ANALYSIS OF MARKOVIAN BATCH SERVICE QUEUE WITH FEEDBACK AND SECOND OPTIONAL SERVICE

P. Vijaya Laxmi*,[1], Hasan A.B.D. Qrewi[2] and Andwilile A. George[3]

Department of Applied Mathematics, Andhra University,
Visakhapatnam, India.
Email: [1]vijayalaxmiau@gmail.com, [2]hasanqrewi@gmail.com,
[3]gandwilile@gmail.com
*Corresponding author.

Abstract

The aim of this paper is to analyze a single server batch service queue model with feedback and second optional service under a transient and steady state environment. The server provides the first essential service (FES) to all customers who arrive at the system and the second optional service (SOS) to those who need it. After completion of FES, if the customer is not satisfied with the service, he may rejoin the queue or may opt for SOS or exit the system with a particular probability. The service times of both FES and SOS follow exponential distribution. We use the probability generating function and the Laplace transform expression to obtain probabilities in the transient state after inverting Laplace transforms into the time domain. Also, we apply the Tauberian property in the Laplace transform expression to get the steady state probabilities. Finally, some performance measures and numerical results are provided.

Keywords: Batch service queue; Transient state; Feedback; First essential service; Second optional service.

I. Introduction

In queue theory, the customers are served one by one or served in batches whose sizes are fixed or variable in size. When the customers are served, they will depart, but when the customer does not seem satisfied they will return to the queue again to be served; this situation is called feedback. Queueing model with feedback have been studied by many researchers such as [1] who is the first to introduce the concept of feedback mechanism in queues which includes the probability of the customer to back the counter and take the service. Later, [2] investigated a single server with feedback wherein the queue is formed in two categories, one is formed in a waiting room and the other is formed in the service room and obtained the queue size, waiting time, and total time spent in the system. [3] presented a single server model with limiting behavior of the waiting time process and having a certain feedback property. Lemoine [3] classified the queue into two types: the primary queue where customers receive a maximum of time units of service. The secondary queue is formed from the customers who are not satisfied with the service in the primary queue. Those customers are attended to only when all customers who enter the system before them have departed and when the primary waiting room is empty.

[4] discussed $M/G/1/\infty$ queueing system with instantaneous Bernoulli feedback. They obtained the time-stationary distribution of the number of customers in the system at an arbitrary epoch. The queueing system with delayed feedback has been studied by [5]. They considered two servers; a lower server with a general service and an upper server with an exponential service. The decision to feedback or not depends on the queue length at the two servers. Other existing works on feedback are found in [7], [10], [9], [11], [12], [14], [15], [16], [17], etc.

In many real service systems, customers want both essential and optional services provided by the server. More precisely, we may consider a system where a single server's service is classified

into two phases. The first phase is required for all customers and only some of them are routed to the second phase service. [6] studied a single server batch arrival and arbitrary service time distribution queue system with second multi-optional service and a finite number of immediate Bernoulli feedback. They provided a steady state analysis of the model, including the asymptotic behavior under a high rate of retrials.
[8] studied the priority retrial queue with immediate Bernoulli feedback and second multi-optional service. In this case, the customers' feedback after completing both phases of services. They use the embedded Markov chain technique and probability generating function to obtain the queue size and orbit size. [13] and [17] presented the queue system with batch arrival and multi-optional service with feedback. They consider two kinds of independent customers' arrival: positive and negative arrival. Positive arrivals in batches with the Poisson process and negative arrived singly with the Poisson process. The arrival of a negative customer removes the positive customer in service from the system and makes down the server.
The batch service queues have potential applications in various areas, including manufacturing, production, computer networks, cargo loading and unloading at a harbor, etc. In this situation, the number of items is processed in batches with a limit on the number of items taken at a time for processing. A number of researchers have made substantial contributions to batch service models. [19] investigated a batch service retrial system with feedback in a steady state. They considered two kinds of the arrival of the customers; the positive customer who is served in batch and the negative customer who arrives in the system and removes the positive customer. Some additional works on queuing with batch service are found in [20], [21], etc. One can study a queue system in two states viz., : transient and steady state. However, most of the literature works involved steady-state only. Steady state results are inappropriate in cases where the time horizon of operation is finite. In this situation, we need time-dependent (transient state) to analyze the system behavior by tracking down the system operation at any instant of time. The transient state has been widely studied in [23], [22], and the references therein.
This paper extends the work of [18] by including the feedback policy. The inclusion of feedback, batch service and SOS makes the model more adaptable. This motivates us to explore the Markovian batch service queue with feedback and SOS under transient and steady state environments. The rest of the paper is organized in the following way: Section 2 contains a discussion of the model as well as the mathematical formulation of differential difference equations. We use the Laplace transform and probability generating functions with Rouche's theorem in Section 3 to develop the system transient state equations. Section 4 presents the steady-state analysis, followed by Section 5 with performance measures. Section 6 presents the numerical investigation and conclusions are made in Section 7.

