Estimation of reliability characteristics for linear consecutive $\boldsymbol{k}$-out-of-$\boldsymbol{n}$: $\boldsymbol{F}$ systems based on exponentiated Weibull distribution

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Abstract

The focus of this paper is to estimate the reliability characteristics of a linear consecutive k-out-of-n: F system with n linearly ordered components. The components are independent and identically distributed with exponentiated Weibull lifetimes. The system fails if and only if at least k successive components fail. In such a system, the reliability function and mean time to system failure are obtained by maximum likelihood estimation method using uncensored failure observations. The asymptotic confidence interval is determined for the reliability function. The results are obtained by Monte Carlo simulation to compare the performance of the systems using various sample sizes and combination of parameters. The procedure is also illustrated through a real data set.

Keywords: Maximum Likelihood Estimation (MLE), Reliability function, Mean Time to System Failure (MTSF), Asymptotic confidence interval and Exponentiated Weibull distribution

1. Introduction

Redundancy can be used to increase the system reliability. The most popular type of redundancy is k-out-of-n system structure which find the wide applications in both industry and defense systems. The consecutive k-out-of-n system is a special type of redundancy in fault-tolerant systems such as oil pipeline systems, street illumination systems, street parking, communication relay stations batch sampling-based quality control systems, computer networks and multi-pump system in hydraulic control system. These systems are characterized as physical or logical connections between the system components that are arranged in line or circle. Pham [16] proposed two basic aspects that have been used to obtain better reliability of a system. The first is to use redundancy such as parallel system, k-out-of-n system and the second one is a manufactured a high reliable system product. Let n components be linearly connected in such a way that the system fails if and only if at least k consecutive components fail. Figure 1 shows a linear consecutive 3-out-of-7: F system. Whenever the number of consecutive failures reaches 3 the signal flow is interrupted from source to sink and the system fails. Chao et al. [3] emphasized that $Lin/Con/k/n/F$ system has a much higher reliability than series system and is often cheaper than the parallel system. In this paper, we are considering

$Lin/Con/k/n$: F system redundancy and develop ways to obtain the maximum likelihood estimate of reliability and MTSF of the proposed system, where the components are independent and identically distributed ($i.i.d$) with exponentiated Weibull lifetimes

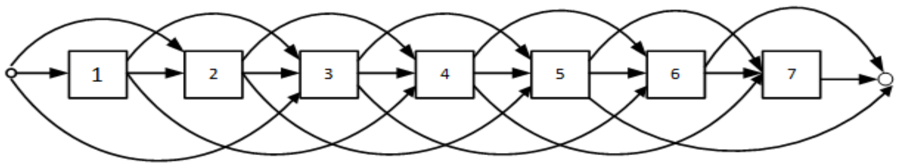

Figure 1: *Linear consecutive* 3-*out-of*-7: F *system*

First, the consecutive k-out-of-n system have been studied by Kontoleon [11]. Chiang and Ni [4] have giving special attention to the reliability of this system. Extensive review of consecutive k-out-of-n and related systems can be found in Hwang [9], Derman and Ross [6], Kuo and Zuo [20] and Eryilmaz [7]. The reliability estimation of a consecutive k-out-of-n: F system has received little attention in the literature. Shi et al. [18] discussed the classical and Bayes approach to study the performance of m-consecutive-k-out-of-n: F system with Burr XII components. Madhumitha and Vijayalakshmi [12] have proposed the Bayesian estimation for reliability and mean time to system failure for Linear (Circular) $Con/k/n$: F using exponential distribution. Recently, Kalaivani and Kannan [10] estimated the reliability function and MTSF of k-out-of-n system using Weibull failure time model by MLE and Bayes estimation. Demiray and Kizilaslan [5] investigated the point and interval estimates of stress-strength reliability in a consecutive k-out-of-n: G system when stress and strength variable follow the proportional hazard rate model. The reliability estimation is studied under both classical and Bayes estimates. In reliability analysis, Weibull family of distribution is mostly used for modeling consecutive k-out-of-n systems with monotone failure rates.

The exponentiated Weibull distribution introduced by Mudholkar and Srivastava [14] provides a good fit to lifetime datasets that exhibit bathtub shaped as well as unimodal failure rates. The performance of the product may involve high initial failure rate and possible high failure rates due to wear out and aging, reflecting a bathtub failure rate. Pathak and Chaturvedi [15] obtained the ML estimator of the reliability function $P(X > t)$ and $P(X > Y)$ using exponentiated Weibull distribution. Srinivasa Rao et al. [19] have estimated the multicomponent stress-strength of a system when stress and strength follow two parameter exponentiated Weibull distribution with different shape parameters and common scale parameter. Alghamdi and Percy [2] studied the reliability equivalence factors of a series-parallel system with each component has an exponentiated Weibull distribution. Méndez-González et al. [13] analyzed the reliability of an electronic component using exponentiated Weibull model and inverse power law.

According to Hong and Meeker [8], components and system structure determine the reliability of the system. When the component level data is available, it can be used to estimate system reliability. Confidence intervals (CIs) are essential to assess the statistical uncertainty in the estimations. Yet, the best estimation of the redundant consecutive k-out-of-n systems would still be of interest due to recent practical applications of the complex systems. This paper establishes the reliability function, mean time to system failure and asymptotic confidence interval at 95% level of significance for linear consecutive k-out-of-n: F system based on three parameter exponentiated Weibull model that provides a better approach to fit monotone as well as non-monotone failure rates which are quite common in reliability analysis.

