EPQ MODELS WITH GENERALIZED PARETO RATE OF PRODUCTION AND WEIBULL DECAY HAVING DEMAND AS FUNCTION OF ON HAND INVENTORY

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Abstract

Economic production quantity (EPQ) models are more important for scheduling production processes in particular batch production in which the production uptime and production downtime are decision variables. This paper addresses the development and analysis of an EPQ model with random production and Weibull decay having stock dependent demand. The random production is more appropriate in several production processes dealing with deteriorated items. The instantaneous state of on hand inventory is derived. With appropriate cost considerations the total cost function is derived and minimized for obtaining optimal production uptime, production downtime and production quantity. The model sensitivity with respect to changes in parameters and costs is also studied and observe that the production distribution parameters and deteriorating distribution parameters have significant influence on optimal operating policies of the model. This model is extended to the case of without shortages and observed that allowing shortages reduce total product cost. It is further observed that the demand being a function of on hand inventory can reduce inventory cost than other patterns of demand.

Keywords: Stochastic production, on hand inventory, Weibull decay, Generalized Pareto distribution, Production Schedules, Sensitivity analysis.

I. Introduction

In production scheduling problems the on hand inventory plays a dominant role. To have efficient decisions on when to start production and when to stop production the EPQ models provide the basic frame work. In developing the EPQ models the major string is on nature of the product. The product may be subjective deterioration or decay depending upon various random factors. Much work has been reported in literature regarding EPQ models for deteriorating items. The literature on inventory models for deteriorating items are reviewed by Pentico and Drake [1], Ruxian Lie et al [2], Goyal and Giri [3], Raafat [4] and Nahmias [5].

In addition to the nature of the commodity another important factor for developing EPQ models is demand. Several authors developed various models with different patterns of demand. Among them the inventory models with stock dependent demand gained importance due to this applicability in many places. Silver et al. [6] mentioned that the demand for many consumer items is directly proportional to the stock on hand. Gupta et al. [7] have pointed the inventory models with inventory models with stock dependent demand. Later Panda et al. [8], Roy et al. [9], Uma Maheswara Rao et al. [10] and others have developed inventory models for deteriorating items with stock dependent demand. Yang et al. [11], Srinivasa Rao et al. [12], Santanu Kumar Ghosh et al. [13] have developed an inventory model for deteriorating items with Weibull replenishment and generalized Pareto decay. Brojeswar et al. [14], Srinivasa Rao et al[15], Lakshmana Rao et al [16], Srinivasa Rao et al [17], Ardak and Borade [18], Anindya Mandal, Brojeswar Pal and Kripasindhu Chaudhuri [19], Sunit Kumar, Sushil Kumar and Rachna Kumari [20]. and Jyothsna et al. [21] studied a production inventory system for deteriorating items. In all these papers the authors assumed that the production is either instantaneous or finite rate.

However, in many production processes the production is random due to various random factors such as availability of raw material, skill level of the manpower, tool wear, environmental conditions and availability of power (electricity). Very little work has been reported regarding EPQ models with random production for deteriorating items except the works of Sridevi et al. [22], Srinivasa Rao et al [23] who developed EPQ models with Weibull production and constant rate of deterioration. In reality, many products may not have constant rate of deterioration but will have a variable rate of deterioration can be well characterized by Weibull decay. The generalized Pareto rate of production can characterize the time dependent production. Hence, in this paper we develop and analyze an EPQ model with the assumption that the production process is characterized by generalized Pareto distribution and the lifetime of the commodity follows a Weibull distribution. It is further assumed that the demand is a linear function of on hand inventory. This type of model is much useful in textile industry where the lifetime of the government is random and may have decreasing or increasing or constant rates of deterioration and production is random.

Using the differential equations the instantaneous state of inventory at different states of production are derived. The total cost function is obtained with appropriate cost considerations. Assuming shortages are allowed and fully backlogged, the optimal operating policies of the production schedule such as production downtime and production uptime are derived. The optimal production quantity is also obtained. The effect of change in parameters on optimal production schedule and optimal production quantity are studied in sensitivity analysis. The case of without shortages is also discussed. The conclusions are given at the end.

II. Assumptions

For developing the model the following assumptions are made:

- The demand rate is a function of on hand inventory.
  i.e. $\lambda(t) = \phi_1 + \phi_2 I(t)$ (1)
- The production is random and follows a Generalized Pareto distribution. The instantaneous rate of production is
  $$K(t) = \frac{1}{\alpha-\gamma t} \quad ; 0 < t < \frac{\alpha}{\gamma} \tag{2}$$
- Lead time is zero.
- Cycle length is T. It is known and fixed.
- Shortages are allowed and fully backlogged.
- A deteriorated unit is lost.
- The life time of the item is random and follows a two parameter Weibull distribution with probability density function
  $$f(t) = \theta\eta t^{\eta-1} e^{-\theta t^{\eta}} \quad ; \theta, \eta > 0, \quad t > 0$$

Therefore the instantaneous rate of deterioration is
$$h(t) = \frac{f(t)}{1-F(t)} = \theta\eta t^{\eta-1} \quad ; \theta, \eta > 0, \quad t > 0 \tag{3}$$

The following notations are used for developing the model.

Q: Production quantity.

A: Setup cost.

C: Cost per unit.

h: Inventory holding cost per unit per unit time.
π: Shortages cost per unit per unit time.

III. EPQ Model with Shortages

Consider a production system in which the stock level is zero at time $t = 0$. The stock level increases during the period $(0, t_1)$, due to production after fulfilling the demand and deterioration. The production stops at time t_1 when stock level reaches *S*. The inventory decreases gradually due to demand and deterioration in the interval (t_1, t_2). At time t_2 the inventory reaches zero and backorders accumulate during the period (t_2, t_3). At time t_3 the production again starts and fulfills the backlog after satisfying the demand. During (t_3, T) the backorders are fulfilled and inventory level reaches zero at the end of cycle *T*. The Schematic diagram representing the inventory level is given in Figure 1.

