By using the expression (6) together with the probability vector

$$z(0) = (z_0(0), z_1(0), z_2(0), ...),$$

we can determine the first term of (5). From (1), we have

$$w(0) = \sum_{i=1}^{\infty} z_i(0) V^{i-1}, \tag{8}$$

where $V = [-P]^{-1} P_2$. Since V is a stochastic matrix, we get

$$w(0)e_{t_1+t_2+s_1+s_2+3} = 1 - z_0(0)e_2. \tag{9}$$

With the help of (7) and (9) together with the probability vector $z(0) = (z_0(0), z_1(0), z_2(0), ...)$, we can compute the final term of (5).

On differentiating (1) with respect to t and substituting $s = 0$, we get

$$w'(0) = (-1) \sum_{i=1}^{\infty} z_{i+1}(0) \sum_{j=0}^{i-1} V^j [-P]^{-1} V^{i-j}. \tag{10}$$

The stochastic nature of V implies that

$$(-1)w'(0)e_{t_1+t_2+s_1+s_2+3} = (-1) \sum_{i=1}^{\infty} z_{i+1}(0) \sum_{j=0}^{i-1} V^j [-P]^{-1} e_{t_1+t_2+s_1+s_2+3}. \tag{11}$$

We can evaluate the value of $(-1)w'(0)e_{t_1+t_2+s_1+s_2+3}$, by using the method specified in Neuts [22].

Now, let us consider the stochastic matrix V_2 satisfying two conditions namely $I - V + V_2$ is non-singular and the generalized inverse of the form $I - V$. Then, the matrix V_2 may be chosen as $V_2 = v_0 e_{t_1+t_2+s_1+s_2+3}$, where v_0 is the stationary probability vector of V such that

$$v_0 V = v_0 \text{ and } v_0 e_{t_1+t_2+s_1+s_2+3} = 1.$$

In view of the property

$$VV_2 = V_2 V = V_2.$$

we get

$$\sum_{j=0}^{i-1} V^j (I - V + V_2) = I - V^i + iV_2, \quad \text{for } i \geq 1. \tag{12}$$

Substituting (12) in (11) and carrying out some simplifications, we get

$$\begin{aligned}(-1)w'(0)e_{t_1+t_2+s_1+s_2+3} = &\big[x_1[I - R]^{-1}[I_{t_1+t_2+s_1+s_2+3} \otimes \frac{D_1 e_m}{\lambda}] \\ &- w(0) + x_1 R[I - R]^{-2}[I_{t_1+t_2+s_1+s_2+3} \otimes \frac{D_1 e_m}{\lambda} V_2] \\ &\times [I - V + V_2]^{-1}[-P]^{-1} e_{t_1+t_2+s_1+s_2+3}.\end{aligned} \tag{13}$$

Thus, we have determined all the terms of (5). By using (5), the expected waiting time can be easily evaluated.

10. Numerical Results

In this section, we examine the outcome of our system with the utilisation of numerical and graphical methods. The five different MAP representations are given below with distinct variance and correlation structure and their mean values are 1. These values are suggested by Chakravarthy [4] . In the first three process of arrival, like $ERL - A$, $EXP - A$ and $HYP.EXP - A$ correspond to renewal process and thus the correlation is zero.

Arrival in Erlang of order 2 (ERL-A):

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

Arrival in Exponential (EXP-A):

$$D_0 = [-1], D_1 = [1]$$

Arrival in Hyper exponential (HYP-EXP-A):

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$$

MAP-NC-A: Arrival in MAP - Negative Correlation:

$$D_0 = \begin{bmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.99220 \\ 223.4925 & 0 & 2.2575 \end{bmatrix}$$

MAP-PC-A: Arrival in MAP - Positive Correlation:

$$D_0 = \begin{bmatrix} 1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.99220 & 0 & 0.01002 \\ 2.2575 & 0 & 223.4925 \end{bmatrix}.$$

Let us consider the following PH-distributions for the service and repair process which are suggested by Chakravarthy [4].