This paper is organized as follows. In the introductory section the motivation for the present study and brief review on $Lin/Con/k/n$: F systems. In Section 2, description of system reliability characteristics and assumptions are given. Section 3 devoted to reliability function, mean time to failure and the asymptotic confidence interval of the proposed system. In section 4, the results based on simulation study and real data set are illustrated. Finally, the paper ends with a concluding remark are presented in Section 5.

2. Background of System Reliability Characteristics

Assumptions

- Each component and the system are either good or failed state
- The components of the system fail statistically independently of each other
- All component lifetimes are independently and identically distributed
- Life time of the component follows exponentiated Weibull distribution with unknown parameters α, β and θ
- The system fails if and only if at least k consecutive components fail, where $1 \leq k \leq n$

Notation

n	Number of components in a system
k	Minimum number of consecutive components whose failures cause system to failure
$Lin/Con/k/n{:}\,F$	Linear consecutive k-out-of-n: F
$R(t)$	Component reliability function. All the components have iid lifetimes
$R_S^L(t)$	System reliability function of $Lin/Con/k/n{:}\,F$ system
μ	Component mean time to failure
μ_S^L	Mean time to system failure
$\hat{R}(t)$	MLE of $R(t)$
$\hat{R}_S^L(t)$	MLE of $R_S^L(t)$
$\hat{\mu}$	MLE of μ
$\hat{\mu}_S^L$	MLE of μ_S^L
$\lfloor a \rfloor$	Largest integer less than or equal to a
$i.i.d.$	Independent and identically distributed

A consecutive k-out-of-n system consists of n linearly ordered components where the system fails if and only if a minimum of k components fail. This type of structure is called the linear $con/k/n{:}\,F$ system shortly represented by $Lin/con/k/n{:}\,F$. Here it is commonly assumed that $1 \leq k \leq n$, and for $k = 1$, the $Lin/con/k/n{:}\,F$ system becomes the series system and when $k = n$, the proposed system becomes a parallel system.

The reliability function $R_S^L(t)$ of a $Lin/Con/k/n{:}\,F$ system is

$$R_S^L(t) = \sum_{j=0}^{m} N(j,k,n)\ p^{n-j}\, q^{j} \tag{1}$$

When the components are $i.i.d.$ replacing p and q with $R(t)$ and $1 - R(t)$ in (1), we get the following reliability function for $Lin/Con/k/n{:}\,F$ system

$$R_S^L(t) = \sum_{j=0}^{m} \sum_{i=0}^{j} (-1)^i\, N(j,k,n) \binom{j}{i}\ (R(t))^{n-j+i} \tag{2}$$

where

$$m = \begin{cases} n - \left(\frac{n+1}{k}\right) + 1 & if\ n+1\ is\ multiple\ of\ k \\ n - \left\lfloor \frac{n+1}{k} \right\rfloor & if\ n+1\ is\ not\ multiple\ of\ k \end{cases}$$

and

$$N(j,k,n) = \begin{cases} \binom{n}{j}, & 0 \le j \le k-1 \\ \sum_{l=0}^{\left\lfloor \frac{j}{k} \right\rfloor} (-1)^i \binom{n-j+1}{l} \binom{n-lk}{n-j}, & k \le j \le n \\ 0, & j > m \end{cases}$$

and m represents the maximum number of failed components that may exist in the system without causing the system to fail. But $N(j,k,n)$ is the number of ways arranging j failed components in a line such that no k or more failed components are consecutive.

Let T be the lifetime of each component following Exponentiated Weibull Distribution (EWD) with probability density function (pdf)

$$f(t,\alpha,\beta,\theta) = \frac{\alpha\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}\left[1 - e^{-\left(\frac{t}{\theta}\right)^{\beta}}\right]^{\alpha-1} e^{-\left(\frac{t}{\theta}\right)^{\beta}}, \qquad t > 0, \alpha, \beta, \theta > 0 \tag{3}$$

where θ is scale parameter, α and β are shape parameters and are unknown.

Failure rate (hazard rate) function, is an important function in lifetime modeling is given by

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{\alpha\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}\left[1 - e^{-\left(\frac{t}{\theta}\right)^{\beta}}\right]^{\alpha-1} e^{-\left(\frac{t}{\theta}\right)^{\beta}}}{1 - \left[1 - e^{-\left(\frac{t}{\theta}\right)^{\beta}}\right]^{\alpha}} \tag{4}$$

It is pertinent to note that the $h(t)$ is:

i. Constant $= \beta^{-1}$ if $\beta = \theta = 1$
ii. Increasing (decreasing) FR if $\beta \ge 1$ and $\alpha\beta \ge 1$ ($\beta \le 1$ and $\alpha\beta \le 1$)
iii. Bathtub shaped FR if $\beta > 1$ and $\alpha\beta < 1$
iv. Upside-down bathtub shaped (unimodal) FR if $\beta < 1$ and $\alpha\beta > 1$

From (3), we know that the EWD includes many distributions as special cases. If $\beta = 1$, it reduced to exponentiated exponential distribution (EED). If $\beta = 2$, it becomes exponentiated Rayleigh distribution (ERD). If $\theta = 1$, it deduced as the standard 2-parameter Weibull distribution (WD). The particular case for $\beta = 2$ and $\theta = 1$ is the Rayleigh distribution. If $\beta = 1$ and $\theta = 1$, it becomes the one-parameter exponential distribution.