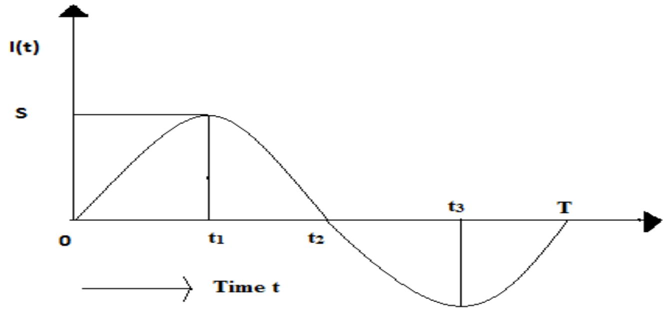

Figure 1: *Schematic Diagram representing the inventory level*

Let *I(t)* be the inventory level of the system at time '*t*' (*0 ≤ t ≤ T*). The differential equations governing the instantaneous state of *I(t)* over the cycle of length are:

$$\frac{d}{dt}I(t) + h(t)I(t) = \frac{1}{\alpha-\gamma t} - \left(\phi_1 + \phi_2 I(t)\right) \quad ; \quad 0 \le t \le t_1 \tag{4}$$
$$\frac{d}{dt}I(t) + h(t)I(t) = -\left(\phi_1 + \phi_2 I(t)\right) \quad ; \quad t_1 \le t \le t_2 \tag{5}$$
$$\frac{d}{dt}I(t) = -\left(\phi_1 + \phi_2 I(t)\right) \quad ; \quad t_2 \le t \le t_3 \tag{6}$$
$$\frac{d}{dt}I(t) = \frac{1}{\alpha-\gamma t} - \left(\phi_1 + \phi_2 I(t)\right) \quad ; \quad t_3 \le t \le T \tag{7}$$

Where, h(t) is as given in equation (3), with the initial conditions I(0) = 0, $I(t_1) = S$, $I(t_2) = 0$ and I(T) = 0. Substituting h(t) in equations (4) and (5) and solving the differential equations, the on hand inventory at time ' t ' is obtained as

$$I(t) = Se^{\theta(t_1^\eta - t^\eta)+\phi_2(t_1-t)} - e^{-(\theta t^\eta+\phi_2 t)} \int_t^{t_1} \left(\frac{1}{\alpha-\gamma u} - \phi_1\right) e^{(\theta u^\eta+\phi_2 u)}\, du \, ; \quad 0 \le t \le t_1 \tag{8}$$
$$I(t) = Se^{\theta(t_1^\eta - t^\eta)+\phi_2(t_1-t)} - \phi_1 e^{-(\theta t^\eta+\phi_2 t)} \int_{t_1}^{t} e^{(\theta u^\eta+\phi_2 u)}\, du \quad ; \quad t_1 \le t \le t_2 \tag{9}$$
$$I(t) = \frac{\phi_1}{\phi_2}\left(e^{\phi_2(t_2-t)} - 1\right) \quad ; \quad t_2 \le t \le t_3 \tag{10}$$
$$I(t) = e^{-\phi_2 t}\left[\int_{t_3}^{t} \left(\frac{1}{\alpha-\gamma u} - \phi_1\right) e^{\phi_2 u}\, du + \int_{t_3}^{T} \left(\frac{1}{\alpha-\gamma u} - \phi_1\right) e^{\phi_2 u}\, du\right] \quad ; \quad t_3 \le t \le T \tag{11}$$

Production quantity Q in the cycle of length T is

$$Q = \int_0^{t_1} K(t)dt + \int_{t_3}^{T} K(t)dt \qquad = \frac{1}{\gamma}\log\left(\frac{\alpha\,(\alpha-\gamma t_3)}{(\alpha-\gamma t_1)(\alpha-\gamma T)}\right) \tag{12}$$

From equation (8) and using the initial condition I(0) = 0, we obtain the value of 'S' as

$$S = e^{-(\theta t_1^{\eta}+\phi_2 t_1)} \int_0^{t_1}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{(\theta u^{\eta}+\phi_2 u\,)}du \tag{13}$$

When t = t_3, then equations (10) and (11) become

$$I(t_3) = \frac{\phi_1}{\phi_2}\left(e^{\phi_2(t_2-t_3)}-1\right) \text{ and} \tag{14}$$

$$I(t_3) = e^{-\phi_2 t_3}\int_{t_3}^{T}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{\phi_2 u}\,du \qquad \text{respectively.} \tag{15}$$

Equating the equations (14) & (15) and on simplification, one can get

$$t_2 = t_3 + \frac{1}{\phi_2}ln\left[1+\frac{\phi_2}{\phi_1}e^{-\phi_2 t_3}\int_{t_3}^{T}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{\phi_2 u}\,du\right] = x(t_3) \quad (say) \tag{16}$$

Let $K(t_1,t_2,t_3)$ be the total production cost per unit time. Since the total production cost is the sum of the set up cost, cost of the units, the inventory holding cost. Hence the total production cost per unit time becomes

$$K(t_1,t_2,t_3) = \frac{A}{T}+\frac{CQ}{T}+\frac{h}{T}\left[\int_0^{t_1} I(t)dt+\int_{t_1}^{t_2} I(t)dt\right]+\frac{\pi}{T}\left[\int_{t_2}^{t_3} -I(t)dt+\int_{t_3}^{T} -I(t)dt\right] \tag{17}$$

