ANALYSIS OF A TWO-STATE PARALLEL SERVERS RETRIAL QUEUEING MODEL WITH BATCH DEPARTURES

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Abstract

This paper deals with the transient state behavior of an M/M/1 retrial queueing model contains two parallel servers with departures occur in batches. At the arrival epoch, if all servers are busy then customers join the retrial group. Whereas, if the customers find any of one server is free then they join the free server and start its service immediately. Here, we assume that primary customers arrive according to Poisson process. The retrial customers also follow the same fashion. Service time follows an exponential distribution. Explicit time dependent probabilities of exact number of arrivals and exact number of departures when both servers are free or when one server is busy or when both servers are busy are obtained by solving the difference differential equation recursively. Some important verification and conversion of two-state model into single state are also discussed. Some of the existing results in the form of special cases have been deduced.

Keywords: Retrial, Queueing, Arrivals, Departures, Batch

1. Introduction

In recent years, computer networks and data communication systems are the fastest growing technologies, which have led to significant development in applications such as advance in internet, audio data traffic, video data traffic, etc. Recently there have been significant contributions to retrial queueing system in which arriving customer who finds the server busy upon arrival is require leaving the service area and repeating his demand after some time. Between trials, a blocked customer who remains in a retrial group is said to be in orbit. Retrial queue have applications in telephone switching systems, telecommunication networks and computers are competing to gain service from a central processing unit. Moreover, retrial queues are also used as mathematical models of several computer systems: packet switching networks, shared bus local area networks operating under the carrier-sense multiple access protocol and collision avoidance star local area networks etc. There are enough of literatures available on retrial queues. We referred some of the work like Artalejo and Corral [1], Falin and Templeton [2] and Artalejo [3] etc.

In many queueing systems it is assumed that customers arrive singly at a service facility and depart singly from the service facility. However, this assumption is violated in many other real word situations. Letters arriving at a post office, ships arriving at a port in convoy, people going to a theatre and so on are some examples of queuing in which customers do not arrive and depart singly but in bulk or groups. The size of an arriving group and departing group may be a random variable or a fixed number. Mathematically as well as practically the cases where the size of an arriving group and departing group is a random variable, are more common, and also more difficult to handle.

One can note that the batch arrival queue may not always be given the name 'batch' but instead of this many authors chose to use the term 'bulk'. Predominantly, this reflects two leading strands of applications, where 'bulk' often gives a connotation of transportation settings whereas 'batch' frequently implies applications in communications.

Queueing situations in which arrivals occur singly, but service is in bulk are considered in this research. Bulk service queues have potential applications in many areas e.g. in loading and unloading of cargoes at a seaport, in traffic signal systems, in computer networks where jobs are processed in batches, manufacturing/ production systems, cinema halls, in transportation processes involving buses, airplanes, trains, ships, elevators etc. Bailey [4] introduced the concept of bulk service and the same was later studied by a number of parishioners. Juan [5] obtained a numerical method for the single server bulk service queueing system with variable capacity. Janssen and Leeuwaarden [6] presented an analytic rather than a numerical framework for dealing with discrete time bulk service queue. Goswami et al. [7] analysed a discrete time single server infinite buffer bulk service queues. In this research, the inter-arrival time of successive arrivals and service times of batches are assumed to be independent and geometrically distributed. Al-khedhairi and Tadj [8] investigated the queueing process of a bulk service queueing system under Bernoulli schedule.

The classical transient results for the M/M/1, M/M/c and M/G/1 queue provide little insight into the behavior of a queueing system through a fixed operation time t. The function $P_n(t)$ gives the distribution for the number in the system at time t, but practically provides no information on how the system has regulated up until time t. The question seems to be answered by Pegden and Rosenshine [9]. The analysis of their paper based on M/M/1 queueing model in which the state of the system is given by (i, j), where i is the number of arrivals and j is the number of departures until time t. Kalra and Singla [10] investigated the performance analysis of a two-state retrial queueing model with batch departures. In this paper, they obtained time dependent probabilities of exact number of arrivals in the system and exact number of departures from the system when only one server is free or busy. Garg and Kumar [11] studied a single server retrial queue with impatient customers and obtained time-dependent probabilities of number of exact arrivals and number of exact departures from the orbit.