The component reliability for mission time t is given by

$$R(t) = 1 - \left[1 - e^{-\left(\frac{t}{\theta}\right)^{\beta}}\right]^{\alpha}, t > 0 \tag{5}$$

and mean time to component failure is expressed as

$$\mu = \int_0^\infty R(t)\ dt = \theta\left[\left(1 + \frac{1}{\beta}\right)\sum_{i=0}^{s}(-1)^i \binom{\alpha}{i} i^{-\left(\frac{1}{\beta}\right)} \tag{6}$$

The reliability function of a $Lin/Con/k/n{:}F$ system is obtained as

$$R_S^L(t) = \sum_{j=0}^{m}\sum_{i=0}^{j}(-1)^i N(j,k,n)\binom{j}{i}\left(1 - \left[1 - e^{-\left(\frac{t}{\theta}\right)^{\beta}}\right]^{\alpha}\right)^{n-j+i} \tag{7}$$

Using $R_S^L(t)$, we have the following expression for the mean time to system failure (MTSF) of a $Lin/Con/k/n{:}\,F$ system

$$\mu_S^L = \int_0^\infty R_S^L(t)\,dt = \sum_{j=0}^{m}\sum_{i=0}^{j}(-1)^i N(j,k,n)\binom{j}{i}\int_0^\infty (R(t))^{n-j+i}\,dt$$

Now,

$$\int_0^\infty [R(t)]^{n-j+i}\,dt = \int_0^\infty \left[1-\left(1-e^{-\left(\frac{t}{\theta}\right)^\beta}\right)^\alpha\right]^{n-j+i} dt$$

$$= \sum_{r=0}^{n-j+i}(-1)^r\binom{n-j+i}{r}\int_0^\infty \left[\left(1-e^{-\left(\frac{t}{\theta}\right)^\beta}\right)^{r\alpha}\right]dt$$

$$= \sum_{r=0}^{n-j+i}(-1)^r\binom{n-j+i}{r}\sum_{l=0}^{s}(-1)^l\binom{r\alpha}{l}\int_0^\infty \left(e^{-l\left(\frac{t}{\theta}\right)^\beta}\right)dt$$

Hence from (7), we have

$$\mu_S^L = \theta\left[\left(1+\frac{1}{\beta}\right)\sum_{j=0}^{m}\sum_{i=0}^{j}(-1)^i N(j,k,n)\binom{j}{i}\sum_{r=0}^{n-j+i}(-1)^r\binom{n-j+i}{r}\sum_{l=0}^{s}(-1)^l\binom{r\alpha}{l}l^{-\left(\frac{1}{\beta}\right)} \qquad (8)$$

where $s = \begin{cases} 1,2,3,\ldots\, if\ r\alpha\ \ is\ integer \\ \infty\ \ if\ r\alpha\ is\ non-integer \end{cases}$

3. Reliability and MTSF Estimation

In this section, we have obtained the ML estimator of $R_S^L(t)$ and the MTSF, μ_S^L for a $Lin/Con/k/n{:}\,F$ system. Let n units are put on test and the test ends when all the units have failed. Let $t_1, t_2, \ldots, t_n$ be the random failure times and assume they follow an exponentiated Weibull distribution with density function given in (3). The log likelihood function of the parameters is

$$L = L(\alpha,\beta,\theta) = nlog\alpha + nlog\beta + (\beta-1)\sum_{i=1}^{n}logt_i - n\beta log\theta - \sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^\beta + (\alpha-1)\sum_{i=1}^{n}log\left[1-\exp\left(-\left(\frac{t_i}{\theta}\right)^\beta\right)\right]$$

Then the maximum likelihood estimator (MLE) of α, β and θ say $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ respectively can be obtained by solving the following simultaneous nonlinear equations using numerical methods

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}log\left[1-e^{-\left(\frac{t_i}{\theta}\right)^\beta}\right] = 0 \qquad (9)$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}logx_i - nlog\theta - \sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^\beta log\left(\frac{t_i}{\theta}\right) + (\alpha-1)\sum_{i=1}^{n}\left[\frac{\left(\frac{t_i}{\theta}\right)^\beta log\left(\frac{t_i}{\theta}\right)e^{-\left(\frac{t_i}{\theta}\right)^\beta}}{1-e^{-\left(\frac{t_i}{\theta}\right)^\beta}}\right] = 0 \qquad (10)$$

$$\frac{\partial L}{\partial \theta} = -n + \sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^\beta - (\alpha-1)\sum_{i=1}^{n}\left[\frac{\left(\frac{t_i}{\theta}\right)^\beta e^{-\left(\frac{t_i}{\theta}\right)^\beta}}{1-e^{-\left(\frac{t_i}{\theta}\right)^\beta}}\right] = 0 \qquad (11)$$

By applying the invariance property of MLE, the MLE of the reliability function and mean time to failure of components and $Lin/Con/k/n{:}\,F$ system is obtained by substituting $\hat{\alpha}, \hat{\beta}\ and\ \hat{\theta}$ in (5), (6), (7) and (8).