Substituting the values of I(t) and Q in equation (17), we can obtain $K(t_1,t_2,t_3)$ as

$$\begin{aligned} K(t_1,t_2,t_3) = {} & \frac{A}{T}+\frac{C}{\gamma T}\log\left(\frac{\alpha\,(\alpha-\gamma t_3)}{(\alpha-\gamma t_1)(\alpha-\gamma T)}\right)+\frac{h}{T}\left[\int_0^{t_1}\left[Se^{\theta(t_1^{\eta}-t^{\eta})+\phi_2(t_1-t)}\right.\right. \\ & \left. -e^{-(\theta t^{\eta}+\phi_2 t)}\int_t^{t_1}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{(\theta u^{\eta}+\phi_2 u)}\,du\right]dt \\ & \left. +\int_{t_1}^{t_2}\left[Se^{\theta(t_1^{\eta}-t^{\eta})+\phi_2(t_1-t)}-\phi_1 e^{-(\theta t^{\eta}+\phi_2 t)}\int_{t_1}^{t} e^{(\theta u^{\eta}+\phi_2 u)}\,du\right]dt\right] \\ & +\frac{\pi}{T}\left[\frac{\phi_1}{\phi_2}\int_{t_2}^{t_3}\left(1-e^{\phi_2(t_2-t)}\right)dt-\int_{t_3}^{T}\left[e^{-\phi_2 t}\left[\int_{t_3}^{t}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{\phi_2 u}\,du\right.\right.\right. \\ & \left.\left.\left. +\int_{t_3}^{T}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{\phi_2 u}\,du\right]\right]dt\right] \end{aligned} \tag{18}$$

0n integration and simplification one can get

$$\begin{aligned} K(t_1,t_2,t_3) = {} & \frac{A}{T}+\frac{C}{\gamma T}\log\left(\frac{\alpha\,(\alpha-\gamma t_3)}{(\alpha-\gamma t_1)(\alpha-\gamma T)}\right)+\frac{h}{T}\left[e^{(\theta t_1^{\eta}+\phi_2 t_1)}\int_0^{t_2} Se^{-(\theta t^{\eta}+\phi_2 t)}dt\right. \\ & -\int_0^{t_1}\left[e^{-(\theta t^{\eta}+\phi_2 t)}\int_t^{t_1}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{(\theta u^{\eta}+\phi_2 u)}\,du\right]dt \end{aligned}$$

$$-\phi_1\int_{t_1}^{t_2}\left[e^{-(\theta t^{\eta}+\phi_2 t)}\int_{t_1}^{t}e^{(\theta u^{\eta}+\phi_2 u)}\,du\right]dt\Bigg]+\frac{\pi}{T}\left[\frac{\phi_1}{\phi_2}\left(t_3-t_2-\frac{1}{\phi_2}\left(1-e^{\phi_2(t_2-t_3)}\right)\right)\right.$$
$$\left.-\int_{t_3}^{T}\left[e^{-\phi_2 t}\left[\int_{t_3}^{t}\frac{e^{\phi_2 u}}{\alpha-\gamma u}du+\int_{t_3}^{T}\frac{e^{\phi_2 u}}{\alpha-\gamma u}du\right]dt+\frac{\phi_1}{\phi_2}\left(e^{\phi_2(T-t)}-1\right)\right]dt\right] \tag{19}$$

Substituting the values of S and t_2 in equation (19), one can obtain

$$K(t_1,t_3)=\frac{A}{T}+\frac{C}{\gamma T}\log\left(\frac{\alpha\,(\alpha-\gamma t_3)}{(\alpha-\gamma t_1)(\alpha-\gamma T)}\right)$$
$$+\frac{h}{T}\left[\int_{0}^{x(t_3)}\left[e^{-(\theta t^{\eta}+\phi_2 t)}\int_0^{t_1}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{(\theta u^{\eta}+\phi_2 u)}du\right]dt\right.$$
$$-\int_0^{t_1}\left[e^{-(\theta t^{\eta}+\phi_2 t)}\int_t^{t_1}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{(\theta u^{\eta}+\phi_2 u)}\,du\right]dt$$
$$\left.-\phi_1\int_{t_1}^{x(t_3)}\left[e^{-(\theta t^{\eta}+\phi_2 t)}\int_{t_1}^{t}e^{(\theta u^{\eta}+\phi_2 u)}\,du\right]dt\right]$$
$$+\frac{\pi}{T\phi_2}\left[e^{-\phi_2 t_3}\int_{t_3}^{T}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{\phi_2 u}\,du-\phi_1\left[t_3-T\right.\right.$$
$$\left.\left.-\frac{1}{\phi_2}\left[1-e^{\phi_2(T-t_3)}-ln\left[1+\frac{\phi_2}{\phi_1}e^{-\phi_2 t_3}\int_{t_3}^{T}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{\phi_2 u}\,du\right]\right]\right]\right]$$
$$-\phi_2\int_{t_3}^{T}e^{-\phi_2 t}\left[\int_{t_3}^{t}\frac{e^{\phi_2 u}}{\alpha-\gamma u}du+\int_{t_3}^{T}\frac{e^{\phi_2 u}}{\alpha-\gamma u}du\right]dt\Bigg] \tag{20}$$

IV. Optimal Production Schedules of the Model

In this section we obtain the optimal policies of the system under study. To find the optimal values of t_1 and t_3, we obtain the first order partial derivatives of $K(t_1,t_3)$ given in equation with respect to t_1 and t_3 and equate them to zero. The condition for minimization of $K(t_1,t_3)$ is

Where D is the Hessian matrix

$$D=\begin{vmatrix}\frac{\partial^2 K(t_1,t_3)}{\partial {t_1}^2} & \frac{\partial^2 K(t_1,t_3)}{\partial t_1\partial t_3}\\ \frac{\partial^2 K(t_1,t_3)}{\partial t_1\partial t_3} & \frac{\partial^2 K(t_1,t_3)}{\partial {t_3}^2}\end{vmatrix}>0$$