II. Mathematical Formulation and Description of the Model

Consider a feedback of single batch service $M/M^{[b]}/1$ queue system with SOS, in which the arrival of customers follows a Poison process with parameter λ. During FES and SOS the service times are distributed exponentially with rate μ_1 and μ_2, respectively. The services are rendered in batches that does not exceed the maximum capacity b. This means if the server finds the units that are equal or fewer than b in the waiting line, then he serves them all in the batch. Otherwise, if he sees in excess of b units in the waiting line, he takes a batch of b on a first-come, first-served, and the other units remain waiting in the queue. All arriving units required FES, and after a batch of units complete the FES, the batch (on the same server) may opt for SOS with the probability r_0 or leave the system with probability r_1. Further, if the batch is unsatisfied with the service in FES, they rejoin the queue (feedback) with probability ($r_2 = 1 - r_0 - r_1$).

Model Formulation of Differential Difference Equations

The single batch service queue with SOS can be modeled by continuous time of two dimensional Markov process $\{(K(t), W(t)); t \geq 0\}$; where

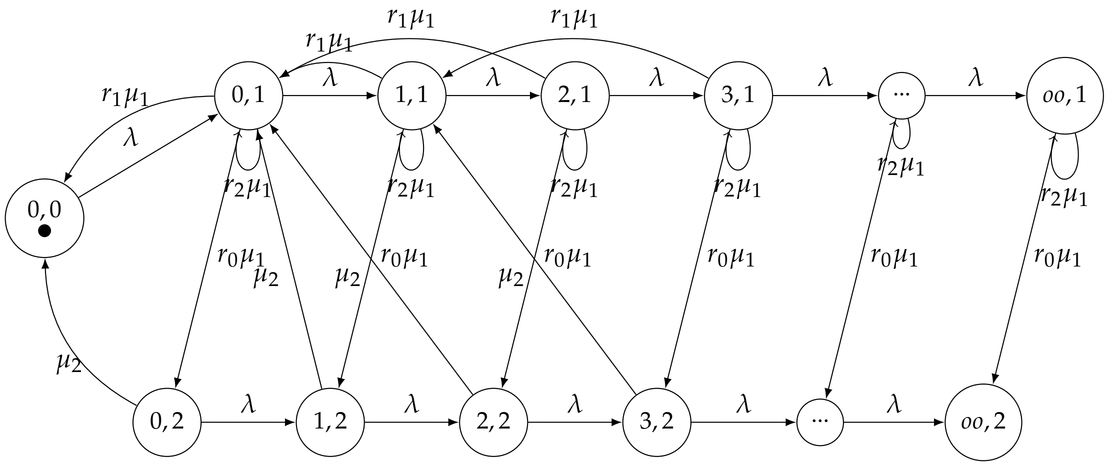

Figure 1: *Transition State diagram for b=2*

$K(t)$ is number of units in a line at time t,
$W(t)$ is server state at time t with

$$W(t) = \begin{cases} 1, & \text{FES providing by server,} \\ 2, & \text{SOS providing by server.} \end{cases}$$

The state-space of a Markov process is given as follows:

$$\Omega = \{(m, j); \quad m \geq 0; \ j = 1, 2\}.$$

The probabilities in transient state are given by

$$P_{m,j}(t) = Pr\{K(t) = m, \quad W(t) = j\}; \quad m \geq 0, \quad j = 1, 2$$

where the probability in transient state is $P_{m,j}(t)$ when m units in the waiting line at time t and the server is rendering FES ($j = 1$) or SOS ($j = 2$).
$O(t)$ is the probability in transient state when the waiting line is empty at time t and the server is idle. The differential difference equations for our model using Markov theory are as follows:

$$O'(t) = -\lambda O(t) + r_1\mu_1 P_{0,1}(t) + \mu_2 P_{0,2}(t), \tag{1}$$

$$P'_{0,1}(t) = -(\lambda + r_0\mu_1 + r_1\mu_1)P_{0,1}(t) + \lambda O(t) + r_1\mu_1 \sum_{i=1}^{b} P_{i,1}(t) + \mu_2 \sum_{i=1}^{b} P_{i,2}(t), \tag{2}$$

$$P'_{m,1}(t) = -(\lambda + r_0\mu_1 + r_1\mu_1)P_{m,1}(t) + \lambda P_{m-1,1}(t) + r_1\mu_1 P_{m+b,1}(t) + \mu_2 P_{m+b,2}(t), \quad m \geq 1, \tag{3}$$

$$P'_{0,2}(t) = -(\lambda + \mu_2)P_{0,2}(t) + r_0\mu_1 P_{0,1}(t), \tag{4}$$

$$P'_{m,2}(t) = -(\lambda + \mu_2)P_{m,2}(t) + \lambda P_{m-1,2}(t) + r_0\mu_1 P_{m,1}(t), \quad m \geq 1. \tag{5}$$