This research studies a time dependent retrial queueing model by obtaining the explicit probabilities of the exact number of arrivals in the system and the exact number of departures from the system by a given time t wherein the departures occur from the orbit in batches of variable size.

The rest of this paper is organized as follows: Section 2 gives a relatively formal description of the queueing model. In Section 3, we defined the two-dimensional state model and derived the difference-differential equations. The time dependent solution for the model is obtained in section 4. Section 5 presents the some useful performance measures of the system and Section 6 discussed some special cases. The last section ends with a suitable conclusion.

2. Model Description

2.1. Assumption and Notation

The two parallel servers retrial queueing system is considered wherein departures take place in batches of variable size whenever these occur from the orbit. The primary calls follow a Poisson distribution with rate λ. If the server is busy at the arrival time, then the arriving call joins the orbit, whereas if the server is free then the service of arriving call gets started. The behavior of customers in orbit is same as in the main model, i.e. every customer in orbit produces a Poisson flow of repeated calls with rate θ. If a batch of repeated calls finds the server free, it is served and leaves the system after service otherwise, if the server is occupied at that time then the system state does not change. Arrivals occur one by one and departures occur from the orbit in batches of variable size with rate μ. The input flow of primary calls, intervals between repeated trials and service times are mutually independent. For distribution of arrivals, service times and retrials, we make use of the following assumptions and notations:

1) The repeated calls for each server follow a Poisson distribution with parameter θ.
2) In this model the departures occur from the orbit is treated as bulk departures whose capacity is determined afresh before each service which is equal to newly determined capacity of the server or units present in the orbit, whichever is less. In this case capacity of the server is a random variable. The size of the batch is determined at beginning of the each service. The probability that the server can serve a batch of γ units is b_γ so that $\sum_{\gamma=1}^{K} b_\gamma = 1$, where K is the maximum capacity of the server.
3) The Service times for each call depart in batches of variable size and follow an exponential distribution with parameter μ.

Laplace transformation $\bar{f}(s)$ of $f(t)$ is given by

$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$, Re (s) > 0

The Laplace inverse of

$\frac{Q(p)}{P(p)}$ is $\sum_{k=1}^{n}\sum_{l=1}^{m_k} \frac{t^{m_k-l} e^{a_k t}}{(m_k-l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}}\frac{Q(p)}{P(p)}$ $(p-a_k)^{m_k}$ $\forall$ p= a_k, $a_i \neq a_k$ for i ≠ k.

where,

$P(p) = (p-a_1)^{m_1}\, (p-a_2)^{m_2} \dots\dots\dots (p-a_n)^{m_n}$

$Q(p)$ is a polynomial of degree < m_1+m_2+m_3 +…………m_n-1.

If $L^{-1}\{p(s)\} = P(t)$ and $L^{-1}\{q(s)\} = Q(t)$, then

$L^{-1}\{p(s)\, q(s)\} = \int_0^t P(u)Q(t-u)du = P * Q$, where $P * Q$ is the convolution of P and Q.

3. The Two-Dimensional State Model

3.1. Definitions

$P_{i,j,0}(t)$= Probability that there are exactly i arrivals in the system and j departures from the system by time t when server is idle.