4. Asymptotic Confidence Interval of $R_S^L(t)$

The Fishers information matrix for $\lambda = (\alpha, \beta, \theta)$ is

$$I = I(\lambda) = -E\begin{bmatrix} \frac{\partial^2 L}{\partial \alpha^2} & \frac{\partial^2 L}{\partial \alpha \partial \beta} & \frac{\partial^2 L}{\partial \alpha \partial \theta} \\ \frac{\partial^2 L}{\partial \beta \partial \alpha} & \frac{\partial^2 L}{\partial \beta^2} & \frac{\partial^2 L}{\partial \beta \partial \theta} \\ \frac{\partial^2 L}{\partial \theta \partial \alpha} & \frac{\partial^2 L}{\partial \theta \partial \beta} & \frac{\partial^2 L}{\partial \theta^2} \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

where

$$I_{22} = -\frac{n}{\beta^2} - \sum_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^{\beta} log^2\left(\frac{t_i}{\theta}\right) + (\alpha - 1) \times \sum_{i=1}^{n} \left[\frac{\left(1 - e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}\right) T_1 - e^{-2\left(\frac{t_i}{\theta}\right)^{\beta}} \left(\frac{t_i}{\theta}\right)^{2\beta} log^2\left(\frac{t_i}{\theta}\right)^{\beta}}{\left(1 - e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}\right)^2} \right]$$

$$I_{33} = \frac{n\beta}{\theta^2} - \sum_{i=1}^{n} \frac{\beta}{\theta^2}\left(\frac{t_i}{\theta}\right)^{\beta} + \left(\frac{\beta}{\theta}\right)^2 \left(\frac{t_i}{\theta}\right)^{\beta} - (\alpha - 1) \sum_{i=1}^{n} \left[\frac{\left(1 - e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}\right) T_2 + \left(\frac{\beta}{\theta}\right)^2 \left(\frac{t_i}{\theta}\right)^{2\beta} e^{-\left(\frac{t_i}{\theta}\right)^{2\beta}}}{\left(1 - e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}\right)^2} \right]$$

$$I_{12} = I_{21} = \sum_{i=1}^{n} \left[\frac{\left(\frac{t_i}{\theta}\right)^{\beta} \left(\frac{t_i}{\theta}\right)^{\beta} log\left(\frac{t_i}{\theta}\right)}{1 - e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}} \right]$$

$$I_{13} = I_{31} = -\sum_{i=1}^{n} \frac{\left(\frac{\beta}{\theta}\right) e^{-\left(\frac{t_i}{\theta}\right)^{\beta}} \left(\frac{t_i}{\theta}\right)^{\beta}}{1 - e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}}$$

$$I_{23} = I_{32} = -\frac{n}{\theta} + \sum_{i=1}^{n} \left[\frac{1}{\theta}\left(\frac{t_i}{\theta}\right)^{\beta} + \left(\frac{\beta}{\theta}\right)\left(\frac{t_i}{\theta}\right)^{\beta} log\left(\frac{t_i}{\theta}\right) \right] + \sum_{i=1}^{n} \left[\frac{\left(1 - e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}\right) T_3 + \left(\frac{\beta}{\theta}\right)\left(\frac{t_i}{\theta}\right)^{2\beta} log\left(\frac{t_i}{\theta}\right) e^{-2\left(\frac{t_i}{\theta}\right)^{\beta}}}{\left(1 - e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}\right)^2} \right]$$

and

$$T_1 = \left[e^{-\left(\frac{t_i}{\theta}\right)^{\beta}} \left(\frac{t_i}{\theta}\right)^{\beta} log^2\left(\frac{t_i}{\theta}\right)^{\beta} - e^{-\left(\frac{t_i}{\theta}\right)^{\beta}} \left(\frac{t_i}{\theta}\right)^{2\beta} log^2\left(\frac{t_i}{\theta}\right)^{\beta} \right]$$

$$T_2 = \left(\frac{\beta}{\theta}\right)^2 e^{-\left(\frac{t_i}{\theta}\right)^{\beta}} \left(\frac{t_i}{\theta}\right)^{2\beta} - \frac{\beta}{\theta^2}\left(\frac{t_i}{\theta}\right)^{\beta} e^{-\left(\frac{t_i}{\theta}\right)^{\beta}} - \left(\frac{\beta}{\theta}\right)^2 e^{-\left(\frac{t_i}{\theta}\right)^{\beta}} \left(\frac{t_i}{\theta}\right)^{\beta}$$

$$T_3 = \left(\frac{\beta}{\theta}\right)\left(\frac{t_i}{\theta}\right)^{2\beta} log\left(\frac{t_i}{\theta}\right) e^{-\left(\frac{t_i}{\theta}\right)^{\beta}} + \frac{1}{\theta}\left(\frac{t_i}{\theta}\right)^{\beta} e^{-\left(\frac{t_i}{\theta}\right)^{\beta}} - \left(\frac{\beta}{\theta}\right)\left(\frac{t_i}{\theta}\right)^{\beta} log\left(\frac{t_i}{\theta}\right) e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}$$

