RT&A, No 2 (68)
Volume 17, June 2022

Generalized Transmuted Exponential-Exponential Distribution and its Applications

A. S. MOHAMMED[1,2*], F. I. UGWUOWO[2γ], T. S. PATRICE[3], H. MUHAMMAD[4]

•

[1]Department of Statistics, Ahmadu Bello University, Zaria
[2]Department of Statistics, University of Nigeria, Nsukka
[γ]Email: fidelis.ugwuowo@unn.edu.ng
[3]Department of Mathematics, University of Yaounde I, Cameroon
Email: patricetakam@gmail.com
[4]Division of Agriculture, Ahmadu Bello University, Zaria
Email: hamzamuhammad@abu.edu.ng
*Corresponding author: Email: mohammedas@abu.edu.ng

Abstract

Modeling of datasets requires knowledge of their appropriate distributional assumptions. In this research, we generalized the transmuted exponential-exponential distribution, and it was observed that the addition of the shape parameter to the model proved to be helpful in improving the flexibility of the model. Different characteristics, as well as structural properties of the model, were investigated and presented in an explicit form. The probability density function of the order statistics and numerical results for some descriptive statistics were obtained. A 95% confidence interval and interval widths, together with biases and mean square errors (MSEs) of the mean estimates, were equally evaluated using the Monte-Carlo simulation approach. To validate the flexibility of the model, we used real datasets and the generalized transmuted exponential-exponential distribution (GTE-ED) outperformed the competing distributions.

Keywords: Transmuted Exponential-Exponential distribution, descriptive statistics, Order statistics, Confidence Interval

1. Introduction

The procedure of parameter(s) induction to a parent distribution has fascinated the attention of numerous researchers in the recent years [1]. The addition of one or more shape parameter(s) to a given baseline model strengthens the distribution, especially when studying its tail characteristics. The parameter induction method has proved useful for boosting the fitness of a proposed model [2].

In statistics, the modelling of datasets requires knowledge of appropriate distributional assumptions about the datasets. In theory, the tractability of a probability distribution can be helpful since it is easier to manipulate when modelling a dataset. The concept of generalizing distributions was proposed by [3] which concern basically with raising the distribution function of the baseline distribution say $A(y)$ to the power of an arbitrary parameter $c > 0$ which give rise to a new model or distribution of the form $B(y) = (A(y))^c \; for\, c > 0$. The parameter (c) plays an important role in adding skewness to the function $A(y)$. In the early 1990s, generalized models were found to be useful in numerous areas of statistics and medical sciences due to their ability to model different forms of data. These distributions were proposed by statisticians from various fields. The concept of generating generalized distributions was used by [4] to developed a new

Weibull distribution named the exponentiated Weibull distribution. Furthermore, [5] studied the general characteristics of the exponentiated Weibull distribution. A new two-parameter model called exponentiated exponential distribution which outperformed other competing distribution in the study when applied to a real dataset was studied by [6]. Notable authors like [7], [8] and [9] applied the same methodology and developed the exponentiated type distributions, and exponentiated generalized inverse Gaussian distribution respectively. The properties of exponentiated transmuted generalized Rayleigh distribution was proposed and studied by [10] and [11] studied the exponentiated generalized class of distributions. Consequently, the properties and MLEs of generalized odd generalized exponential- exponential distribution was presented in an explicit form by [12]. The properties and applications of the transmuted exponential-exponential distribution (TE-ED) which has two scale parameters and a transmuted parameter. In practice, to find the distribution that captures the sensitive part of a given dataset, there are many possibilities was studied by [13]. We can either estimate non-parametrically the density function as well as the distribution function and compare them with the existing distributions to see which one is closest to the empirical distribution. However, in some situations for which we are obliged to consider some characteristics such as hazard rate, many of the existing distributions cannot adequately model a dataset with non-monotone hazard rates, and as such, these distributions are limited in applications.
The current kinds of literature in mathematical statistics as highlighted by [14] pay more attention to proposing more flexible distributions but give less concern to the hazard function of the distributions. It is critical to generate distributions with varying failure rates because the hazard rate function guides model selection [11]. Furthermore, many of the existing exponential extended distributions cannot adequately describe some of the existing datasets, particularly the ones with monotone and non-monotone hazard rates. For example, exponentiated exponential, transmuted exponential-exponential, and Weibull exponential distribution, among others. This has opened the room for more research that can account for monotone and non-monotone hazard rate data.

In this research, we are motivated by the above-mentioned rationale to develop a new exponential extended distribution called the generalized transmuted exponential-exponential distribution (GTE-ED). As compared to the existing exponential extended distributions, the GTE-ED is more flexible and can model both monotone and non-monotone hazard rate data.

Table 1: *Hazard rates behaviour for GTE-E and the competing distributions*

Distribution	Constant	Increasing	Decreasing	Unimodal
GTE-E	Yes	Yes	Yes	Yes
TE-E	Yes	Yes	No	No
EE	Yes	Yes	Yes	No
E	Yes	No	No	No

From table 1, we can deduce that GTE-ED has more advantages over the competing distributions in the study and, as such, it will be more robust in analysing data with different hazard rates.

2. The Generalized Transmuted Exponential-Exponential Distribution

Consider the density function $a(y;\lambda,\theta,\alpha) = \lambda\alpha\left(1-\theta\right)e^{-\alpha\lambda y} + 2\lambda\theta\alpha e^{-2\alpha\lambda y}$ and distribution function $A(y;\lambda,\theta,\alpha) = \left(1-e^{-\lambda\alpha y}\right)\left(1+\theta e^{-\lambda\alpha y}\right)$ of the transmuted exponential-exponential distribution with scale parameter $\alpha,\lambda > 0$, transmuted parameter $-1 \leq \theta \leq 1$ and $y \geq 0$ (Mohammed and Ugwuowo 2021). The cumulative distribution function and density of generalized transmuted exponential distribution (GTE-ED) are respectively derived from the following functions:

$$B(y) = \left(A(y)\right)^{c} \ for\, c > 0 \tag{1}$$

and,

$$b(y) = ca(y)\,(A(y))^{c-1} \tag{2}$$

The GTE-ED is then defined as;

$$B(y) = \left[\left(1 - e^{-\lambda\alpha y}\right)\left(1 + \theta e^{-\lambda\alpha y}\right)\right]^{c} \tag{3}$$