Differentiating $K(t_1,t_3)$ given in equation (20) with respect to t_1 and equating to zero, we get

$$\frac{C}{\alpha-\gamma t_1}+he^{(\theta t_1^{\eta}+\phi_2 t_1)}\left[\left(\frac{1}{\alpha-\gamma t_1}-\phi_1\right)\left[\int_0^{x(t_3)}e^{-(\theta t^{\eta}+\phi_2 t)}\,dt\right.\right.$$
$$\left.\left.-\int_0^{t_1}e^{-(\theta t^{\eta}+\phi_2 t)}\,dt\right]+\phi_1\int_{t_1}^{x(t_3)}e^{-(\theta t^{\eta}+\phi_2 t)}\,dt\right]\quad=0 \tag{21}$$

Differentiating $K(t_1,t_3)$ with respect to t_3 and equating to zero, we get

$$\frac{-C}{\alpha-\gamma t_3}+he^{-\left[\theta(x(t_3))^{\eta}+\phi_2 x(t_3)\right]}y(t_3)\left[\int_0^{t_1}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{(\theta u^{\eta}+\phi_2 u)}du\right.$$
$$\left.-\phi_1\int_{t_1}^{x(t_3)}e^{(\theta u^{\eta}+\phi_2 u)}du\right]+\frac{\pi}{\phi_2}\left[\left(\frac{3-2e^{\phi_2(t_3-T)}}{\alpha-\gamma t_3}\right)\right.$$

$$-\phi_1\left[1-e^{\phi_2(T-t_3)}+\frac{1}{(\alpha-\gamma t_3)\left[\phi_1+\phi_2 e^{-\phi_2 t_3}\int_{t_3}^{T}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{\phi_2 u}\,du\right]}\right]\Bigg]=0$$

(22)

where, $x(t_3)=t_2=\ t_3+\frac{1}{\phi_2}ln\left[1+\frac{\phi_2}{\phi_1}e^{-\phi_2 t_3}\int_{t_3}^{T}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{\phi_2 u}\,du\right]$

$$y(t_3)=\frac{\partial}{\partial t_3}x(t_3)=\frac{1}{(\alpha-\gamma t_3)\left[\phi_1+\phi_2 e^{-\phi_2 t_3}\int_{t_3}^{T}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{\phi_2 u}\,du\right]}-1$$

Solving the equations (21) and (22) simultaneously, we obtain the optimal time at which production is stopped t_1^* of t_1 and the optimal timet_3^* of t_3 at which the production is restarted after accumulation of backorders. The optimum production quantity Q^* of Q in the cycle of length T is obtained by substituting the optimal values of t_1^*, t_3^* in equation (12) as

$$Q^*=\frac{1}{\gamma}\log\left(\frac{\alpha\,(\alpha-\gamma t_3{}^*)}{(\alpha-\gamma t_1{}^*)(\alpha-\gamma T)}\right) \qquad (23)$$

V. Numerical Illustration

The numerical illustration is carried to explore the effect of changes in model parameters and costs on the optimal policies, by varying each parameter (-15%, -10%, -5%, 0%, 5%, 10%, 15%) at a time for the model under study. The results are presented in Table 1. The relationships between the parameters and the optimal values of the production schedule are shown in Figure 2.