III. The Model's Transient Solution

In this part, the Laplace transform (L.T) and probability generating function are employed to get the transient probability of the number of units in the waiting line during server idle and busy. We consider that time has been calculated from the moment that the server has taken a batch for service, without leaving none in the queue. i.e, $P_{0,1}(0) = 1$. Let us denote the Laplace transform for $O(t), P_{m,1}(t)$ $P_{m,2}(t)$ as $O^*(s), P^*_{m,1}(s)$ and $P^*_{m,2}(s)$, respectively. Taking L.T of equations (1), (2), (3), (4), (5), we get

$$(s+\lambda)O^*(s) = r_1\mu_1 P^*_{0,1}(s) + \mu_2 P^*_{0,2}(s), \tag{6}$$

$$(s+\lambda+r_0\mu_1+r_1\mu_1)P^*_{0,1}(s) = 1 + \lambda O^*(s) + r_1\mu_1\sum_{i=1}^{b} P^*_{i,1}(s) + \mu_2\sum_{i=1}^{b} P^*_{i,2}(s), \tag{7}$$

$$(s+\lambda+r_0\mu_1+r_1\mu_1)P^*_{m,1}(s) = \lambda P^*_{m-1,1}(s) + r_1\mu_1 P^*_{m+b,1}(s) + \mu_2 P^*_{m+b,2}(s), \quad m \geq 1, \tag{8}$$

$$(s+\lambda+\mu_2)P^*_{0,2}(s) = r_0\mu_1 P^*_{0,1}(s), \tag{9}$$

$$(s+\lambda+\mu_2)P^*_{m,2}(s) = \lambda P^*_{m-1,2}(s) + r_0\mu_1 P^*_{m,1}(s), \quad m \geq 1. \tag{10}$$

The probability generating functions are defined as:

$$P_1(s,z) = \sum_{m=0}^{\infty} P^*_{m,1}(s)z^m, \quad P_2(s,z) = \sum_{m=0}^{\infty} P^*_{m,2}(s)z^m.$$

Multiplying equations (7) and (8) by z^m then summing over $m = 0$ to $m = \infty$, adding to (6) and re-arranging the terms, we have

$$P_1(s,z) = \frac{z^b(sO^*(s)-1) + (1-z^b)\left(r_1\mu_1\sum_{m=0}^{b-1} P^*_{m,1}(s)z^m + \mu_2\sum_{m=0}^{b-1} P^*_{m,2}(s)z^m\right) - \mu_2 P_2(s,z)}{\lambda z^{b+1} - (s+\lambda+\mu_1)z^b + r_1\mu_1}. \tag{11}$$

Similarly, from equations (9) and (10), we get

$$P_2(s,z) = \frac{-r_0\mu_1 P_1(s,z)}{\lambda z - (s+\lambda+\mu_2)}. \tag{12}$$

Substituting equation (12) in equation (11), we have

$$P_1(s,z) = \frac{\left[z^b(sO^*(s)-1) + (1-z^b)\left[r_1\mu_1\sum_{m=0}^{b-1} P^*_{m,1}(s)z^m + \mu_2\sum_{m=0}^{b-1} P^*_{m,2}(s)z^m\right]\right](\lambda z - (s+\lambda+\mu_2))}{\lambda^2 z^{b+2} - \lambda(2s+2\lambda+r_0\mu_1+r_1\mu_2+\mu_2)z^{b+1} + (s+\lambda++r_0\mu_1+r_1\mu_2)(s+\lambda+\mu_2)z^b + \lambda r_1\mu_1 z - (s+\lambda+\mu_2)r_1\mu_1 - r_0\mu_1\mu_2}. \tag{13}$$