The MLE of $R_S^L(t)$, $\hat{R}_S^L(t)$ is asymptotically normal with mean $R_S^L(t)$, and variance

$$\sigma^2_{R_S^L(t)} = \sum_{j=1}^{3}\sum_{i=1}^{3} \frac{\partial R_S^L(t)}{\partial \lambda_i} \frac{\partial R_S^L(t)}{\partial \lambda_j} I_{ij}^{-1}$$

where I_{ij}^{-1} is the $(i,j)th$ element of $I(\lambda)$ (see Rao[17]). Then,

$$\sigma^2_{R^L_{S(t)}} = \left(\frac{\partial R_S^L(t)}{\partial\alpha}\right)^2 I_{11}^{-1} + \left(\frac{\partial R_S^L(t)}{\partial\beta}\right)^2 I_{22}^{-1} + \left(\frac{\partial R_S^L(t)}{\partial\theta}\right)^2 I_{33}^{-1} + 2\frac{\partial R_S^L(t)}{\partial\alpha}\frac{\partial R_S^L(t)}{\partial\beta} I_{12}^{-1} + 2\frac{\partial R_S^L(t)}{\partial\alpha}\frac{\partial R_S^L(t)}{\partial\theta} I_{13}^{-1} + 2\frac{\partial R_S^L(t)}{\partial\beta}\frac{\partial R_S^L(t)}{\partial\theta} I_{23}^{-1}$$

where

$$\frac{\partial R_S^L(t)}{\partial\alpha} = -log\left[1-e^{-\left(\frac{t}{\theta}\right)^\beta}\right]\left[1-e^{-\left(\frac{t}{\theta}\right)^\beta}\right]^\alpha h$$

$$\frac{\partial R_S^L(t)}{\partial\beta} = -\alpha\left(\frac{t}{\theta}\right)^\beta e^{-\left(\frac{t}{\theta}\right)^\beta} log\left(\frac{t}{\theta}\right)\left[1-e^{-\left(\frac{t}{\theta}\right)^\beta}\right]^{\alpha-1} h$$

$$\frac{\partial R_S^L(t)}{\partial\beta} = \frac{\alpha\beta}{\theta}\left(\frac{t}{\theta}\right)^\beta e^{-\left(\frac{t}{\theta}\right)^\beta}\left[1-e^{-\left(\frac{t}{\theta}\right)^\beta}\right]^{\alpha-1} h$$

and

$$h = \sum_{j=0}^{m}\sum_{i=0}^{j}(-1)^i N(j,k,n)\binom{j}{i}(n-j+i)(R(t))^{n-j+i-1}$$

Therefore, an asymptotic $100(1-\gamma)\%$ confidence interval of $R^L_{S(t)}$ is given by

$$R_S^L(t) \in \left(\hat{R}_S^L(t) \pm Z_{\gamma/2}\right)$$

where $Z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the standard normal distribution and $\hat{\sigma}_{R_S^L(t)}$ is the value of $\sigma_{R_S^L(t)}$ at the MLE of the parameters.

5. Simulation Study and Data Analysis

In this section, a simulation study is carried out along with the application of the $Lin/Con/k/n$: F system and a real data set to the estimate system reliability and mean time to system failure when samples are drawn from EWD.

5.1. Simulation Study

We study some results based on Monte Carlo simulation to compare the performance of $R_S^L(t)$ μ_S^L and asymptotic confidence interval using different sample sizes $n = 30$ and 50 for combination of parameters $(\alpha,\beta,\theta) = (0.5,0.5,1)$, $(0.5,1.5,1)$, $(0.5,2.5,1)$, $(1.5,0.5,1)$ and $(2.5,0.5,1)$ and are evaluated using R software.

i. For each combination of α,β,θ and sample size n, we can derive the random samples from the EWD by inverting the cumulative distribution of (3).
$X = \theta\left[-log\left(1-U^{1/\alpha}\right)\right]^{1/\beta}$, $U \sim U(0,1)$
ii. Based on the data and using (9), (10) and (11), we estimate the MLE of α,β and θ, $R_S^L(t)$, μ_S^L and asymptotic confidence interval.
iii. Repeat step (i) and (ii) over 3000 times and the mean square errors for the estimators are calculated.
iv. The above steps are repeated for $Lin/Con/k/n$: F system by taking $n = 10$ and $k = 3$ and 6. The results are presented in Table 1, 2 and 3.