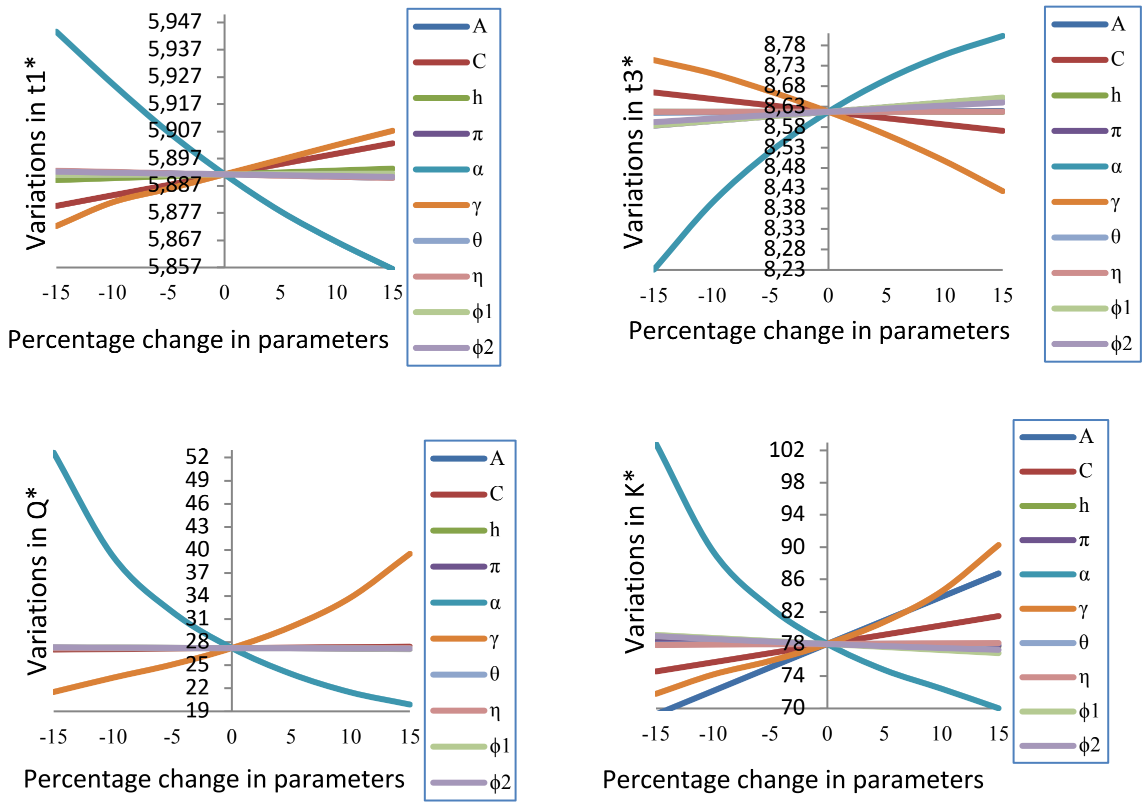

Figure 2: *Relationship between parameters and optimal values with shortages*

Table 1: *Numerical Illustration of the Model - With Shortages*

Variation Parameters	Optimal Policies	-15%	-10%	-5%	0%	5%	10%	15%
A	t_1^*	5.8924	5.892	5.8916	5.8912	5.8908	5.8903	5.8899
	t_3^*	8.6156	8.6162	8.6169	8.6175	8.6181	8.6187	8.6193
	Q^*	27.2344	27.2313	27.2281	27.225	27.2219	27.2187	27.2156
	K^*	69.2564	72.1721	75.0877	78.0034	80.919	83.8347	86.7503
C	t_1^*	5.8796	5.8835	5.8873	5.8912	5.895	5.8988	5.9026
	t_3^*	8.6648	8.6489	8.6332	8.6175	8.6018	8.5863	8.5708
	Q^*	27.0359	27.0993	27.1623	27.225	27.2872	27.3491	27.4106
	K^*	74.5927	75.7248	76.8617	78.0034	79.1498	80.3009	81.4566
h	t_1^*	5.8891	5.8898	5.8905	5.8912	5.8919	5.8926	5.8933
	t_3^*	8.6181	8.6179	8.6177	8.6175	8.6173	8.6171	8.6169
	Q^*	27.218	27.2203	27.2226	27.225	27.2273	27.2297	27.232
	K^*	78.2736	78.1835	78.0934	78.0034	77.9133	77.8233	77.7334
π	t_1^*	5.8909	5.891	5.8911	5.8912	5.8913	5.8914	5.8914
	t_3^*	8.5835	8.5948	8.6061	8.6175	8.6288	8.6402	8.6515
	Q^*	27.3403	27.3019	27.2635	27.225	27.1864	27.1477	27.109
	K^*	78.1961	78.1288	78.0646	78.0034	77.9451	77.8898	77.8375
α	t_1^*	5.9436	5.9245	5.9067	5.8912	5.8776	5.8664	5.8564
	t_3^*	8.2309	8.3944	8.5196	8.6175	8.6961	8.7571	8.8032
	Q^*	52.6288	39.1438	31.9239	27.225	23.8626	21.4632	19.8741
	K^*	102.832	89.4855	82.5151	78.0034	74.7664	72.4419	70.0115
γ	t_1^*	5.8722	5.8809	5.8859	5.8912	5.8966	5.902	5.9072
	t_3^*	8.7439	8.7112	8.6671	8.6175	8.5612	8.4968	8.423
	Q^*	21.5134	23.3696	25.0854	27.225	29.9954	33.7873	39.4837
	K^*	71.8131	74.203	75.8954	78.0034	80.7391	84.512	90.2701
θ	t_1^*	5.8921	5.8918	5.8915	5.8912	5.8909	5.8906	5.8904
	t_3^*	8.6173	8.6174	8.6174	8.6175	8.6175	8.6176	8.6177
	Q^*	27.2277	27.2268	27.2259	27.225	27.2241	27.2233	27.2225
	K^*	77.9196	77.9484	77.9763	78.0034	78.0296	78.0549	78.0796
η	t_1^*	5.8925	5.8921	5.8916	5.8912	5.8907	5.8903	5.8899
	t_3^*	8.6172	8.6173	8.6174	8.6175	8.6176	8.6177	8.6178
	Q^*	27.2292	27.2278	27.2264	27.225	27.2236	27.2222	27.2208
	K^*	77.8796	77.9207	77.962	78.0034	78.0449	78.0864	78.128
ϕ_1	t_1^*	5.8909	5.891	5.8911	5.8912	5.8913	5.8914	5.8915
	t_3^*	8.5841	8.5951	8.6062	8.6175	8.6289	8.6405	8.6523
	Q^*	27.338	27.3009	27.2633	27.225	27.1861	27.1465	27.1062
	K^*	79.1187	78.7492	78.3773	78.0034	77.6275	77.25	76.8709
ϕ_2	t_1^*	5.8922	5.8918	5.8915	5.8912	5.8909	5.8906	5.8903
	t_3^*	8.592	8.6009	8.6094	8.6175	8.6253	8.6328	8.6402
	Q^*	27.3141	27.2828	27.2532	27.225	27.1978	27.1713	27.1455
	K^*	78.9696	78.6124	78.2924	78.0034	77.7405	77.4996	77.2776

VI. Observations

The major observations drawn from the numerical study are:

- It is observed that the costs are having a significant influence on the optimal production quantity and production schedules.
- As the setup cost 'A' decreases, the optimal production downtime t_1^* and the optimal production quantity Q^* are increasing and the total production cost per unit time K^* and the optimal production up time t_3^* are decreasing.
- As the cost per unit 'C' decreases, the optimal production up time t_3^* increases and the optimal production downtime t_1^*, the optimal production quantity Q^* and the total cost per unit time K^* are decreasing.
- As the holding cost 'h' decreases, the optimal production up time t_3^* and the total production cost per unit time K^* are increasing, the optimal production downtime t_1^* and the optimal production quantity Q^* are decreasing.
- As shortage cost 'π' decreases the optimal production downtime t_1^*, the optimal production quantity Q^* and the total production cost per unit time K^* are increasing and the optimal production up time t_3^* decreases.
- As the production rate parameter 'γ' decreases, the optimal production up time t_3^* increases and the optimal production downtime t_1^*, the optimal production quantity Q^* and the total production cost per unit time K^* are decreasing.
- Another production rate parameter 'α' decreases, the optimal production downtime t_1^*, the optimal production quantity Q^* and the total production cost per unit time K^* are increasing and the optimal production up time t_3^* decreases.
- As deteriorating parameter θ decreases, the optimal production downtime t_1^* and the total production cost per unit time K^* are increasing and the optimal production up time t_3^* and the optimal production quantity Q^* are decreasing.
- Another deteriorating parameter η decreases the optimal production downtime t_1^* and the optimal production quantity Q^* are increasing and the optimal production up time t_3^* and the total cost production per unit time K^* are decreasing.
- As demand rate parameter ϕ_1 decreases, the optimal production quantity Q^* and the total production cost per unit time K^* are increasing and the optimal production downtime t_1^* and the optimal production up time t_3^* are decreasing.
- Another demand rate parameter ϕ_2 increases, the optimal production quantity Q^* and the total production cost per unit time K^* are decreasing, the optimal production downtime t_1^* and the optimal production up time t_3^* are increasing.