The expression for $P_1(s,z)$ has the characteristic of converging within the unit circle. We can see that the denominator of $P_1(s,z)$ has exactly $b+2$ zeros. Applying the Rouche's theorem for denominator of $P_1(s,z)$ to find the number of zeros on and within the unit circle of the analytic function, we observe that b of these roots lie inside or on the unit circle, one of zeros is $z = 1$ and others $b-1$, lies inside the circle and must agree with zeros of the numerator of $P_1(s,z)$ to be converged. As a result, one zero in the denominator of $P_1(s,z)$ is canceled by $P_1(s,z)$ of the numerator, so that the two remaining zeros of the denominator are found outside the unit circle and let us consider them as z_0 and z_1. Since two polynomial differ by at most a multiplicative function (constant), let us call it $A(s)$ and is independent of z. Therefore, equation (13) can be written as

$$P_1(s,z) = \frac{(\lambda z - (s+\lambda+\mu_2))(1-z^b)A(s)}{(z-1)(z-z_0)(z-z_1)}. \tag{14}$$

Applying the rule of L'Hospital and setting $z = 1$ in (14), we obtain

$$P_1(s,1) = \frac{(s+\mu_2)bA(s)}{(1-z_0)(1-z_1)}, \tag{15}$$

Taking $z = 1$ in (12), we get

$$P_2(s,1) = \frac{r_0\mu_1 P_1(s,1)}{(s+\mu_2)}. \tag{16}$$

Applying the normalization condition

$$P_1(s,1) + P_2(s,1) + O^*(s) = \frac{1}{s},$$

we get

$$P_1(s,1) = \frac{(1 - sO^*(s))(s + \mu_2)}{s(s + r_0\mu_1 + \mu_2)}. \tag{17}$$

Using (15) and (17), we can determined the function $A(s)$ as

$$A(s) = \frac{(1 - sO^*(s))(1 - z_0)(1 - z_1)}{s(s + r_0\mu_1 + \mu_2)b}. \tag{18}$$

Substituting (18) into (14), we have

$$P_1(s,z) = \frac{(1 - sO^*(s))(1 - z_0)(1 - z_1)(\lambda z - (s + \lambda + \mu_2))(1 - z^b)}{s(s + r_0\mu_1 + \mu_2)b(z - 1)(z - z_0)(z - z_1)}. \tag{19}$$

When $z = 0$, equation (19) becomes

$$P^*_{0,1}(s) = \frac{(1 - sO^*(s))(q_0 - 1)(q_1 - 1)(s + \lambda + \mu_2)}{s(s + r_0\mu_1 + \mu_2)b}, \tag{20}$$

where $q_0 = \frac{1}{z_0}$, $q_1 = \frac{1}{z_1}$. Note that $P^*_{0,1}(s)$ is the L.T of the probability of empty queue and server providing FES.

From equation (12), when $z = 0$ and using (20), we have

$$P^*_{0,2}(s) = \frac{r_0\mu_1(1 - sO^*(s))(q_0 - 1)(q_1 - 1)}{s(s + r_0\mu_1 + \mu_2)b}. \tag{21}$$

$P^*_{0,2}(s)$ is the L.T of the probability of empty queue and server providing SOS.

Using (6), (20) and equation (21), we can determine $O^*(s)$ as below:

$$O^*(s) = \frac{r_1\mu_1(q_0 - 1)(q_1 - 1)(s + \lambda + \mu_2)r_0\mu_1\mu_2(q_0 - 1)(q_1 - 1)}{s\left[(s + \lambda)(s + r_0\mu_1 + \mu_2)b + (r_1\mu_1(s + \lambda + \mu_2) + r_0\mu_1\mu_2)(q_0 - 1)(q_1 - 1)\right]}. \tag{22}$$

Equation (22) indicates the Laplace transform of the state probability that the server is idle and the queue is empty.

IV. Steady State Analysis

Using the Tauberian property, we get the closed form expressions of the stationary probability for the number of items in the waiting line while the server is idle or active in FES and SOS. We define the stationary probabilities.

$$O = \lim_{t\to\infty} O(t) = \lim_{s\to 0} sO^*(s), \tag{23}$$

$$P_{m,1} = \lim_{t\to\infty} P_{m,1}(t) = \lim_{s\to 0} sP^*_{m,1}(s), \tag{24}$$

$$P_{m,2} = \lim_{t\to\infty} P_{m,2}(t) = \lim_{s\to 0} sP^*_{m,2}(s). \tag{25}$$

Assuming that the steady state probabilities exist, the equations (20), (21) and (22) are, respectively given by

$$P_{0,1} = \frac{(1 - O)(q_0 - 1)(q_1 - 1)(\lambda + \mu_2)}{(r_0\mu_1 + \mu_2)b},$$

$$P_{0,2} = \frac{r_0\mu_1(1 - O)(q_0 - 1)(q_1 - 1)}{(r_0\mu_1 + \mu_2)b},$$

$$O = \frac{r_1\mu_1(q_0 - 1)(q_1 - 1)(\lambda + \mu_2) + r_0\mu_1\mu_2(q_0 - 1)(q_1 - 1)}{\lambda(r_0\mu_1 + \mu_2)b + (r_1\mu_1(\lambda + \mu_2) + r_0\mu_1\mu_2)(q_0 - 1)(q_1 - 1)}.$$

As a special case, if $r_2 = 0$, the model reduces to $M/M^b/1$ queueing system with second optional service [18].