ERL-S (Service in Erlang of order 2):

$$\alpha_1 = \alpha_2 = (1, 0), \quad T_1 = T_2 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

ERL-R (Repair in Erlang of order 2):

$$\beta_1 = \beta_2 = (1, 0), \quad S_1 = S_2 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

EXP-S (Service in Exponential):

$$\alpha_1 = \alpha_2 = (1), \quad T_1 = T_2 = [-1]$$

EXP-R (Repair in Exponential):

$$\beta_1 = \beta_2 = (1), \quad S_1 = S_2 = [-1]$$

HYP.EXP-S (Service in Hyper exponential):

$$\alpha_1 = \alpha_2 = (0.8, 0.2), \quad T_1 = T_2 = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}$$

HYP.EXP-R (Repair in Hyper exponential):

$$\beta_1 = \beta_2 = (0.8, 0.2), \quad S_1 = S_2 = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}.$$

10.1. Illustrative Example 1

From the Tables 1-5, we explore the effect of the fundamental arrival rate (λ) on the Expected system size (ES) and Expected waiting time (EW). Fix $\mu_1 = 15$, $\mu_2 = 12$, $\sigma_1 = 8$, $\sigma_2 = 6$, $\eta = 8$, $\tau = 1$, $\varphi = 10$, $\psi = 10$, $p = 0.2$, $q = 0.8$, $b = 0.1$.

Table 1: *Fundamental arrival rate (λ) vs ES and EW - EXP-A*

	$EXP-S$		$ERL-S$		$HYP-S$	
λ	ES	EW	ES	EW	ES	EW
1.0	0.15853	0.30160	0.15741	0.30182	0.16332	0.29898
1.2	0.16477	0.36572	0.16339	0.36566	0.17067	0.36396
1.4	0.17091	0.43122	0.16925	0.43075	0.17796	0.43085
1.6	0.17696	0.49815	0.17502	0.49714	0.18522	0.49972
1.8	0.18296	0.56659	0.18071	0.56490	0.19247	0.57067
2.0	0.18892	0.63661	0.18635	0.63409	0.19973	0.64383
2.2	0.19486	0.70832	0.19196	0.70480	0.20703	0.71930
2.4	0.20080	0.78180	0.19755	0.77712	0.21438	0.79723
2.6	0.20677	0.85719	0.20316	0.85116	0.22181	0.87777
2.8	0.21278	0.93460	0.20879	0.92704	0.22935	0.96107

Table 2: *Fundamental arrival rate (λ) vs ES and EW - ERL-A*

	$EXP-S$		$ERL-S$		$HYP-S$	
λ	ES	EW	ES	EW	ES	EW
1.0	0.14153	0.29431	0.14086	0.29510	0.14543	0.29040
1.2	0.14657	0.35594	0.14567	0.35661	0.15163	0.35268
1.4	0.15177	0.41860	0.15063	0.41905	0.15806	0.41660
1.6	0.15706	0.48236	0.15566	0.48243	0.16466	0.48225
1.8	0.16241	0.54724	0.16072	0.54681	0.17138	0.54970
2.0	0.16778	0.61332	0.16579	0.61222	0.17818	0.61904
2.2	0.17317	0.68065	0.17085	0.67871	0.18507	0.69038
2.4	0.17856	0.74932	0.17590	0.74636	0.19203	0.76384
2.6	0.18396	0.81942	0.18094	0.81523	0.19908	0.83955
2.8	0.18939	0.89105	0.18598	0.88543	0.20621	0.91765

Table 3: *Fundamental arrival rate (λ) vs ES and EW - HYP-EXP-A*

	$EXP-S$		$ERL-S$		$HYP-S$	
λ	ES	EW	ES	EW	ES	EW
1.0	0.17911	0.31245	0.17708	0.31172	0.18736	0.31350
1.2	0.18868	0.38164	0.18618	0.38018	0.19877	0.38521
1.4	0.19808	0.45341	0.19507	0.45097	0.21005	0.46028
1.6	0.20740	0.52794	0.20386	0.52429	0.22127	0.53894
1.8	0.21671	0.60549	0.21262	0.60036	0.23251	0.62140
2.0	0.22610	0.68632	0.22143	0.67945	0.24383	0.70793
2.2	0.23564	0.77074	0.23037	0.76186	0.25529	0.79880
2.4	0.24540	0.85911	0.23952	0.84794	0.26695	0.89431
2.6	0.25546	0.95182	0.24894	0.93807	0.27886	0.99479
2.8	0.26588	1.04931	0.25872	1.03271	0.29107	1.10060