Finally, from the numerical illustrations we can observe that the parameters are having tremendous influence on the optimal policies of the system.

VII. EPQ Model without Shortages

In this section the inventory model for deteriorating items without shortages is developed and analyzed. Here, it is assumed that shortages are not allowed and the stock level is zero at time $t=0$. The stock level increases during the period $(0, t_1)$ due to excess production after fulfilling the demand and deterioration. The production stops at time t_1 when the stock level reaches S. The inventory decreases gradually due to demand and deterioration in the interval (t_1, T) . At time T the inventory reaches zero. The schematic diagram representing the instantaneous state of inventory is given in Figure 3.

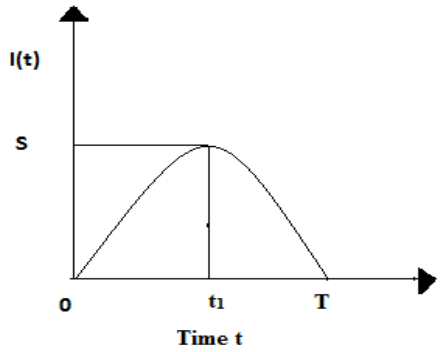

Figure 3: *Schematic diagram representing the inventory level*

Let $I(t)$ be the inventory level of the system at time 't' $(0 \le t \le T)$. Then the differential equations governing the instantaneous state of $I(t)$ over the cycle of length T are:

$$\frac{d}{dt}I(t) + h(t)I(t) = \frac{1}{\alpha-\gamma t} - \left(\phi_1 + \phi_2 I(t)\right) \quad ; \quad 0 \le t \le t_1 \tag{24}$$

$$\frac{d}{dt}I(t) + h(t)I(t) = -\left(\phi_1 + \phi_2 I(t)\right) \quad ; \quad t_1 \le t \le T \tag{25}$$

where, h(t) is as given in equation (3), with the initial conditions I(0) = 0, $I(t_1) = S$ and I(T) = 0.

Substituting h(t) in equations (24) and (25) and solving the differential equations, the on hand inventory at time 't ' is obtained as

$$I(t) = Se^{\theta(t_1^{\eta}-t^{\eta})+\phi_2(t_1-t)} - e^{-(\theta t^{\eta}+\phi_2 t)}\int_t^{t_1}\left(\frac{1}{\alpha-\gamma u} - \phi_1\right)e^{(\theta u^{\eta}+\phi_2 u)}\,du \quad ; \quad 0 \le t \le t_1 \tag{26}$$

$$I(t) = Se^{\theta(t_1^{\eta}-t^{\eta})+\phi_2(t_1-t)} - \phi_1 e^{-(\theta t^{\eta}+\phi_2 t)}\int_{t_1}^{t} e^{(\theta u^{\eta}+\phi_2 u)}\,du \quad ; \quad t_1 \le t \le T \tag{27}$$

Production quantity Q in the cycle of length T is

$$Q = \int_0^{t_1} K(t)dt = \frac{1}{\gamma}\log\left(\frac{\alpha}{(\alpha-\gamma t_1)}\right) \tag{28}$$

From equation (26) and using the initial conditions I(0) = 0, we obtain the value of 'S' as

$$S = e^{-(\theta t_1^{\eta}+\phi_2 t_1)}\int_0^{t_1}\left(\frac{1}{\alpha-\gamma u} - \phi_1\right)e^{(\theta u^{\eta}+\phi_2 u)}du \tag{29}$$

Let $K(t_1)$ be the total production cost per unit time. Since the total production cost is the sum of the set up cost, cost of the units, the inventory holding cost. Therefore the total production cost per unit time becomes

$$K(t_1) = \frac{A}{T} + \frac{CQ}{T} + \frac{h}{T}\left[\int_0^{t_1} I(t)dt + \int_{t_1}^{T} I(t)dt\right] \tag{30}$$

Substituting the values of I(t) and Q in equation (30), one can obtain K(t_1) as

$$K(t_1) = \frac{A}{T} + \frac{C}{\gamma T}\log\left(\frac{\alpha}{\alpha-\gamma t_1}\right)$$

$$+ \frac{h}{T}\left[\int_0^{t_1}\left[Se^{\theta(t_1^{\eta}-t^{\eta})+\phi_2(t_1-t)} - e^{-(\theta t^{\eta}+\phi_2 t)}\int_t^{t_1}\left(\frac{1}{\alpha-\gamma u} - \phi_1\right)e^{(\theta u^{\eta}+\phi_2 u)}\,du\right]dt\right.$$

$$\left. + \int_{t_1}^{T}\left[Se^{\theta(t_1^{\eta}-t^{\eta})+\phi_2(t_1-t)} - \phi_1 e^{-(\theta t^{\eta}+\phi_2 t)}\int_{t_1}^{t} e^{(\theta u^{\eta}+\phi_2 u)}\,du\right]dt\right] \tag{31}$$