and by taking the differential of $B(y)$, we have;

$$b(y) = c\left[\lambda\alpha\,(1-\theta)\,e^{-\alpha\,\lambda y} + 2\lambda\theta\alpha e^{-2\alpha\,\lambda y}\right]\left[\left(1 - e^{-\lambda\,\alpha\,y}\right)\left(1 + \theta e^{-\lambda\,\alpha\,y}\right)\right]^{c-1} \tag{4}$$

where, $y \geq 0,\ \alpha, \lambda, c > 0\, and\ -1 \leq \theta \leq 1.$

Equation (4) can be written in the following contracted form;

$$b(y) = me^{-\lambda\,\alpha(f+g+1)y}\left((1-\theta) + 2\theta\,e^{-\alpha\lambda y}\right) \tag{5}$$

where, $m = c\,\lambda\,\alpha \sum_{f,g=0}^{\infty} (-1)^{f} \binom{c-1}{f}\binom{c-1}{g}$

Figure 1 displays some possible shapes of density and distribution function of the GTE-ED for chosen values of the parameters $a = \alpha,\ \ b = \lambda,\ d = \theta\, and\, c$. Moreover, the density changes in shape when the parameters take different values.

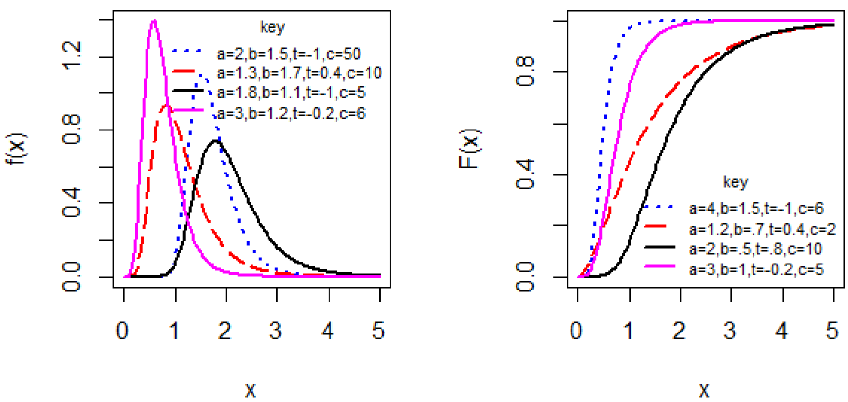

Figure 1: *density and distribution function of GTE-ED*

3. Statistical Properties of the model

Here, some statistical properties of GTE-ED including survival and hazard functions, quantile function, moments, moment generating function, limiting behaviour, and order statistics are considered and presented in an explicit form.

3.1. Survival and Hazard function

If Y has $GTE - E(\alpha, \lambda, c, \theta)$ model, then the survival and hazard function are respectively given as;

The survival function is defined mathematically as;

$S(x) = 1 - B(y)$

$$S(y) = 1 - \left(\left(1 - e^{-\lambda\alpha y}\right)\left(1 + \theta e^{-\lambda\alpha y}\right)\right)^{c} \tag{6}$$

The hazard function is defined mathematically as;

$h(y) = \frac{b(y)}{1-B(y)}$

$$h(y) = \frac{c\left(\lambda\alpha\left(1-\theta\right)e^{-\alpha\lambda y} + 2\lambda\theta\alpha e^{-2\alpha\lambda y}\right)\left(\left(1-e^{-\lambda\alpha y}\right)\left(1+\theta e^{-\lambda\alpha y}\right)\right)^{c-1}}{\left[1-\left(\left(1-e^{-\lambda\alpha y}\right)\left(1+\theta e^{-\lambda\alpha y}\right)\right)^{c}\right]} \tag{7}$$

Figures 3 and 4 displays some possible shapes of hazard and survival function (hf) of the GTE-ED for chosen values of the parameters $a = \alpha,\ \ b = \lambda,\ d = \theta\ and\ c$. The hf can take the shape of either increasing, decreasing, and unimodal as the parameter keep changing.

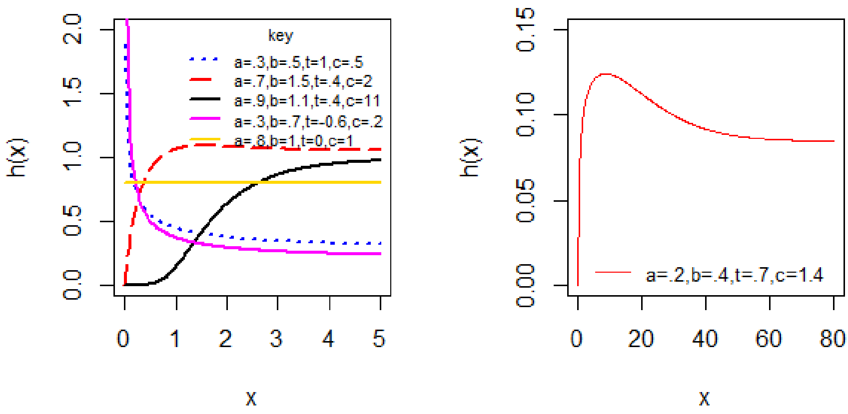

Figure 2: *Hazard function of GTE-ED*

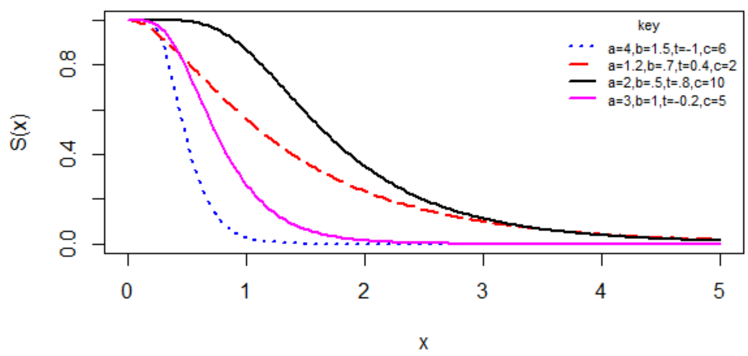

Figure 3: *Survival function of GTE-ED*

Limiting behaviour of the distribution

In this section, the asymptotic behaviour of the model is investigated by taking the limit $as\, y \to 0\, and\, y \to \infty$ of the distribution function.

$$\lim_{y\to 0} B(y) = \lim_{y\to 0} \left(\left(1-e^{-\lambda\alpha y}\right)\left(1+\theta e^{-\lambda\alpha y}\right)\right)^c = 0$$

and,

$$\lim_{y\to \infty} B(y) = \lim_{y\to \infty} \left(\left(1-e^{-\lambda\alpha y}\right)\left(1+\theta e^{-\lambda\alpha y}\right)\right)^c = 1$$

The results show that the GTE-ED is a valid distribution since $\lim_{y\to 0} B(y) = 0\, and \lim_{y\to\infty} B(y) = 1$.