V. Performance Measures

The measures of performance are key aspects of queueing models as they demonstrate the models efficient and effective index. In this section, we present the probability when the server is active with FES or SOS and the anticipated number of units in the waiting line during the server busy with FES or SOS.

I. The Measures of Performance in Transient State

The probability while the server is active with FES

$$P[FES](s) = \sum_{m=0}^{\infty} P^*_{m,1}(s) = \frac{(1 - sO^*(s))(s + \mu_2)}{s(s + r_0\mu_1 + \mu_2)}. \tag{26}$$

Similarly, the probability while the server is active with SOS

$$P[SOS](s) = \sum_{m=0}^{\infty} P^*_{m,2}(s) = \frac{r_0\mu_1(1 - sO^*(s))}{s(s + r_0\mu_1 + \mu_2)}. \tag{27}$$

When the server is active with FES or SOS, the probability is

$$P_b(s) = \sum_{m=0}^{\infty} P^*_{m,1}(s) + \sum_{m=0}^{\infty} P^*_{m,2}(s). \tag{28}$$

When the server is active with FES the anticipated number of items in the waiting line is

$$E[FES](s) = \sum_{m=0}^{\infty} mP^*_{m,1}(s).$$

Taking derivative of (19) with respect to z, putting $z = 1$, then applying the rule of L'Hospital, the expected number of units in the waiting line when the server is active with FES is obtained as

$$\sum_{m=0}^{\infty} mP^*_{m,1}(s) = \frac{[[((s+\mu_2)b(b-1) - 2\lambda b](q_0-1)(q_1-1) - [(s+\mu_2)b(4q_0q_1 - 2(q_0+q_1))]] \, (1 - sO^*(s))}{2b\,(s(s + r_0\mu_1 + \mu_2)(q_0-1)(q_1-1))}. \tag{29}$$

Similarly, from equations (12) and (29), the anticipated number of units in the waiting line when the server is active with SOS is given by

$$\begin{aligned} E[SOS](s) &= \sum_{m=0}^{\infty} mP^*_{m,2}(s) \\ &= \frac{[[(b(b-1) - 2\lambda b](q_0-1)(q_1-1) - [b(4q_0q_1 - 2(q_0+q_1))]] \, r_0\mu_1(1 - sO^*(s))}{2b\,(s(s + r_0\mu_1 + \mu_2)(q_0-1)(q_1-1))} \\ &+ \frac{\lambda r_0\mu_1(1 - sO^*(s))}{s(s+\mu_2)(s + r_0\mu_1 + \mu_2)}. \end{aligned} \tag{30}$$

The overall queue length is given by

$$L_q(s) = \sum_{m=1}^{\infty} mP_{m,1}(s) + \sum_{m=1}^{\infty} mP_{m,2}(s) = E[FES](s) + E[SOS](s). \tag{31}$$

The overall amount of time a unit spends in the waiting line is determined by

$$W_q(s) = \frac{\sum_{m=1}^{\infty} mP_{m,1}(s) + \sum_{m=1}^{\infty} mP_{m,2}(s)}{\lambda} = \frac{E[FES](s) + E[SOS](s)}{\lambda}. \tag{32}$$

II. The Measures of Performance in Steady State

Suppose the limit of (23), (24) and (25) exist, the steady state quantities from (26) to (32), respectively are denoted by