0n simplification, one can get

$$K(t_1) = \frac{A}{T} + \frac{C}{\gamma T}\log\left(\frac{\alpha}{\alpha-\gamma t_1}\right) + \frac{h}{T}\left[e^{(\theta t_1^{\eta}+\phi_2 t_1)}\int_0^T Se^{-(\theta t^{\eta}+\phi_2 t)}dt - \int_0^{t_1}\left[e^{-(\theta t^{\eta}+\phi_2 t)}\int_t^{t_1}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{(\theta u^{\eta}+\phi_2 u)}\,du\right]dt - \phi_1\int_{t_1}^{T}\left[e^{-(\theta t^{\eta}+\phi_2 t)}\int_{t_1}^{t}e^{(\theta u^{\eta}+\phi_2 u)}\,du\right]dt\right] \quad (32)$$

Substituting the value of S in equation (32), one can obtain

$$K(t_1) = \frac{A}{T} + \frac{C}{\gamma T}\log\left(\frac{\alpha}{\alpha-\gamma t_1}\right) + \frac{h}{T}\left[\int_0^T\left[e^{-(\theta t^{\eta}+\phi_2 t)}\int_0^{t_1}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{(\theta u^{\eta}+\phi_2 u)}du\right]dt - \int_0^{t_1}\left[e^{-(\theta t^{\eta}+\phi_2 t)}\int_t^{t_1}\left(\frac{1}{\alpha-\gamma u}-\phi_1\right)e^{(\theta u^{\eta}+\phi_2 u)}\,du\right]dt - \phi_1\int_{t_1}^{T}\left[e^{-(\theta t^{\eta}+\phi_2 t)}\int_{t_1}^{t}e^{(\theta u^{\eta}+\phi_2 u)}\,du\right]dt\right] \quad (33)$$

VIII. Optimal Production Schedules of the Model

In this section we obtain the optimal policies of the inventory system under study. To find the optimal values of t_1, we equate the first order partial derivatives of $K(t_1)$ with respect to t_1 equate them to zero. The condition for minimum of $K(t_1)$ is

$$\frac{\partial^2 K(t_1)}{\partial {t_1}^2} > 0$$

Differentiating $K(t_1)$ with respect to t_1 and equating to zero, we get

$$\frac{C}{\alpha-\gamma t_1} + he^{(\theta t_1^{\eta}+\phi_2 t_1)}\left[\left(\frac{1}{\alpha-\gamma t_1}-\phi_1\right)\left[\int_0^T e^{-(\theta t^{\eta}+\phi_2 t)}\,dt - \int_0^{t_1}e^{-(\theta t^{\eta}+\phi_2 t)}\,dt\right] + \phi_1\int_{t_1}^{T}e^{-(\theta t^{\eta}+\phi_2 t)}\,dt\right] = 0 \quad (34)$$

Solving the equation (34), we obtain the optimal time t_1^* of t_1 at which the production is to be stopped.
The optimal production quantity Q^* of Q in the cycle of length T is obtained by substituting the optimal values of t_1 in equation (28).

IX. Numerical Illustration

The numerical illustration is carried to explore the effect of changes in model parameters and costs on the optimal policies, by varying each parameter (-15%, -10%, -5%, 0%, 5%, 10%, 15%) at a time for the model under study. The results are presented in Table 2.

The relationship between the parameters and the optimal values of the production schedule is shown in Figure 4.