3.2. The r^{th} moments and moment generating function

If $Y\, has\, GTEE(\alpha, \lambda, c, \theta)$ then, the r^{th} moments is given as;

$$E\left(y^r\right) = \frac{c\,\Gamma(r+1)}{(\lambda\,\alpha)^r} \sum_{f,g=0}^{\infty} (-1)^f \binom{c-1}{f}\binom{c-1}{g} \left\{\frac{(1-\theta)}{(f+g+1)^{r+1}} + \frac{2\,\theta}{(f+g+2)^{r+1}}\right\} \tag{8}$$

By using (8) the first two moments about the origin are obtained which can pave way in finding the variance and the coefficient of variation.

When r =1,

$$E\left(y\right) = \frac{c}{(\lambda\,\alpha)} \sum_{f,g=0}^{\infty} (-1)^f \binom{c-1}{f}\binom{c-1}{g} \left\{\frac{(1-\theta)}{(f+g+1)^{2}} + \frac{2\,\theta}{(f+g+2)^{2}}\right\}$$

When r =2,

$$E\left(y^2\right) = \frac{2\,c}{(\lambda\,\alpha)^2} \sum_{f,g=0}^{\infty} (-1)^f \binom{c-1}{f}\binom{c-1}{g} \left\{\frac{(1-\theta)}{(f+g+1)^{3}} + \frac{2\,\theta}{(f+g+2)^{3}}\right\}$$

$$Let,\ A_1 = \sum_{f,g=0}^{\infty} (-1)^f \binom{c-1}{f}\binom{c-1}{g} \left\{\frac{(1-\theta)}{(f+g+1)^{2}} + \frac{2\,\theta}{(f+g+2)^{2}}\right\}$$

$$A_2 = \sum_{f,g=0}^{\infty} (-1)^f \binom{c-1}{f}\binom{c-1}{g} \left\{\frac{(1-\theta)}{(f+g+1)^{3}} + \frac{2\,\theta}{(f+g+2)^{3}}\right\}$$

Therefore, $E\left(y\right) = \frac{c}{(\lambda\,\alpha)} A_1\, and\, E\left(y^2\right) = \frac{2\,c}{(\lambda\,\alpha)^2} A_2$

If $Y\, has\, GTE - E(\alpha, \lambda, c, \theta)$ then, the variance and the coefficient of variation of GTE-ED are respectively given as;

$$Var(y) = \frac{2\,c}{(\lambda\,\alpha)^2} A_2 - \frac{c}{(\lambda\,\alpha)} A_1 \text{ and } C\,V = \frac{\sqrt{\frac{2\,c}{(\lambda\,\alpha)^2} A_2 - \frac{c}{(\lambda\,\alpha)} A_1}}{\frac{c}{(\lambda\,\alpha)} A_1}$$

Moment Generating Function

If $Y\, has\, GTE - E(\alpha, \lambda, c, \theta)$ distribution then, the moment generating function (MGF) is given as;

$$K_y(t) = c\,\lambda\,\alpha \sum_{f,g=0}^{\infty} (-1)^f \binom{c-1}{f}\binom{c-1}{g} \left\{\frac{(1-\theta)}{\lambda\,\alpha(f+g+1)-t} + \frac{2\,\theta}{\lambda\,\alpha(f+g+2)-t}\right\} \tag{9}$$

3.3. The r^{th} moments about the mean

If $Y\, has\, GTE - E(\alpha, \lambda, c, \theta)$ distribution then, the r^{th} moments about the mean is given;

$$E(y-\mu)^r = c \sum_{f,g=0}^{\infty} \sum_{h=0}^{r} (-1)^{f+h} \binom{c-1}{f} \binom{c-1}{g} \binom{r}{h} \mu^h \times \left\{ \frac{(1-\theta)\Gamma(r-h+1)}{((f+g+1))^{r-h+1}(\lambda\,\alpha)^{r-h}} + \frac{2\,\theta\,\Gamma(r-h+1)}{((f+g+2))^{r-h+1}(\lambda\,\alpha)^{r-h}} \right\} \tag{10}$$

If $\mu = 0$, the result will give us the moment about the origin.

$$E(y^r) = c \sum_{f,g=0}^{\infty} (-1)^f \binom{c-1}{f} \binom{c-1}{g} \left\{ \frac{(1-\theta)\Gamma(r+1)}{((f+g+1))^{r+1}(\lambda\,\alpha)^r} + \frac{2\,\theta\,\Gamma(r+1)}{((f+g+2))^{r+1}(\lambda\,\alpha)^r} \right\}$$

In order to find the skewness and kurtosis, we have to find the expressions for r =1,2,3 and 4. The expressions are given below;

If r =1,

$$E(y-\mu) = c \sum_{f,g=0}^{\infty} \sum_{h=0}^{1} (-1)^{f+h} \binom{c-1}{f} \binom{c-1}{g} \binom{1}{h} \mu^h \times \left\{ \frac{(1-\theta)\Gamma(2-h)}{((f+g+1))^{2-h}(\lambda\,\alpha)^{1-h}} + \frac{2\,\theta\,\Gamma(2-h)}{((f+g+2))^{2-h}(\lambda\,\alpha)^{1-h}} \right\}$$

If r =2,

$$E(y-\mu)^2 = c \sum_{f,g=0}^{\infty} \sum_{h=0}^{2} (-1)^{f+h} \binom{c-1}{f} \binom{c-1}{g} \binom{1}{h} \mu^h \times \left\{ \frac{(1-\theta)\Gamma(3-h)}{((f+g+1))^{3-h}(\lambda\,\alpha)^{2-h}} + \frac{2\,\theta\,\Gamma(3-h)}{((f+g+2))^{3-h}(\lambda\,\alpha)^{2-h}} \right\}$$