Table 4: *Fundamental arrival rate (λ) vs ES and EW - MAP-NC-A*

	$EXP-S$		$ERL-S$		$HYP-S$	
λ	ES	EW	ES	EW	ES	EW
1.0	0.21789	0.32963	0.21810	0.33120	0.21685	0.32110
1.2	0.22094	0.39857	0.22097	0.40024	0.22086	0.38946
1.4	0.22416	0.46873	0.22399	0.47041	0.22513	0.45954
1.6	0.22757	0.54021	0.22718	0.54180	0.22966	0.53149
1.8	0.23116	0.61312	0.23053	0.61448	0.23444	0.60542
2.0	0.23493	0.68755	0.23404	0.68855	0.23947	0.68149
2.2	0.23888	0.76360	0.23771	0.76409	0.24474	0.75981
2.4	0.24301	0.84138	0.24153	0.84121	0.25025	0.84055
2.6	0.24732	0.92102	0.24552	0.92001	0.25600	0.92385
2.8	0.25181	1.00264	0.24968	1.00061	0.26200	1.00989

Table 5: *Fundamental arrival rate (λ) vs ES and EW - MAP-PC-A*

	$EXP-S$		$ERL-S$		$HYP-S$	
λ	ES	EW	ES	EW	ES	EW
1.0	4.93292	5.03451	5.02349	5.12623	4.43675	4.53209
1.2	5.06281	6.18903	5.15683	6.30311	4.54890	5.56542
1.4	5.18831	7.38672	5.28604	7.52488	4.65537	6.63320
1.6	5.31349	8.63171	5.41520	8.79585	4.76012	7.73857
1.8	5.44069	9.92837	5.54670	10.12062	4.86538	8.88481
2.0	5.57147	11.28133	5.68210	11.50408	4.97259	10.07536
2.2	5.70694	12.69553	5.82258	12.95143	5.08272	11.31383
2.4	5.84801	14.17626	5.96905	14.46829	5.19652	12.60403
2.6	5.99546	15.72922	6.12233	16.06069	5.31459	13.95000
2.8	6.15001	17.36055	6.28322	17.73519	5.43745	15.35604

From the above tables 1, 2, 3, 4 and 5, the following observations are made:

- As fundamental arrival rate (λ) increases, the expected system size (ES) increases for various probable sequence of arrival and service times.
- As fundamental arrival rate (λ) increases, the expected waiting time of the system (EW) increases for various probable sequence of arrival and service times.
- With the estimation of the values of various arrival times, the average system size increases much faster for hyper exponential service time and slowly for Erlang service time.

10.2. Illustrative Example 2

The two dimensional graphs are illustrated in the following Figures $1-12$. We explore the effect of the vacation rate (η) on the Expected system size (ES) and Expected waiting time (EW). Fix $\lambda=1$, $\mu_1=15$, $\mu_2=12$, $\sigma_1=8$, $\sigma_2=6$, $\tau=1$, $\varphi=10$, $\psi=10$, $p=0.2$, $q=0.8$, $b=0.1$.