Table 1: *Reliability Estimation of* $Lin/Con/3/10{:}\,F$ *system*

Parameters	t	$R_S^L(t)$	$n = 30$				$n = 50$			
			$\hat{R}_S^L(t)$	MSE	Asymptotic CI		$\hat{R}_S^L(t)$	MSE	Asymptotic CI	
					LL	UL			LL	UL
$\alpha = 0.5$, $\beta = 0.5$, $\theta = 1$	0.2	0.297747	0.303069	0.018713	0.249567	0.355806	0.297385	0.011802	0.264664	0.329299
	0.4	0.162046	0.176572	0.011650	0.137303	0.214943	0.169849	0.007063	0.146110	0.192815
	0.6	0.100745	0.116668	0.007394	0.087137	0.145560	0.110241	0.004283	0.092532	0.127299
	0.8	0.067312	0.082330	0.004848	0.059416	0.104628	0.076508	0.002686	0.062907	0.089549
	1.0	0.047150	0.060606	0.003270	0.042428	0.078231	0.055435	0.001737	0.044833	0.065699
$\alpha = 0.5$, $\beta = 1.5$, $\theta = 1$	0.2	0.854684	0.831226	0.008891	0.789136	0.873496	0.833757	0.005326	0.807877	0.859582
	0.4	0.546503	0.537213	0.021241	0.474321	0.599168	0.531110	0.013585	0.492483	0.569434
	0.6	0.281312	0.294728	0.018049	0.241666	0.346063	0.283371	0.011144	0.251503	0.314827
	0.8	0.122807	0.143560	0.009532	0.109112	0.176196	0.132677	0.005272	0.112644	0.152358
	1.0	0.047150	0.063644	0.003753	0.044633	0.081262	0.055754	0.001731	0.045140	0.066163
$\boldsymbol{\alpha = 0.5}$, $\boldsymbol{\beta = 2.5}$, $\boldsymbol{\theta = 1}$	0.2	0.983359	0.973400	0.000761	0.962288	0.984654	0.976043	0.000371	0.969691	0.982422
	0.4	0.830987	0.805319	0.010896	0.759474	0.851011	0.808428	0.006486	0.780539	0.836273
	0.6	0.505575	0.493241	0.022685	0.430379	0.555143	0.490158	0.014306	0.451895	0.528157
	0.8	0.197713	0.210763	0.014190	0.166860	0.253530	0.202739	0.008425	0.176460	0.228638
	1.0	0.047150	0.061131	0.003794	0.042884	0.078609	0.054835	0.001680	0.044332	0.065048
$\alpha = 1.5$, $\beta = 0.5$, $\theta = 1$	0.2	0.934871	0.915864	0.004256	0.889625	0.942087	0.919800	0.002230	0.904357	0.935242
	0.4	0.815863	0.788815	0.013581	0.740635	0.836936	0.792504	0.007750	0.763610	0.821397
	0.6	0.693397	0.667344	0.020847	0.607435	0.727154	0.669078	0.012392	0.632893	0.705262
	0.8	0.582536	0.561714	0.024429	0.497174	0.626124	0.561157	0.014812	0.522036	0.600278
	1.0	0.487040	0.472712	0.025109	0.407868	0.537404	0.470101	0.015345	0.430729	0.509473
$\alpha = 2.5$, $\beta = 0.5$, $\theta = 1$	0.2	0.996454	0.992783	0.000106	0.989188	0.996379	0.993912	0.000045	0.992011	0.995813
	0.4	0.976451	0.965417	0.001186	0.952008	0.978826	0.968151	0.000583	0.960528	0.975774
	0.6	0.937420	0.918920	0.004066	0.892858	0.944981	0.922706	0.002155	0.907477	0.937936
	0.8	0.883603	0.859445	0.008437	0.820951	0.897940	0.863512	0.004693	0.840678	0.886346
	1.0	0.820477	0.793148	0.013325	0.744110	0.842186	0.796827	0.007664	0.767454	0.8262

Table 2: *Reliability Estimation of $Lin/Con/6/10$: F system*

Parameters	t	$R_S^L(t)$	$n=30$				$n=50$			
			$\hat{R}_S^L(t)$	MSE	Asymptotic CI		$\hat{R}_S^L(t)$	MSE	Asymptotic CI	
					LL	UL			LL	UL
$\alpha=0.5$, $\beta=0.5$, $\theta=1$	0.2	0.878187	0.854031	0.009330	0.820145	0.887532	0.861369	0.005120	0.841135	0.881186
	0.4	0.767127	0.742157	0.017567	0.695727	0.787805	0.750602	0.010092	0.722402	0.778081
	0.6	0.676749	0.654458	0.022542	0.602294	0.705814	0.663003	0.013098	0.630975	0.694176
	0.8	0.602410	0.583262	0.025178	0.528435	0.637110	0.591528	0.014681	0.557627	0.62446
	1.0	0.540364	0.524086	0.026268	0.468303	0.578803	0.531901	0.015332	0.497441	0.565671
$\alpha=0.5$, $\beta=1.5$, $\theta=1$	0.2	0.997601	0.995090	4.75E-05	0.992582	0.997590	0.995899	0.000018	0.994557	0.997237
	0.4	0.965283	0.952884	0.001772	0.937312	0.968168	0.955734	0.000925	0.946606	0.964788
	0.6	0.868454	0.849576	0.008993	0.814768	0.883334	0.852847	0.005244	0.831893	0.873555
	0.8	0.714275	0.698971	0.019396	0.648864	0.747063	0.701058	0.011516	0.670576	0.731087
	1.0	0.540364	0.532961	0.025731	0.476643	0.586475	0.533617	0.014971	0.499182	0.567542
$\alpha=0.5$, $\beta=2.5$, $\theta=1$	0.2	0.999975	0.999860	0.000000	0.999752	0.999970	0.999913	0.000000	0.999869	0.999958
	0.4	0.996649	0.993299	0.000084	0.990042	0.996534	0.994407	0.000033	0.992675	0.996134
	0.6	0.956381	0.940101	0.002694	0.921477	0.958424	0.944680	0.001395	0.933970	0.955323
	0.8	0.804664	0.779584	0.014721	0.736400	0.821867	0.786669	0.008520	0.760821	0.81225
	1.0	0.540364	0.524527	0.025982	0.468467	0.579204	0.530556	0.015108	0.496113	0.564469
$\alpha=1.5$, $\beta=0.5$, $\theta=1$	0.2	0.999574	0.998694	0.000008	0.997905	0.999481	0.999065	0.000002	0.998704	0.999426
	0.4	0.995942	0.991603	0.000156	0.987592	0.995609	0.993213	0.000050	0.991174	0.995252
	0.6	0.986857	0.977439	0.000723	0.968280	0.986586	0.980672	0.000282	0.975772	0.985572
	0.8	0.971944	0.957120	0.001876	0.941835	0.972381	0.961952	0.000820	0.953511	0.970392
	1.0	0.951824	0.932068	0.003592	0.910394	0.953705	0.938280	0.001691	0.926041	0.950519
$\alpha=2.5$, $\beta=0.5$, $\theta=1$	0.2	0.999999	0.999986	0.000000	0.999974	0.999999	0.999993	0.000000	0.999989	0.999997
	0.4	0.999949	0.999760	0.000000	0.999586	0.999935	0.999847	0.000000	0.999777	0.999918
	0.6	0.999608	0.998782	0.000007	0.998018	0.999546	0.999126	0.000002	0.998780	0.999473
	0.8	0.998523	0.996374	0.000039	0.994360	0.998388	0.997205	0.000011	0.996229	0.998182
	1.0	0.996166	0.991969	0.000142	0.987952	0.995987	0.993500	0.000047	0.991461	0.99554