If r =3,

$$E(y-\mu)^3 = c \sum_{f,g=0}^{\infty} \sum_{h=0}^{3} (-1)^{f+h} \binom{c-1}{f} \binom{c-1}{g} \binom{1}{h} \mu^h \times \left\{ \frac{(1-\theta)\Gamma(4-h)}{((f+g+1))^{4-h}(\lambda\,\alpha)^{3-h}} + \frac{2\,\theta\,\Gamma(4-h)}{((f+g+2))^{4-h}(\lambda\,\alpha)^{3-h}} \right\}$$

If r =4,

$$E(y-\mu)^4 = c \sum_{f,g=0}^{\infty} \sum_{h=0}^{4} (-1)^{f+h} \binom{c-1}{f} \binom{c-1}{g} \binom{1}{h} \mu^h \times \left\{ \frac{(1-\theta)\Gamma(5-h)}{((f+g+1))^{5-h}(\lambda\,\alpha)^{4-h}} + \frac{2\,\theta\,\Gamma(5-h)}{((f+g+2))^{5-h}(\lambda\,\alpha)^{4-h}} \right\}$$

The coefficient of skewness are kurtosis of the GTE-ED are respectively given as;

$$C\,S = \frac{E(y-\mu)^3}{\left(E(y-\mu)^2\right)^{\frac{3}{2}}}$$

$$C\,S = \frac{c \sum_{f,g=0}^{\infty} \sum_{h=0}^{3} (-1)^{f+h} \binom{c-1}{f} \binom{c-1}{g} \binom{1}{h} \mu^h \left\{ \frac{(1-\theta)\Gamma(4-h)}{((f+g+1))^{4-h}(\lambda\,\alpha)^{3-h}} + \frac{2\,\theta\,\Gamma(4-h)}{((f+g+2))^{4-h}(\lambda\,\alpha)^{3-h}} \right\}}{\left(c \sum_{f,g=0}^{\infty} \sum_{h=0}^{2} (-1)^{f+h} \binom{c-1}{f} \binom{c-1}{g} \binom{1}{h} \mu^h \left\{ \frac{(1-\theta)\Gamma(3-h)}{((f+g+1))^{3-h}(\lambda\,\alpha)^{2-h}} + \frac{2\,\theta\,\Gamma(3-h)}{((f+g+2))^{3-h}(\lambda\,\alpha)^{2-h}} \right\} \right)^{\frac{3}{2}}}$$

and

$$C\,K = \frac{E(y-\mu)^4}{\left(E(y-\mu)^2\right)^2}$$

$$CK = \frac{c \sum_{f,g=0}^{\infty} \sum_{h=0}^{4} (-1)^{f+h} \binom{c-1}{f} \binom{c-1}{g} \binom{1}{h} \mu^h \left\{ \frac{(1-\theta)\Gamma(5-h)}{((f+g+1))^{5-h}(\lambda\alpha)^{4-h}} + \frac{2\theta\Gamma(5-h)}{((f+g+2))^{5-h}(\lambda\alpha)^{4-h}} \right\}}{\left(c \sum_{f,g=0}^{\infty} \sum_{h=0}^{2} (-1)^{f+h} \binom{c-1}{f} \binom{c-1}{g} \binom{1}{h} \mu^h \left\{ \frac{(1-\theta)\Gamma(3-h)}{((f+g+1))^{3-h}(\lambda\alpha)^{2-h}} + \frac{2\theta\Gamma(3-h)}{((f+g+2))^{3-h}(\lambda\alpha)^{2-h}} \right\} \right)^2}$$

Table 2: *Some selected measures of $Y \sim GTE - E$ for some chosen values of the c and $\alpha = 2$, $\lambda = 0.3$, $\theta = 0.5$. The standard errors (SEs) in bracket, where τ_1 and τ_2 stands for the mean deviation about mean and the mean deviation about the median*

	Parameter (c)				
	1	2	3	4	5
Mean	1.2704 (0.1022)	1.9317 (0.1168)	2.3875 (0.1239)	2.7374 (0.1284)	3.0223 (0.1315)
Variance	2.1170 (0.5922)	2.7815 (0.6601)	3.1413 (0.6927)	3.3761 (0.7127)	3.5445 (0.7265)
Skewness	2.4648 (0.7892)	2.0447 (0.6535)	1.8681 (0.5996)	1.7658 (0.5696)	1.6971 (0.5501)
Kurtosis	12.1973 (7.7423)	9.5389 (5.6921)	8.5507 (4.9328)	8.0148 (4.5232)	7.6712 (4.2619)
τ_1	0.4023 (0.0408)	0.4793 (0.0425)	0.5168 (0.0433)	0.5401 (0.0438)	0.5564 (0.0441)
τ_2	0.3666 (0.0349)	0.4490 (0.0376)	0.4888 (0.0388)	0.5134 (0.0395)	0.5306 (0.04)

Table 3: *Some selected measures of $Y \sim GTE - E$ for some chosen values of the c and $\alpha = 2$, $\lambda = 0.3$, $\theta = -0.5$ The standard errors (SEs) in bracket, where τ_1 and τ_2 stands for the mean deviation about mean and the mean deviation about the median*

	Parameter (c)				
	1	2	3	4	5
Mean	2.1051 (0.1317)	3.0448 (0.1414)	3.6465 (0.1450)	4.0886 (0.1468)	4.4381 (0.1479)
Variance	3.5522 (0.7368)	4.1168 (0.7727)	4.3324 (0.7859)	4.4453 (0.7927)	4.5147 (0.7969)
Skewness	1.7785 (0.5441)	1.5157 (0.4921)	1.4258 (0.4758	1.3808 (0.4679)	1.3539 (0.4633)
Kurtosis	7.8311 (4.3142)	6.7415 (3.5522)	6.4140 (3.3160)	6.2586 (3.0221)	6.1681 (3.1351)
τ_1	0.5588 (0.0451)	0.06278 (0.0454)	0.6278 (0.0456)	0.6370 (0.0457)	0.6425 (0.0458)
τ_2	0.5286 (0.0409)	0.5860 (0.0418)	0.6060 (0.0422)	0.6162 (0.0424)	0.6223 (0.0425)

From tables 2 and 3, it can be deduced that as the value of the parameter (c) increases, the mean, variance, mean deviation about mean and the mean deviation about the median also increases. While the skewness and kurtosis decrease.