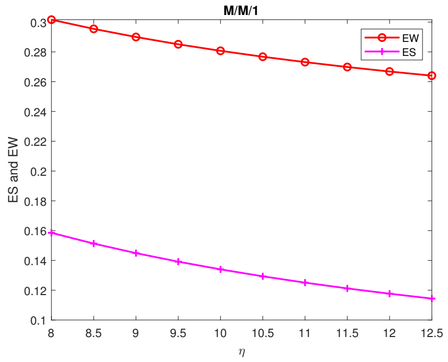

Figure 1: *Vacation rate (η) vs. ES and EW*

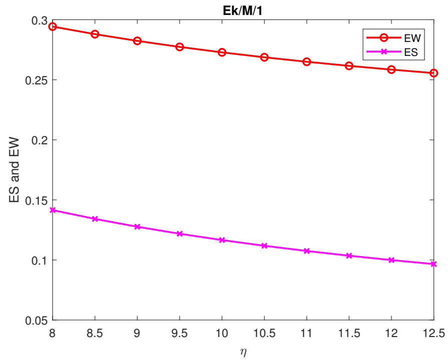

Figure 2: *Vacation rate (η) vs. ES and EW*

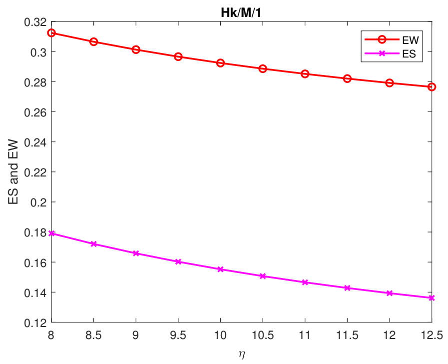

Figure 3: *Vacation rate (η) vs. ES and EW*

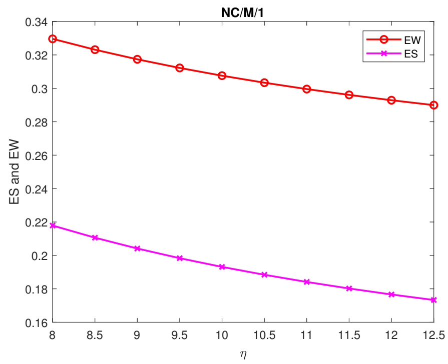

Figure 4: *Vacation rate (η) vs. ES and EW*

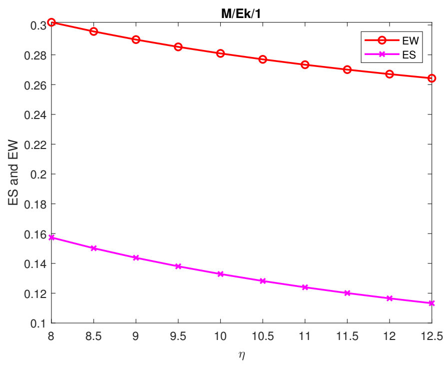

Figure 5: *Vacation rate (η) vs. ES and EW*

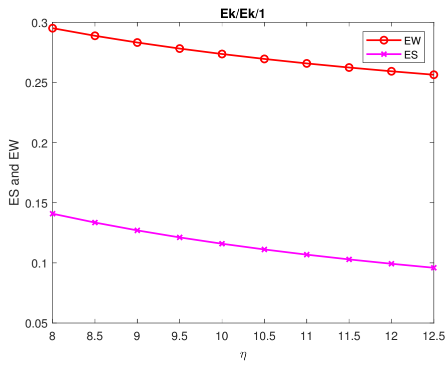

Figure 6: *Vacation rate (η) vs. ES and EW*

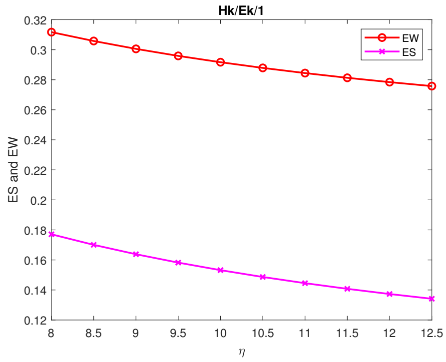

Figure 7: *Vacation rate (η) vs. ES and EW*

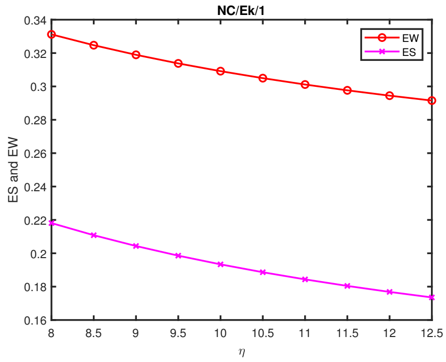

Figure 8: *Vacation rate (η) vs. ES and EW*

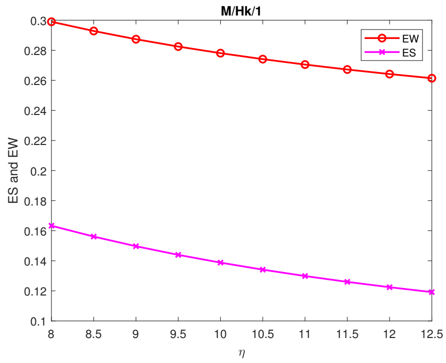

Figure 9: *Vacation rate (η) vs. ES and EW*

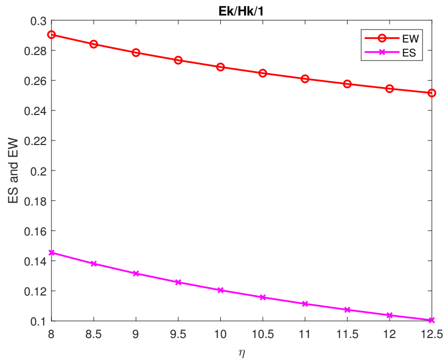

Figure 10: *Vacation rate (η) vs. ES and EW*

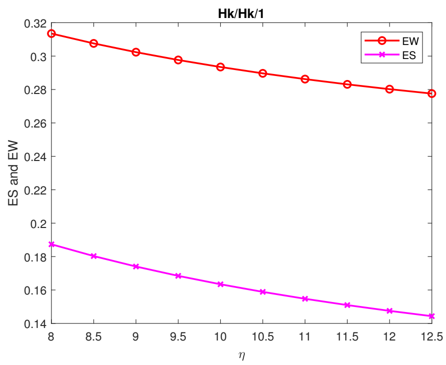

Figure 11: *Vacation rate (η) vs. ES and EW*