$$
\begin{aligned}
P[FES] &= \sum_{n=0}^{\infty} P_{m,1} = \frac{(1-O)\mu_2}{r_0\mu_1+\mu_2}, \\
P[SOS] &= \sum_{m=0}^{\infty} P_{m,2} = \frac{r_0\mu_1(1-O)}{r_0\mu_1+\mu_2}, \\
P_b &= \sum_{m=0}^{\infty} P_{m,1} + \sum_{m=0}^{\infty} P_{m,2}, \\
\sum_{m=0}^{\infty} mP_{m,1} &= \frac{[[\mu_2 b(b-1)-2\lambda b](r_0-1)(r_1-1)-[\mu_2 b(4r_0r_1-2(r_0+r_1))]]\,(1-O)}{2b\,(r_0\mu_1+\mu_2)(r_0-1)(r_1-1))}, \\
\sum_{m=0}^{\infty} mP_{m,2} &= \frac{[[b(b-1)-2\lambda b](q_0-1)(q_1-1)-[b(4q_0q_1-2(q_0+q_1))]]\,r_0\mu_1(1-O)}{2b\,(\mu_2(r_0\mu_1+\mu_2)(q_0-1)(q_1-1))} \\
&\quad + \frac{\lambda r_0\mu_1(1-O)}{\mu_2(r_0\mu_1+\mu_2)}, \\
L_q &= \sum_{m=0}^{\infty} mP_{m,1} + \sum_{m=0}^{\infty} mP_{m,2} = E[FES]+E[SOS], \\
W_q &= \frac{\sum_{m=1}^{\infty} mP_{m,1} + \sum_{m=1}^{\infty} mP_{m,2}}{\lambda} = \frac{E[FES]+E[SOS]}{\lambda}.
\end{aligned}
\tag{33}
$$

VI. Numerical Results and Discussion

In this part, the numerical examples are presented after inverting L.T of equations (20), (21), (22) and (26) to (32) into time domain with the help of Mathematica software. Since an inverted L.T expressions are too long, therefore, we compute the model numerically by using the arbitrary model parameters as $\lambda = 2,\ \mu_1 = 4.5,\ \mu_2 = 3.5,\ r_0 = 0.36,\ b = 5,\ r_1 = 0.4$.
The effect of these parameters on probability and other performance measure with respect to time are shown in terms of graphs and tables.

Table 1: *The effect of r_0 on r_1, L_qFES, L_qSOS and L_q with $\lambda = 2,\ \mu_1 = 4.5,\ \mu_2 = 3.5,\ r_2 = 0.1,\ b = 5$.*

r_0	r_1	$LqFES$	$LqSOS$	L_q
0.1	0.8	1.61413	0.136298	1.75042
0.2	0.7	1.52905	0.258306	1.78736
0.3	0.6	1.44568	0.366441	1.81213
0.4	0.5	1.36723	0.462234	1.82947
0.5	0.4	1.29436	0.54716	1.84152
0.6	0.3	1.22737	0.622792	1.85016
0.7	0.2	1.16602	0.69048	1.8565
0.8	0.1	1.10997	0.751454	1.86143

Table 1 depicts the effect of the probability of opting for SOS (r_0) on the expected queue length (L_q). We observe that for fixed feedback probability (r_2), an increase in r_0 leads to a decrease in the probability of the departure (r_1), resulting in decrease of L_qFES while L_qSOS and overall L_q increases. This is because as r_0 increases, more customers attend for SOS, resulting in increasing the L_q in SOS and overall queue size, as we expect.

Table 2: *Effect of feedback probability (r_2) on r_0, L_qFES, L_qSOS and L_q with $\lambda = 2$, $\mu_1 = 4.5$, $\mu_2 = 3.5$, $r_1 = 0.2$, $b = 5$*

r_2	r_0	$LqFES$	$LqSOS$	L_q
0.2	0.6	1.29082	0.663747	1.95457
0.3	0.5	1.44771	0.630166	2.07788
0.4	0.4	1.65249	0.586778	2.23927
0.5	0.3	1.93281	0.527648	2.46045
0.6	0.2	2.34909	0.441877	2.79096
0.7	0.1	3.05593	0.30109	3.35702

Table 2 shows the effect of feedback probability (r_2) on the expected queue length. For the fixed probability of the departure (r_1), as r_2 increases, the probability of opting for SOS (r_0) decreases, which leads to decrease L_qSOS and opposite trend observed in L_qFES and overall L_q. This implies that as r_2 increases more customers are not satisfied with service (FES). Therefore, the customers feedback by joining the queue tends to increase L_qFES and overall queue size.

Table 3: *The effect of r_1 on feedback probability r_2, L_qFES, L_qSOS and L_q with $\lambda = 2$, $\mu_1 = 4.5$, $\mu_2 = 3.5$, $r_0 = 0.1$, $b = 5$*

r_1	r_2	$LqFES$	$LqSOS$	L_q
0.1	0.8	1.61413	0.136298	1.75042
0.2	0.7	1.52905	0.258306	1.78736
0.3	0.6	1.44568	0.366441	1.81213
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0.5	0.4	1.29436	0.54716	1.84152
0.6	0.3	1.22737	0.622792	1.85016
0.7	0.2	1.16602	0.69048	1.8565
0.8	0.1	1.10997	0.751454	1.86143

From Table 3, for the fixed probability of opting for SOS r_0, as r_1 increase, r_2 decrease, resulting in decreasing L_qFES, L_qSOS and overall L_q, which agrees with our intuition.