3.4. The Quantile function of the model

The quantile function can be defined mathematically as; $Q(u) = Inf\{y \in \Re : u \leq F(y)\}$ for which $0 < u < 1$. Since the function $F(y)$ is continuous and monotonically increasing, then we have $Y = F^{-1}(u)$.

Corollary 1. The quantile function of the GTE-ED is given as;

$$Q(u) = -\frac{1}{\alpha\lambda} \ln\left(\frac{\theta - 1 + \sqrt{(1+\theta)^2 - 4\theta u^{\frac{1}{c}}}}{2\theta}\right), \qquad 0 < u < 1 \tag{11}$$

Note that, when u=0.5 (11) gives the median.

The effect of the shape and Transmuted parameter were examined on the skewness and kurtosis and it was evaluated by using the relationship of Bowley (BS) and Moors (MK). Figures (a) and (b) shows the plot of Bowley (BS) and Moors (MK) for GTE-ED for fixed parameters (α and λ) respectively. The plot for the skewness shows a steady decrease as the parameter (c)increases while for parameter θ shows a steady increase to a minimum point before decreasing as its value increases. Again, the Kurtosis shows a steady decrease to a certain point and decreases as the parameter (c) increases while for parameter θ, shows a steady decrease as its value increases.

$Sk_B = \frac{Q(\frac{3}{4})-2Q(\frac{1}{2})+Q(\frac{1}{4})}{Q(\frac{3}{4})-Q(\frac{1}{4})}$ and $Ku_M = \frac{Q(\frac{7}{8})-Q(\frac{5}{8})+Q(\frac{3}{8})-Q(\frac{1}{8})}{Q(\frac{6}{8})-Q(\frac{2}{8})}$

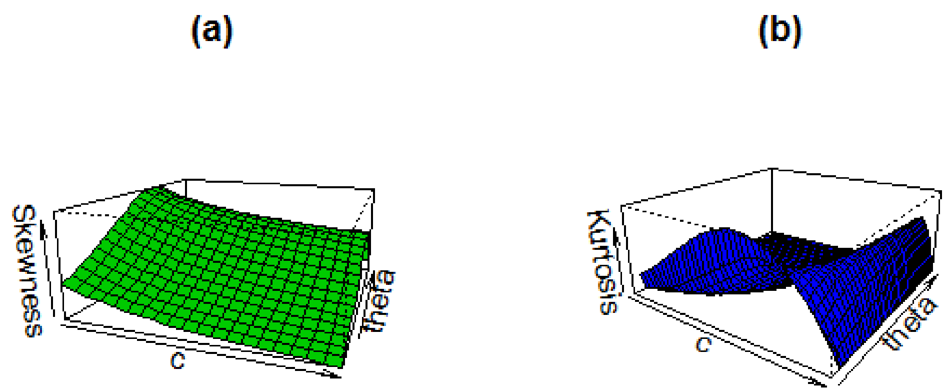

Figure 4: *3D diagram for Skewness and kurtosis*

3.5. Order Statistics of the GTE-ED

The general form of the density of the h^{th} order statistics for a given random samples $y_1, y_2, ..., y_n$ from the distribution function is obtained as; $b_{n,h}(y) = \frac{n!}{(h-1)!(n-h)!} b(y) B(y)^{h-1} (1 - B(y))^{n-h}$. Therefore, by substituting the resulting density as well as the distribution function of the GTE-ED we have;

$$\begin{aligned} b_{n,h}(y) = \tfrac{n!}{(h-1)!(n-h)!} & \left(c \left(\lambda\alpha (1-\theta) e^{-\alpha\lambda y} + 2\lambda\theta\alpha e^{-2\alpha\lambda y} \right) \left(\left(1 - e^{-\lambda\alpha y}\right) \left(1 + \theta e^{-\lambda\alpha y}\right) \right)^{c-1} \right) \\ & \times \left(\left(\left(1 - e^{-\lambda\alpha y}\right) \left(1 + \theta e^{-\lambda\alpha y}\right) \right)^{c} \right)^{h-1} \left(1 - \left(\left(1 - e^{-\lambda\alpha y}\right) \left(1 + \theta e^{-\lambda\alpha y}\right) \right)^{c} \right)^{n-h} \end{aligned} \tag{12}$$

The distribution of the minimum and maximum order statistics for the GTE-ED are respectively given as;

$$\begin{aligned} b_{n,1}(y) = n & \left(c \left(\lambda\alpha (1-\theta) e^{-\alpha\lambda y} + 2\lambda\theta\alpha e^{-2\alpha\lambda y} \right) \left(\left(1 - e^{-\lambda\alpha y}\right) \left(1 + \theta e^{-\lambda\alpha y}\right) \right)^{c-1} \right) \\ & \times \left(1 - \left(\left(1 - e^{-\lambda\alpha y}\right) \left(1 + \theta e^{-\lambda\alpha y}\right) \right)^{c} \right)^{n-1} \end{aligned}$$

and,

$$\begin{aligned} b_{n,n}(y) = n & \left(c \left(\lambda\alpha (1-\theta) e^{-\alpha\lambda y} + 2\lambda\theta\alpha e^{-2\alpha\lambda y} \right) \left(\left(1 - e^{-\lambda\alpha y}\right) \left(1 + \theta e^{-\lambda\alpha y}\right) \right)^{c-1} \right) \\ & \times \left(\left(\left(1 - e^{-\lambda\alpha y}\right) \left(1 + \theta e^{-\lambda\alpha y}\right) \right)^{c} \right)^{n-1} \end{aligned}$$

4. Estimation of the Parameters of GTE-ED

If the parameters of the GTE-ED are unknown, then the maximum likelihood estimates of the parameters are presented below, let $y_1, y_2, ..., y_n$ be the random sample of size (n) from the GTE-ED, then the log-likelihood function of (4) is obtained as;

$$\begin{aligned} ll(\Psi) = n \log \alpha + n \log \lambda + n \log c - \lambda\alpha \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} \log \left(1 - \theta + 2\theta e^{-\alpha\lambda y_i} \right) \\ + (c-1) \sum_{i=1}^{n} \log \left(1 + \theta e^{-\alpha\lambda y_i} - e^{-\alpha\lambda y_i} - \theta e^{-2\alpha\lambda y_i} \right) \end{aligned} \tag{13}$$