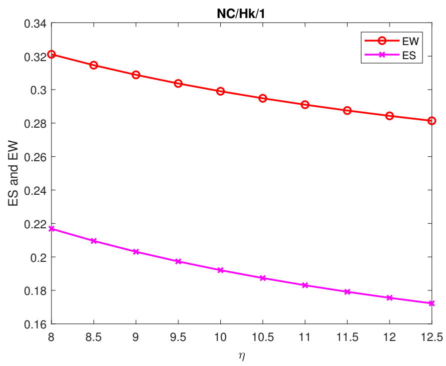

Figure 12: *Vacation rate (η) vs. ES and EW*

We observe from the above Figures $1-12$, that when lifting the vacation rate η, the Expected system size (ES) and Expected waiting time (EW) increase rapidly in the case of arrival by hyper-exponential and slowly in Erlang arrival. Likewise it is high in Erlang service and slow in hyper-exponential. By examining the graphs, we see that the Expected system size (ES) and Expected waiting time (EW) decrease faster for Erlang service rather than those of exponential service and hyper-exponential services.

10.3. Illustrative Example 3

From the three dimensional graphs $13-24$, we explore the effect of the normal service rate (μ_1) and the breakdown rate (τ) on the probability that the server is busy with the normal service

(P_{BNS}). Fix $\lambda = 1$, $\mu_2 = 12$, $\sigma_1 = 8$, $\sigma_2 = 6$, $\eta = 8$, $\varphi = 10$, $\psi = 10$, $p = 0.2$, $q = 0.8$, $b = 0.1$.