Table 4: *The effect of μ_1 on the expected queue length with different values of μ_2 and $r_0 = 0.36, r_1 = 0.4$, $\lambda = 3.8$.*

μ_1	$\mu_2 = 1.8$	$\mu_2 = 2$	$\mu_2 = 2.2$
2.4	6.54206	6.09142	5.7638
2.8	5.71485	5.31505	5.02568
3.2	5.23358	4.85613	4.57894
3.6	4.92135	4.55096	4.27954
4.0	4.7030 8	4.33623	4.06484
4.4	4.54338	4.17542	3.90165

The impact of μ_1 on the expected queue length (L_q) with different values of μ_2 is shown in Table 4. Here L_q deceases as μ_1 increases, as we expect. Furthermore, for fixed μ_1, as μ_2 increases, L_q decreases. The reason is that, by increasing μ_1(μ_2), customers are serviced faster and results in reducing the length of the queue.

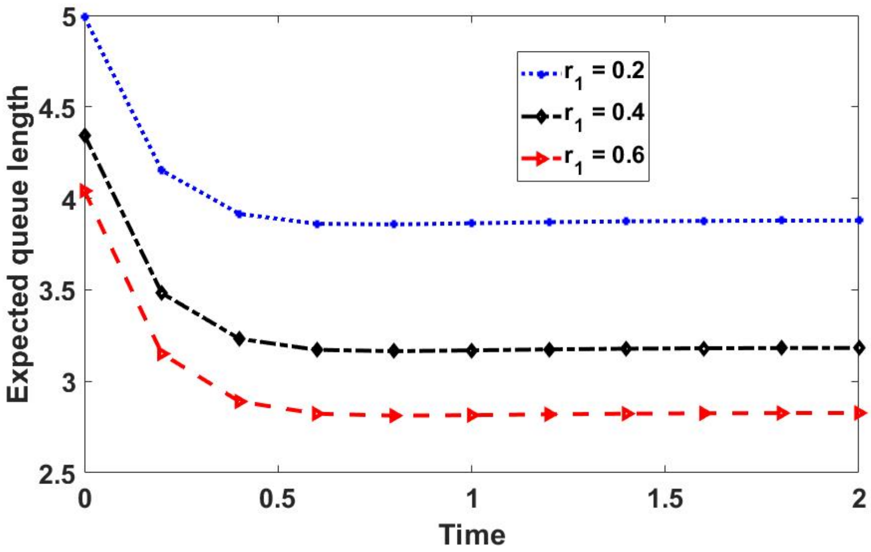

Figure 2: *The effect of departure probability r_1 on the L_q*

Figure 2 shows the effect of departure probability r_1 after completing FES on the expected queue (L_q). We observe that L_q decreases with time until it reaches a steady state. Further, we notice that as r_1 increases , L_q decreases. The reason is that more customers depart from the system after completion of FES, hence reducing L_q.

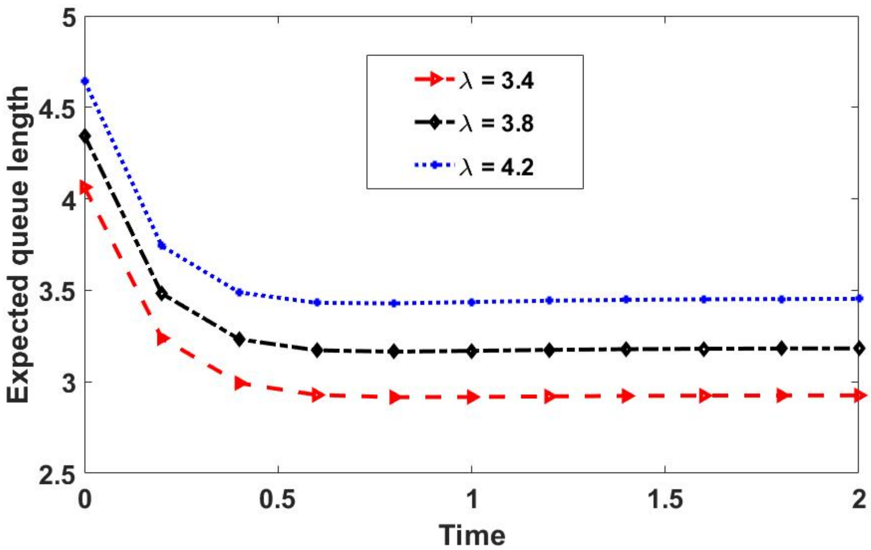

Figure 3: *The effect of λ on the expected L_q as time progresses*

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Figure 3 display the effect of the rate of arrival (λ) on the expected queue length (L_q). As time progresses, L_q decreases until it attains steady state. Moreover, as λ increases, more customers enter the queue, resulting in increase in L_q.