By differentiating the $ll(\Psi)$ with respect to the parameters. The following results were obtained;

$$\begin{aligned} \frac{\delta ll(\Psi)}{\delta\alpha} &= \frac{n}{\alpha} - \lambda \sum_{i=1}^{n} y_i - 2\lambda\theta \sum_{i=1}^{n} \frac{y_i e^{-\alpha\lambda y_i}}{\left(1-\theta+2\theta e^{-\alpha\lambda y_i}\right)} \\ &\quad + (c-1) \sum_{i=1}^{n} \frac{\lambda y_i e^{-\alpha\lambda y_i} - \theta\lambda y_i e^{-\alpha\lambda y_i} + 2\theta\lambda y_i e^{-2\alpha\lambda y_i}}{\left(1+\theta e^{-\alpha\lambda y_i} - e^{-\alpha\lambda y_i} - \theta e^{-2\alpha\lambda y_i}\right)} \\ \frac{\delta ll(\Psi)}{\delta\lambda} &= \frac{n}{\lambda} - \alpha \sum_{i=1}^{n} y_i - 2\alpha\theta \sum_{i=1}^{n} \frac{y_i e^{-\alpha\lambda y_i}}{\left(1-\theta+2\theta e^{-\alpha\lambda y_i}\right)} \\ &\quad + (c-1) \sum_{i=1}^{n} \frac{\alpha y_i e^{-\alpha\lambda y_i} - \theta\alpha y_i e^{-\alpha\lambda y_i} + 2\theta\alpha y_i e^{-2\alpha\lambda y_i}}{\left(1+\theta e^{-\alpha\lambda y_i} - e^{-\alpha\lambda y_i} - \theta e^{-2\alpha\lambda y_i}\right)} \\ \frac{\delta ll(\Psi)}{\delta\theta} &= \sum_{i=1}^{n} \frac{2e^{-\alpha\lambda y_i} - 1}{\left(1-\theta+2\theta e^{-\alpha\lambda y_i}\right)} + (c-1) \sum_{i=1}^{n} \frac{e^{-\alpha\lambda y_i} - e^{-2\alpha\lambda y_i}}{\left(1+\theta e^{-\alpha\lambda y_i} - e^{-\alpha\lambda y_i} - \theta e^{-2\alpha\lambda y_i}\right)} \\ \frac{\delta ll(\Psi)}{\delta c} &= \frac{n}{c} + \sum_{i=1}^{n} \log \left(1 + \theta e^{-\alpha\lambda y} - e^{-\alpha\lambda y} - \theta e^{-2\alpha\lambda y} \right) \end{aligned}$$

The ML Estimator $\widehat{\Phi} = \left(\widehat{\alpha}, \widehat{\lambda}, \widehat{c}, \widehat{\theta} \right)^{T}$ of the parameter vector is gotten by finding the solution of the set of nonlinear system of equations. The results will give the MLEs $\widehat{\alpha}, \widehat{\lambda}, \widehat{c}$ *and* $\widehat{\theta}$. We applied an optimization technique to numerically maximize the log- likelihood (LL) function given in (13).

5. Simulation Study

5.1. The Design

Here, a simulation study is conducted to assess the performance of the maximum likelihood estimation method as expressed above. So also, 10,000 random samples were generated for different sizes, n= 20, 50, 100, 200, 300, and 500 from GTE-ED. Furthermore, the estimates, Biases, MSEs, Confidence Interval (C. Is) at 95%, widths are evaluated. The steps are:

i Choose the initial values of the parameters and the sample size say (n).

ii Draw a random sample of size (n) from the GTE-ED using the quantile function given in (11).

iii Evaluate the estimates of the parameters using the approach of maximum likelihood.

iv Repeat steps i and ii for N=10,000 times to evaluate the bias, mean square error, 95% C.I and the interval width of the given estimates.

Table 4: *Results for the MLEs, Biases and MSEs, 95% C. Is, and Widths of the GTE-ED for* $\alpha = 3,\ \lambda = 2,\ c = 0.5, \theta = 0.5$

					C I		
Sample	Parameter	Esimate	Bias	MSE	LC	UC	width
	α	3.1657	0.1657	0.1334	2.9580	3.3734	0.4154
n=20	λ	2.2169	0.2169	0.2664	1.7868	2.6469	0.8601
	c	0.5379	0.0379	0.0345	0.4732	0.6026	0.1294
	θ	0.3583	-0.1417	0.3579	-0.3038	1.0203	1.13241
	α	3.1072	0.1072	0.0794	2.9741	3.2403	0.2662
n=50	λ	2.1433	0.1433	0.1680	1.8543	2.4324	0.5782
	c	0.5022	0.0022	0.0105	0.4817	0.5227	0.0410
	θ	0.3480	-0.1520	0.2873	-0.1697	0.8658	1.0355
	α	3.0888	0.0888	0.0640	2.9787	3.1989	0.2206
n=100	λ	2.1280	0.1280	0.1416	1.8827	2.3734	0.4907
	c	0.4952	-0.0048	0.0050	0.4854	0.5051	0.0196
	θ	0.3514	-0.1486	0.2227	-0.0419	0.7447	0.7866
	α	3.0670	0.0670	0.0510	2.9757	3.1582	0.1825
n=200	λ	2.0978	0.0978	0.1165	1.8883	2.3073	0.4190
	c	0.4943	-0.0057	0.0023	0.4898	0.4988	0.0090
	θ	0.3841	-0.1159	0.1552	0.1063	0.6620	0.5557
	α	3.0539	0.0539	0.0429	2.9754	3.1323	0.1568
n=300	λ	2.0787	0.0787	0.0962	1.9023	2.2551	0.3528
	c	0.4942	-0.0058	0.0015	0.4913	0.4972	0.0058
	θ	0.4035	-0.0965	0.1228	0.1811	0.6258	0.4448
	α	3.0430	0.0430	0.0346	2.9788	3.1072	0.1284
n=500	λ	2.0580	0.0580	0.0770	1.9135	2.2024	0.2888
	c	0.4.951	-0.0049	0.0008	0.4935	0.4967	0.0032
	θ	0.4267	-0.0733	0.0920	0.2569	0.5964	0.3395

The bias and MSE are respectively calculated as;

$B\left(\Psi_j\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\Psi}_j - \Psi_j\right)$ and $MSE\left(\Psi_j\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\Psi}_j - \Psi_j\right)^2$ where, $\widehat{\Psi}_j$ stands for the estimate of Ψ_j for $j = 1, ..., 4$.