Table 3: *Estimation of mean time to system failure (MTSF)*

Parameters		$\alpha=0.5$, $\beta=0.5$, $\theta=1$	$\alpha=0.5$, $\beta=1.5$, $\theta=1$	$\alpha=0.5$, $\beta=2.5$, $\theta=1$	$\alpha=1.5$, $\beta=0.5$, $\theta=1$	$\alpha=2.5$, $\beta=0.5$, $\theta=1$
				$\boldsymbol{Lin/Con/3/10: F}$ system		
Sample size	μ_S^L	0.233168	0.475392	0.614456	1.338123	2.431575
$\boldsymbol{n=30}$	$\hat{\mu}_S^L$	0.265047	0.483305	0.620654	1.364669	2.435709
	MSE	0.0278	0.014594	0.151323	0.234167	0.460743
$\boldsymbol{n=50}$	$\hat{\mu}_S^L$	0.250012	0.473758	0.608595	1.336523	2.405559
	MSE	0.014272	0.006604	0.004345	0.13133	0.264359
				$\boldsymbol{Lin/Con/6/10: F}$ system		
Sample size	μ_S^L	2.564288	1.134486	1.051981	6.084472	8.497633
$\boldsymbol{n=30}$	$\hat{\mu}_S^L$	3.520812	1.140934	1.112734	6.36528	8.728028
	MSE	373.2682	0.173304	10.29648	7.102796	9.147431
$\boldsymbol{n=50}$	$\hat{\mu}_S^L$	2.844258	1.12965	1.044834	6.247015	8.627549
	MSE	7.607097	0.022023	0.007051	3.34101	4.402205