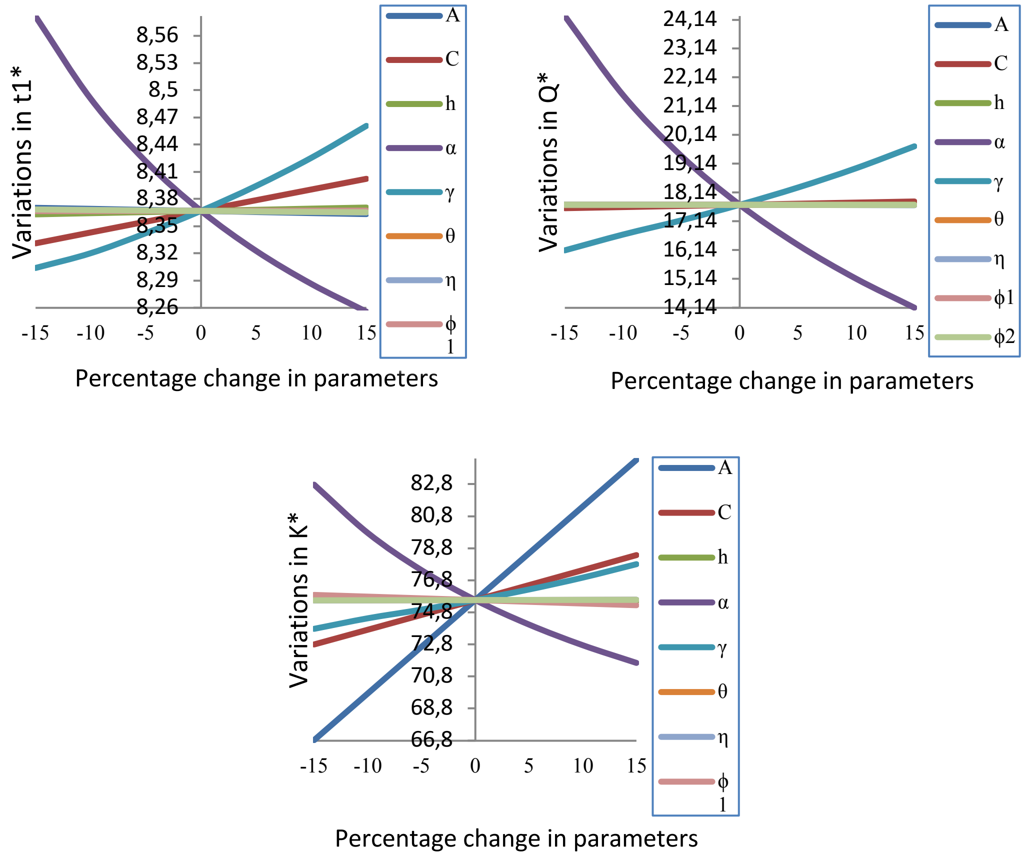

Figure 4: *Relationship between parameters and optimal values without shortages*

Table 2: Numerical illustration *of the model - Without Shortages*

Variation Parameters	Optimal Policies	-15%	-10%	-5%	0%	5%	10%	15%
A	t_1^*	8.3703	8.3692	8.368	8.3669	8.3657	8.3645	8.3634
	Q^*	17.7291	17.7253	17.7215	17.7177	17.7139	17.7101	17.7062
	K^*	66.83	69.7424	72.6549	75.5674	78.4798	81.3923	84.3047
C	t_1^*	8.3311	8.3431	8.355	8.3669	8.3787	8.3905	8.4023
	Q^*	17.6011	17.64	17.6789	17.7177	17.7565	17.7952	17.8339
	K^*	72.7991	73.718	74.6408	75.5674	76.4978	77.432	78.37
h	t_1^*	8.3631	8.3644	8.3656	8.3669	8.3681	8.3694	8.3706
	Q^*	17.7055	17.7096	17.7136	17.7177	17.7218	17.7258	17.7299
	K^*	75.6268	75.6069	75.5871	75.5674	75.5476	75.528	75.5083
α	t_1^*	8.5814	8.4906	8.4214	8.3669	8.3227	8.2863	8.2556
	Q^*	24.275	21.5265	19.4117	17.7177	16.3216	15.1462	14.1399
	K^*	82.7598	79.7446	77.4251	75.5674	74.0371	72.7491	71.6467
γ	t_1^*	8.3039	8.3202	8.3423	8.3669	8.3944	8.4254	8.4607
	Q^*	16.1354	16.6849	17.1767	17.7177	18.3176	18.9892	19.7498
	K^*	73.7729	74.4309	74.9721	75.5674	76.2279	76.967	77.8039
θ	t_1^*	8.3679	8.3676	8.3672	8.3669	8.3665	8.3662	8.3659
	Q^*	17.7212	17.72	17.7188	17.7177	17.7166	17.7155	17.7145
	K^*	75.55	75.5561	75.5619	75.5674	75.5726	75.5776	75.5823
η	t_1^*	8.3685	8.368	8.3674	8.3669	8.3663	8.3657	8.3651
	Q^*	17.7231	17.7213	17.7195	17.7177	17.7158	17.7139	17.712
	K^*	75.5454	75.5529	75.5602	75.5674	75.5743	75.5811	75.5877
ϕ_1	t_1^*	8.3667	8.3667	8.3668	8.3669	8.367	8.3671	8.3672
	Q^*	17.7173	17.7174	17.7176	17.7177	17.7178	17.718	17.7181
	K^*	75.888	75.7811	75.6742	75.5674	75.4605	75.3536	75.2467
ϕ_2	t_1^*	8.3685	8.3679	8.3674	8.3669	8.3663	8.3658	8.3654
	Q^*	17.7231	17.7213	17.7194	17.7177	17.716	17.7144	17.7128
	K^*	75.5528	75.5578	75.5626	75.5674	75.572	75.5765	75.5809

X. Observations

The major observations drawn from the numerical study are:

- It is observed that the costs are having a significant influence on the optimal production quantity and production schedules.
- As the setup cost 'A' decreases, the optimal production time t_1^* and the optimal production quantity Q^* are increasing and the total production cost per unit time K^* decreases.
- As the cost per unit 'C' decreases, the optimal production time t_1^*, the optimal production quantity Q^* and the total production cost per unit time K^* are decreasing.
- As the holding cost 'h' decreases, the total production cost per unit time K^* increases and the optimal production time t_1^* and the optimal production quantity Q^* are decreasing.
- As the production rate parameter 'γ' decreases, the optimal production time t_1^*, the optimal production quantity Q^* and the total production cost per unit time K^* are decreasing.
- As the production rate parameter 'α' decreases, the optimal production time t_1^*, the optimal production quantity Q^* and the total production cost per unit time K^* are increasing.
- As deteriorating rate parameter θ decreases, the total production cost per unit time K^* decreases and the optimal production time t_1^* and the optimal production quantity Q^* are increasing.
- As deteriorating parameter η decreases, the total cost per unit time K^* decreases and the optimal production time t_1^* and the optimal production quantity Q^* are increasing.
- As demand rate parameter ϕ_1 decreases, the total production cost per unit time K^* increases and the optimal production time t_1^* and the optimal production quantity Q^* are decreasing.
- Another demand rate parameter ϕ_2 decreases, the total production cost per unit time K^* decreases and the optimal production time t_1^* and the optimal production quantity Q^* are increasing.

Finally, from the numerical illustration we can observe that the parameters are having tremendous influence on the optimal policies of the system.

XI. Conclusions

This paper addresses the derivation of optimal ordering policies of an EPQ model with the assumption that the production process is random and follows a generalized Pareto distribution. Further it is assumed that the lifetime of the commodity is random and follows a Weibull distribution. The generalized Pareto distribution characterizes the production process more close to the reality. The Weibull rate deterioration can include increasing/decreasing/constant rates of deterioration for different values of parameters. The sensitivity analysis of the model reveals that the replenishment distribution parameters have significant influence on the optimal values of the production uptime, production downtime, production quantity and total cost per a unit time. The deterioration distribution parameters also influencing the optimal values of the model. The replenishment and deterioration distributions can be estimated by using historical data. With the distributional data the production and deterioration distributions parameters can be estimated and the analysis of the production process can obtain the optimal production downtime and uptime. This model also includes some of the earlier models as particular cases for specific or limiting values of the parameters. This model can be extended for the cases of changing money value (inflation) and multicommodity production systems which will be taken elsewhere.

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Declaration of Conflicting Interests

The Authors declare that there is no conflict of interest.

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