Table 5: *Results for the MLEs, Biases and MSEs, 95% C. Is, and Widths of the GTE-ED for $\alpha = 0.3,\ \lambda = 0.5,\ c = 1,\ \theta = 0.7$*

					C I		
Sample	Parameter	Esimate	Bias	MSE	LC	UC	width
	α	0.3361	0.0361	0.0108	0.3175	0.3547	0.0372
n=20	λ	0.5095	0.0095	0.0126	0.4850	0.5340	0.0490
	c	1.1427	0.1427	0.2028	0.7852	1.5002	0.7150
	θ	0.6781	-0.0219	0.1180	0.4478	0.9084	0.9084
	α	0.3225	0.0225	0.0057	0.0057	0.3327	0.0204
n=50	λ	0.5003	0.0003	0.0007	0.4867	0.5140	0.0273
	c	1.0390	0.0390	0.0486	0.9467	1.1314	0.1847
	θ	0.6606	-0.0394	0.1028	0.4622	0.8590	0.3968
	α	0.3168	0.0168	0.0038	0.3098	0.3237	0.0139
n=100	λ	0.4993	-0.0007	0.0046	0.4903	0.5082	0.0180
	c	1.0111	0.0111	0.0211	0.9701	1.0523	0.0822
	θ	0.6523	-0.0477	0.0905	0.4795	0.8252	0.3458
	α	0.3101	0.0101	0.0026	0.3051	0.3150	0.0099
n=200	λ	0.5007	0.0007	0.0028	0.4952	0.4952	0.0110
	c	0.9992	-0.0008	0.0099	0.9798	1.0186	0.0389
	θ	0.6608	-0.0392	0.0705	0.5257	0.7960	0.2703
	α	0.3083	0.0083	0.0023	0.3040	0.3126	0.0086
n=300	λ	0.5016	0.0016	0.0022	0.4972	0.5060	0.0088
	c	0.9948	-0.0052	0.0067	0.9817	1.0080	0.0263
	θ	0.6599	-0.0401	0.0626	0.5403	0.7794	0.2390
	α	0.3067	0.0067	0.0019	0.3031	0.3103	0.0071
n=500	λ	0.5015	0.0015	0.0018	0.4979	0.5051	0.0072
	c	0.9935	-0.0065	0.0042	0.9854	1.0015	0.0162
	θ	0.6649	-0.0351	0.0522	0.5651	0.7647	0.1997

Interpretation of the results for tables 3 and 4:

a . The difference between the true and the estimated values of the parameters are relatively small.

b . As the sample size increases the estimates converge toward the true values of the parameters.

c . The interval widths, biases and MSEs decrease with an increase in sample size.

6. Applications

6.1. Datasets

The first data is on the remission times (in months) of a randomly selected (128) bladder cancer patients, which can be found in [15]. The second data was used by [16] and it represents the number of million revolutions before failure for each of the twenty-three ball bearings in the life tests. These datasets are used in order to check the flexibility of the proposed distribution over the competing distribution in the study.

Table 6: *Summary Statistics for the first dataset*

Min.	Mean	Variance	Max.	Skewness	Kurtosis
0.080	9.366	110.425	79.050	3.2866	18.4831

Table 7: *Summary Statistics for the second dataset*

Min.	Mean	Variance	Max.	Skewness	Kurtosis
17.88	72.23	1404.783	173.40	0.9419	3.4889

6.2. The Criteria

The generalized transmuted exponential-exponential (GTE-ED), transmuted exponential-exponential distribution (TE-ED), exponentiated exponential distribution (EED) and exponential distribution (ED) are compared using some goodness-of-fit statistics, including Akaike Information Criterion (AIC), Cramer-von Mises Criterion (W), Anderson-Darling Criterion (AD) and Kolmogorov Smirnov (KS). Furthermore, the model with the smallest values of these criteria indicates better fit. The R software (AdequacyModel package) is employed to evaluate these statistics.

Table 8: *Estimated parameters for the first data*

Model	$\widehat{\alpha}$	$\widehat{\lambda}$	$\widehat{\theta}$	$\widehat{c}$
GTE-E	0.8313	0.1007	0.7914	1.3506
TE-E	0.6401	0.0922	0.8898	-
EE	0.1199	-	-	1.2222
E	0.1066	-	-	-

Table 9: *Goodness-of-fit statistics for the first dataset*

Model	-LL	AIC	AD	W	KS
GTE-E	**410.8999**	**829.7998**	**0.3314**	**0.0523**	**0.0523**
TE-E	413.5223	833.0447	0.4945	0.0824	0.0725
EE	413.0901	830.1802	0.6705	0.1116	0.0778
E	414.3419	830.6841	0.7156	0.1192	0.0844

Tables 8 and 10 give the estimates of the parameters for the GTE-ED and the competing models in the study. The values of the computed goodness-of-fits statistics are given in tables 9 and 11. It was observed that GTE-ED has the lowest values of these statistics among the competing distributions in this study. Hence, the GTE-ED provides a better fit to the datasets.

Figures 5 and 6 show that the GTE-ED fits both the datasets well compared to the competing distributions.