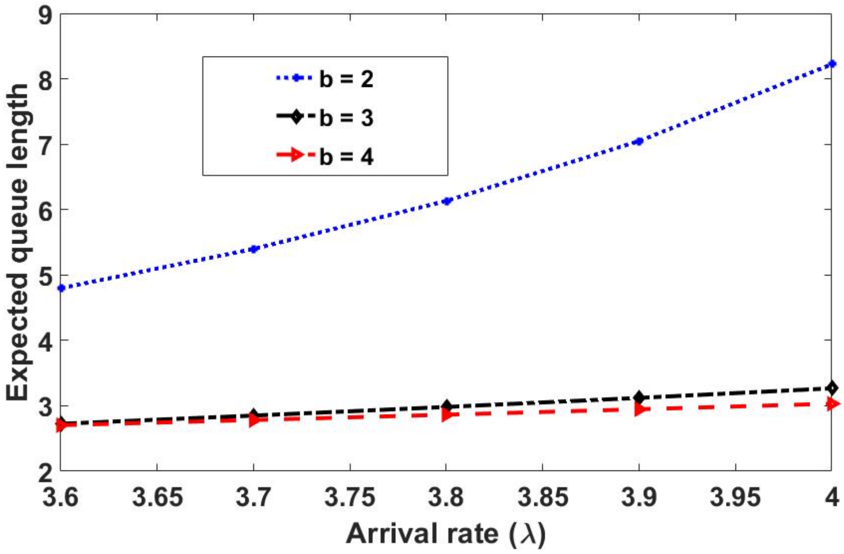

Figure 4: *The effect of arrival rate on L_q with different batch size*

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Figure 4 displays the effect of λ on the expected length of queue (L_q) with different batch size. For fixed λ, we observe that as b increases, L_q decreases. This is because, more customers are served in batch at a time, which results in reducing the queue length. Further, as b keeps constant, L_q increases as λ increases, as intuitively expected.

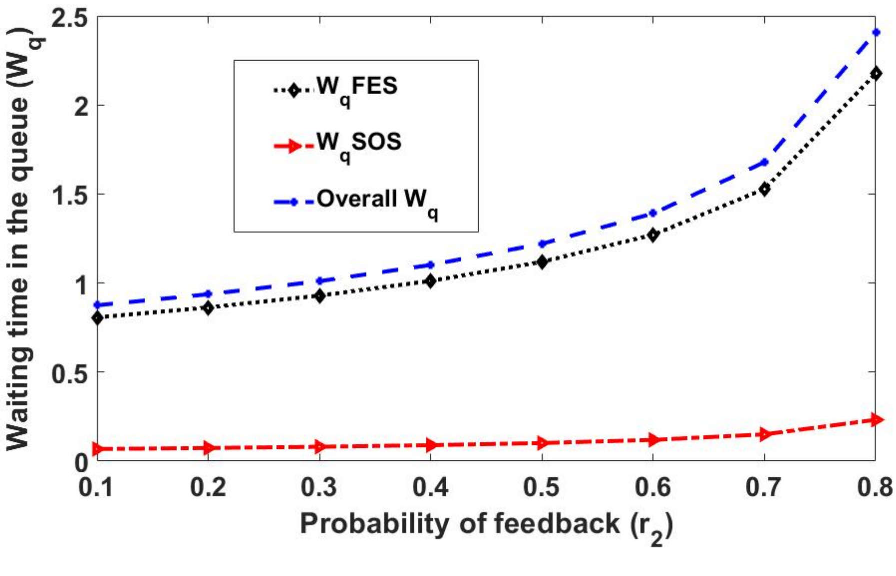

Figure 5: *The effect of r_2 on W_q*

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The effect of the probability of feedback (r_2) on the expected waiting time in the queue (W_q) is presented in Figure 5 . We can observe from the graph that as r_2 increases, W_qFES, W_qSOS, and overall W_q increase. Because as r_2 increases, more customers rejoin the queue (feedback), resulting in increasing W_qFES, W_qSOS and overall W_q.

VII. Conclusion

We investigate the transient and steady state behavior of a single server batch service queue with SOS and feedback. We use the probability generating function, Laplace transforms, and Rouche's theorem to obtain the transient state probabilities after inverting Laplace transform into the time domain. Also, the Tauberian theorem is applied in Laplace transform expression to obtain the steady state probabilities. Finally, we present numerical results as tables and figures to show the effects of various parameters on the model performance measures.

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