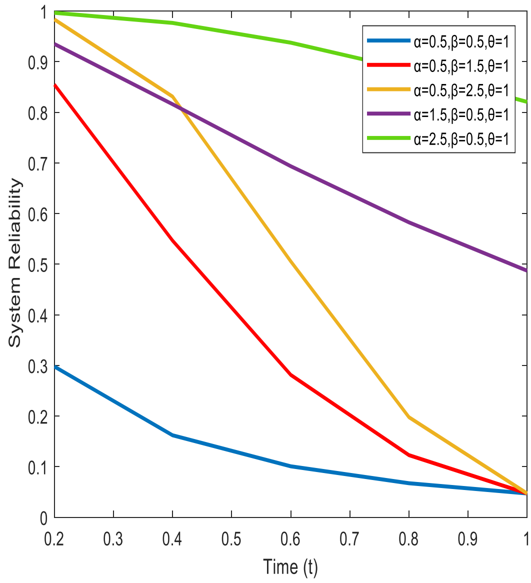

Figure 2: *Reliability of* $Lin/Con/3/10: F$ *system*

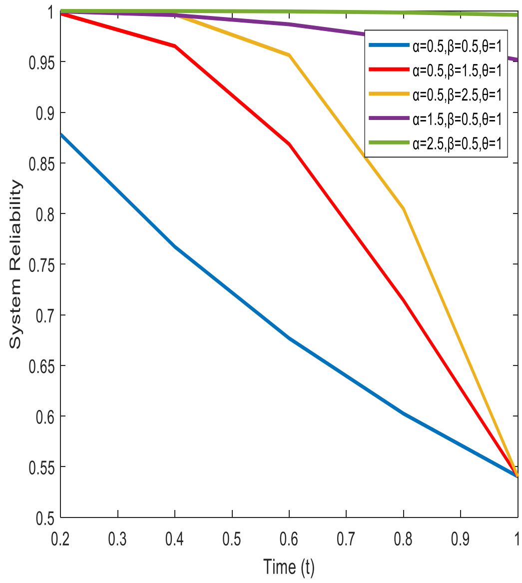

Figure 3: *Reliability of* $Lin/Con/6/10: F$ *system*

According to Table 1, 2 and 3, the MSEs for estimate of reliability and average lifetime of the system decreases as the sample size increases. The expected length of the confidence interval reduces as sample size increases at 95% level of significance for all combinations of parameters. By comparing the system reliabilities and MTSFs for $k=3$ and $k=6$, we can see that the system reliability and MTSF are increase when the number of consecutive failed components, k increases from 3 to 6 and other parameters are kept unchanged. The results are consistent with the definition of the consecutive k-out-of-n: F system. Further, it is observed that the time increases the reliability of the system declines, as expected. In addition, the length of the confidence interval of the $Lin/Con/6/10: F$ larger than the $Lin/Con/3/10: F$ system for the all combinations of parameters.

Figure 2 and 3 show the true value of reliability in $Lin/Con/3/10: F$ and $Lin/Con/6/10: F$. $R_S^L(t)$ is low when $\alpha<1$, $\beta<1$ and $\alpha\beta<1$, the components of the system having initial failure rate. On the other hand, for $\alpha<1, \beta>1$ and $\alpha\beta<1$, the performance of the system is improved, the system's components are in bath tub failure mode. When, $\alpha < 1, \beta > 1$ and $\alpha\beta > 1$, $R_S^L(t)$ declines quickly because the components are in the increasing failure rate. If $\alpha > 1, \beta < 1$ and $\alpha\beta < 1$, the reliability improved well when compare with first one even though the components are in the decreasing failure rate. Suppose $\alpha > 1, \beta < 1$ and $\alpha\beta > 1$, the performance of the system is high, because the system's components possessing unimodal failure rate.

From Table 3, it is seen that the MSEs of MTSF of L$in/Con/6/10{:}\,F$ is more as compare with L$in/Con/3/10{:}\,F$ system. This is due the number of consecutive failed components is large. The MTTF of each component are same. Therefore, it can be concluded that the failure rate of the distribution will affect the average failure time of the system. However, the influence is dependent on the values of the parameters.

5.2. Data Analysis

In this section, the model proposed in (3) is applied to estimate the lifetimes of 18 electronic devices shown in Table 4. The presented data were taken from Ahmad and Ghazal [1] as a lifetime distribution having bathtub shaped failure rate. The MLE of α, β, θ and their standard errors are given below

$$\hat{\alpha} = 0.144884\ (0.008049) \qquad \hat{\beta} = 5.285692\ (0.275845) \qquad \hat{\theta} = 373.218\ (9.72593)$$

The ML estimates of reliability and MTSF for a $Lin/Con/k/10{:}\,F$ system has been evaluated with $n = 10$ and $k = 3\ and\ 6$.

Table 4: *Lifetime of 18 electronic devices*

5	11	21	31	46	75	98	122	145	165
196	224	245	293	321	330	350	420		

Table 5: *Reliability, MTSF and asymptotic confidence intervals at 95% level of significance*

t	$Lin/Con/3/10{:}\,F$ system $\hat{\mu}_S^L = 154.3145$			$Lin/Con/6/10{:}\,F$ system $\hat{\mu}_S^L = 289.8265$		
	$\hat{R}_S^L(t)$	Asymptotic CI		$\hat{R}_S^L(t)$	Asymptotic CI	
		LL	UL		LL	UL
0	1	-	-	1	-	-
50	0.936522	0.895956	0.977088	0.999596	0.999051	1.000142
100	0.747427	0.652674	0.84218	0.991666	0.984417	0.998914
150	0.497669	0.387101	0.608237	0.95448	0.927456	0.981504
200	0.265821	0.179307	0.352334	0.858565	0.801264	0.915866
250	0.106003	0.05878	0.153227	0.686347	0.602146	0.770548
300	0.028072	0.010935	0.045209	0.457671	0.366109	0.549232
350	0.004127	0.000463	0.007791	0.236461	0.161964	0.310958
400	0.000257	-0.00011	0.00062	0.08773	0.043646	0.131815

From Table 5, it can be seen that the reliability and MTSF of the consecutive k-out-of-n the failure rate of the distribution when time increases the reliability of the system decreases, as expected. The system parameter k increases the reliability and expected life time increase. In addition, the length of the confidence interval decreases when the number of consecutive failure components decreases. These can be seen in the simulation study results.

6. Conclusions

In this paper, we have proposed a $Lin/Con/k/n{:}\,F$ system which composed of n independent and identically distributed components having exponentiated Weibull lifetimes with three unknown parameters and studied the reliability characteristics. This distribution has the ability to model the non-monotonic and monotonic failure rate. The reliability and mean time to system failure are estimated based on simulated observations by maximum likelihood estimation for various combination of parameters. Asymptotic confidence intervals were also constructed. The MSEs of $R_S^L(t)$, MTSF and length of the confidence interval decrease as sample size increases for $Lin/Con/3/10{:}\,F$ and $Lin/Con/6/10{:}\,F$ In addition, MSEs of MTSF as well as the mean length of the confidence interval increases as the number consecutive failed components k increases. A real-life data set was used to show the entire approach.

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