Table 10: *Estimated parameters for the second data*

Model	$\widehat{\alpha}$	$\widehat{\lambda}$	$\widehat{\theta}$	$\widehat{c}$
GTE-E	0.1516	0.2095	0.1407	5.8194
TE-E	0.1053	0.2037	-0.9996	-
EE	0.0150	-	-	1.3535
E	0.0139	-	-	-

Table 11: *Goodness-of-fit statistics for the second dataset*

Model	-LL	AIC	AD	W	KS
GTE-E	**112.9714**	**233.9428**	**0.1868**	**0.0315**	**0.1037**
TE-E	116.0201	238.0402	0.2064	0.0367	0.2172
EE	118.8677	241.7355	0.2110	0.0377	0.2226
E	121.4366	244.8731	0.2157	0.0386	0.3072

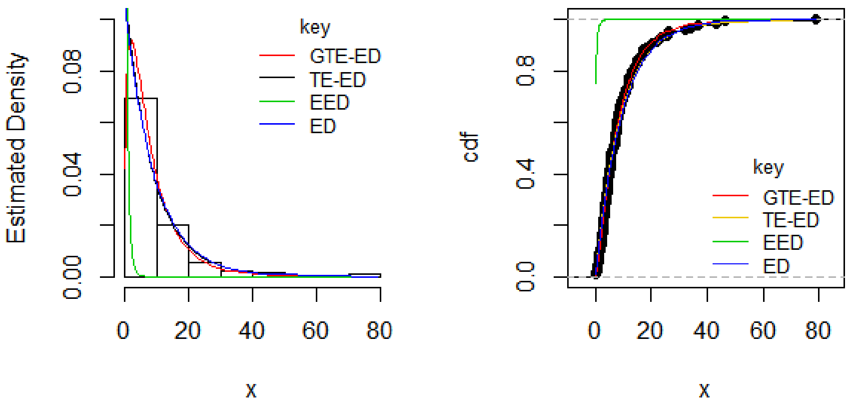

Figure 5: *Shows the estimated densities and ecdf for first data*

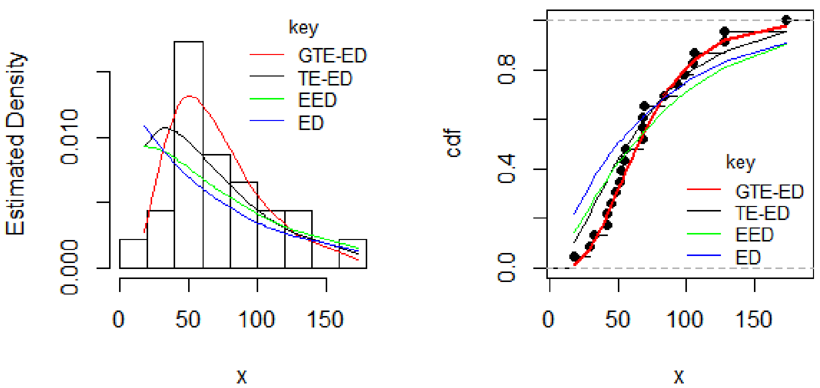

Figure 6: *Shows the estimated densities and ecdf for second data*

7. Conclusion

This research proposed an extension of the transmuted exponential-exponential distribution named the generalized transmuted exponential-exponential distribution. Expressions for some of its statistical properties, including the moments, moment generating function, limiting behaviour, and quantile function, were explicitly derived. A simulation study was conducted, and numerical values for some of the descriptive statistics were obtained and presented. The method of the maximum likelihood is adopted in estimating the unknown parameters of GTE-ED. A 95% confidence interval and interval widths together with biases, mean square errors (MSEs) of the mean estimates were equally presented on a table for different parameter values. An application to real datasets proved that the GTE-ED outperformed the competing distributions with lower values of the goodness-of-fit statistics used in this research.

Disclosure statement
On behalf of The authors, I declare that no potential conflict of interest was reported.
Funding
No funding was provided for the work.

References

[1] Tahir, M. H., & Cordeiro, G. M. (2016). *Journal of Statistical Distributions and Applications,* 3(1), 1-35.

[2] Bourguignon, M., Ghosh, I., & Cordeiro, G. M. (2016). General results for the transmuted family of distributions and new models. *Journal of Probability and Statistics*, Article ID 7208425, http://dx.doi.org/10.1155/2016/7208425

[3] Gupta, R. C., Gupta, P. L., & Gupta, R. D. (1998). Modeling failure time data by Lehman alternatives. *Communications in Statistics-Theory and methods*, 27(4), 887-904.

[4] Mudholkar, G. S., Srivastava, D. K., & Freimer, M. (1995). The exponentiated Weibull family: A reanalysis of the bus-motor-failure data.

[5] Mudholkar, G. S., & Hutson, A. D. (1996). The exponentiated Weibull family: some properties and a flood data application. *Communications in Statistics-Theory and Methods*, 25(12), 3059-3083.

[6] Gupta, R. D., & Kundu, D. (1999). Theory and methods: Generalized exponential distributions. *Australian and New Zealand Journal of Statistics,* 41(2), 173-188.

[7] Nadarajah, S., & Kotz, S. (2006). The exponentiated type distributions. *Acta Applicandae Mathematicae*, 92(2), 97-111.

[8] Silva, R. B., Barreto-Souza, W., & Cordeiro, G. M. (2010). A new distribution with decreasing, increasing and upside-down bathtub failure rate. *Computational Statistics and Data Analysis*, 54(4), 935-944.

[9] Lemonte, A. J., & Cordeiro, G. M. (2011). . The exponentiated generalized inverse Gaussian distribution. *Statistics and Probability Letters,* 81(4), 506-517.

[10] Afify, A. Z., Nofal, Z. M. & Ebraheim, A. E. D. (2015).Exponentiated transmuted generalized Rayleigh Distribution. *Pakistan Journal of Statistics and Operation Research,* 11(1), 115-134.

[11] Cordeiro, G. M., Ortega, E. M., & da Cunha, D. C. (2013). The exponentiated generalized class of distributions. *Journal of data science*, 11(1), 1-27.

[12] Abba, B., Mohammed, A. S., & Musa, A. G. (2020). On the properties and mles of generalized odd generalized exponential- exponential distribution. *Fudma Journal of Sciences*,4(4), 155 - 165.

[13] Mohammed, A. S., & Ugwuowo, F. I. (2021). The Transmuted Exponential-Exponential Distribution with Application to Breast Cancer Data. *Pakistan Journal of Statistics*, 37(1), 1-18.

[14] Mohammed, A. S., & Ugwuowo, F. I. (2021).On Transmuted Exponential-Topp Leon Distribution with Monotonic and Non-Monotonic Hazard Rates and its Applications. *Reliability: Theory & Applications*, 16(4 (65)), 197-209.

[15] Lee, E. T. & Wang, J. W. (2003). Statistical methods for survival data analysis (3rd Edition), John Wiley and Sons, New York, USA, 535 Pages, ISBN 0-471-36997-7.

[16] Lawless, J.F (1982) Statistical models and methods for lifetime data. John Wiley and Sons, New York, USA.