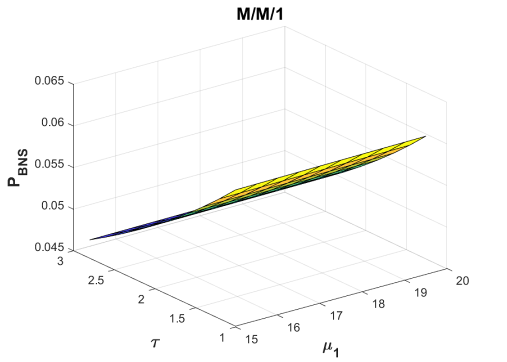

Figure 13: *Normal service rate (μ_1) and Breakdown rate(τ) vs. P_{BNS}*

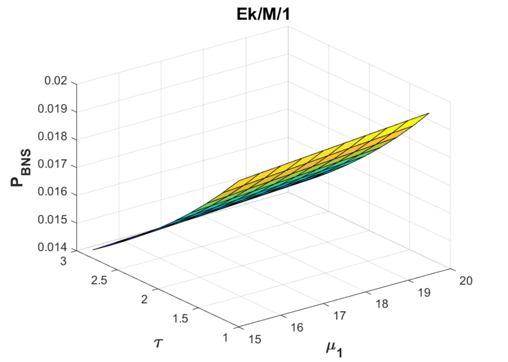

Figure 14: *Normal service rate (μ_1) and Breakdown rate(τ) vs. P_{BNS}*

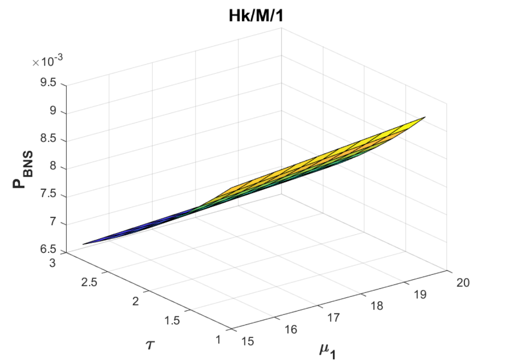

Figure 15: *Normal service rate (μ_1) and Breakdown rate(τ) vs. P_{BNS}*

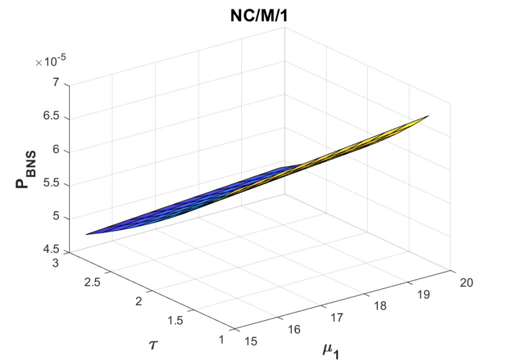

Figure 16: *Normal service rate (μ_1) and Breakdown rate(τ) vs. P_{BNS}*

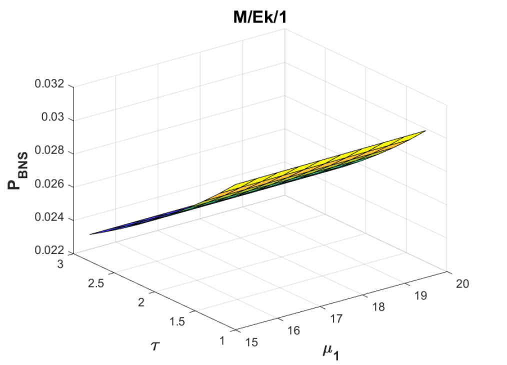

Figure 17: *Normal service rate (μ_1) and Breakdown rate(τ) vs. P_{BNS}*

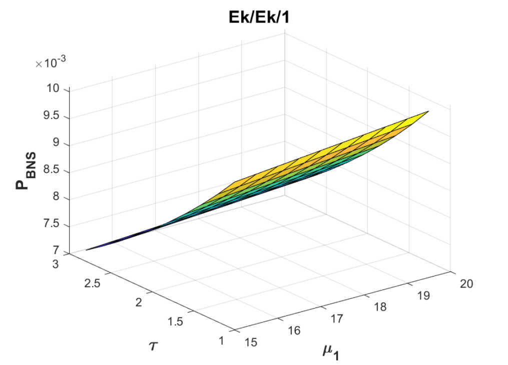

Figure 18: *Normal service rate (μ_1) and Breakdown rate(τ) vs. P_{BNS}*

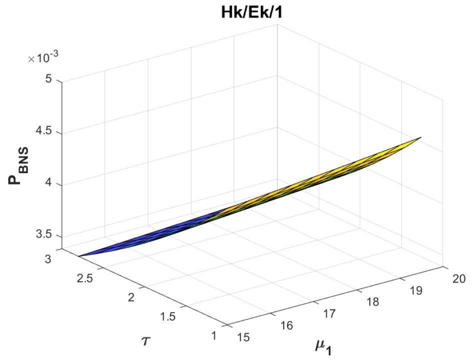

Figure 19: *Normal service rate* (μ_1) *and Breakdown rate*(τ) *vs.* P_{BNS}

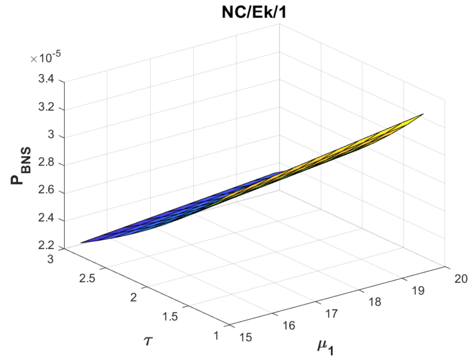

Figure 20: *Normal service rate* (μ_1) *and Breakdown rate*(τ) *vs.* P_{BNS}

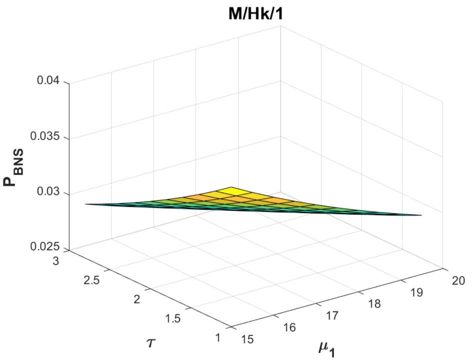

Figure 21: *Normal service rate* (μ_1) *and Breakdown rate*(τ) *vs.* P_{BNS}

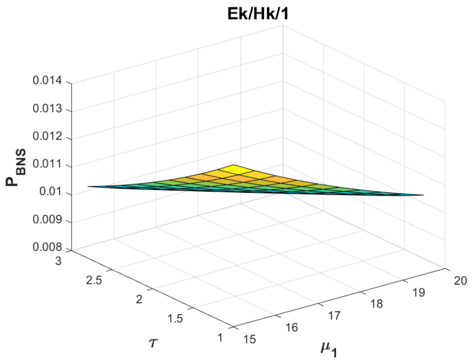

Figure 22: *Normal service rate* (μ_1) *and Breakdown rate*(τ) *vs.* P_{BNS}

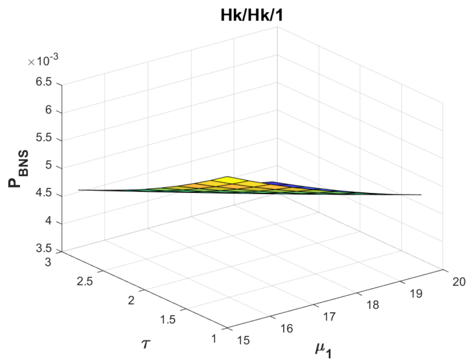

Figure 23: *Normal service rate* (μ_1) *and Breakdown rate*(τ) *vs.* P_{BNS}

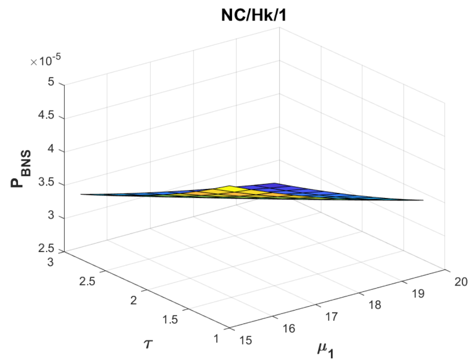

Figure 24: *Normal service rate* (μ_1) *and Breakdown rate*(τ) *vs.* P_{BNS}

We observe from the Figures $13-24$ that when lifting both the normal service rate (μ_1) and the breakdown rate (τ), the probability that the server is busy with the normal service (P_{BNS}) decreases for various arrival and service patterns. The negative arrival (NC) of MAP increases rapidly rather than that for hyper-exponential arrival. Likewise, the increment rate reduces in Erlang service and increases in hyper-exponential services.

11. Conclusion

In this paper, we have dealt with a single server queueing model with multiple vacations, optional service, close-down, setup, balking, breakdown and phase type repair. The arrival time follows MAP and service and repair times follow phase type distributions. We have presented the busy period analysis and waiting time distribution of this model. We have evaluated the probability of the situation that the server is busy, under repair and on vacation. Particular case has also been discussed. Numerical illustrations and graphical representations are presented in this paper. The future direction of our work can be the investigation of the queueing model by using BMAP for the arrival process with N-policy.

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