$P_{i,j,k}(t)$= Probability that there are exactly i arrivals in the system and j departures from the system by time t when k servers are busy. $k = 1,2.$

$P_{i,j}(t)$= Probability that there are exactly i arrivals in the system and j departures from the system by time t.

$$P_{i,j}(t) = P_{i,j,0}(t) + P_{i,j,1}(t) \qquad \forall i, j; \quad i \ge j.$$

Also

$$P_{i,j,1}(t) = 0, i \le j; \; P_{i,j,0}(t) = 0, i < j.$$

Initially

$$P_{0,0,0}(0) = 1; \; P_{i,j,0}(0) = 0 \; \& \; P_{i,j,k}(0) = 0; \;\; \forall i,j \ne 0. \; k = 1,2.$$

3.2. The difference – differential equations governing the system are

$$\frac{d}{dt}P_{i,i,0}(t) = -\lambda P_{i,i,0}(t) + \mu \sum_{\gamma=1}^{K}\left(\sum_{l=\gamma}^{K} b_l\right) P_{i,i-\gamma,1}(t) \qquad i \ge 0, i \ge \mathrm{K} \tag{1}$$

$$\frac{d}{dt}P_{i,j,0}(t) = -(\lambda + (i-j)\theta)P_{i,j,0}(t) + \mu \sum_{\gamma=1}^{K} b_\gamma P_{i,j-\gamma,1}(t) \qquad i > j, i > 0; j \ge \mathrm{K} \tag{2}$$

$$\frac{d}{dt}P_{1,0,1}(t) = -(\lambda + \mu)P_{1,0,1}(t) + \lambda P_{0,0,0}(t) \tag{3}$$

$$\frac{d}{dt}P_{2,0,2}(t) = -(\lambda + \mu)P_{2,0,2}(t) + \lambda P_{1,0,1}(t) \tag{4}$$

$$\frac{d}{dt}P_{i,j,1}(t) = -(\lambda + \mu + (i-j-1)\theta)P_{i,j,1}(t) + \lambda P_{i-1,j,0}(t) + (i-j)\theta P_{i,j,0}(t) + 2\mu P_{i,j-1,2}(t) \qquad i > 1, i > j \ge 0 \tag{5}$$

$$\frac{d}{dt}P_{i,j,2}(t) = -(\lambda + 2\mu)P_{i,j,2}(t) + \lambda P_{i-1,j,1}(t) + \lambda\left(1 - \delta_{i-2,j}\right)P_{i-1,j,2}(t) + (i-j-1)\theta P_{i,j,1}(t) \qquad i > 2, i > j \ge 0 \tag{6}$$

$$\text{where } \delta_{i-2,j} = \begin{cases} 1, & when\ i-2=j \\ 0, & otherwise \end{cases}$$

Using the Laplace transformation $\bar{f}$ (s) of $f(t)$which is given by

$\bar{f}$ (s) = $\int_0^\infty e^{-st} f(t)$ dt, Re (s) > 0

in the equations (1) - (6) along with the initial conditions, the following equations are obtained:

$$\left.\begin{aligned} (s+\lambda)\bar{P}_{0,0,0}(s) &= P_{0,0,0}(0) \\ (s+\lambda)\bar{P}_{i,i,0}(s) &= \mu \sum_{\gamma=1}^{K}\left(\sum_{l=\gamma}^{K} b_l\right)\bar{P}_{i,i-\gamma,1}(s) \end{aligned}\right\} \qquad i > 0, i \ge \mathrm{K} \tag{7}$$

$$(s + \lambda + (i-j)\theta)\bar{P}_{i,j,0}(s) = \mu \sum_{\gamma=1}^{K} b_\gamma \bar{P}_{i,j-\gamma,1}(s) \qquad i > j, i > 0, \; j \ge \mathrm{K} \tag{8}$$

$$(s + \lambda + \mu)\bar{P}_{1,0,1}(s) = \lambda \bar{P}_{0,0,0}(s) \tag{9}$$

$$(s + \lambda + \mu)\bar{P}_{2,0,2}(s) = \lambda \bar{P}_{1,0,1}(s) \tag{10}$$

$$(s + \lambda + \mu + (i-j-1)\theta)\bar{P}_{i,j,1}(s) = \lambda \bar{P}_{i-1,j,0}(s) + (i-j)\theta\bar{P}_{i,j,0}(s) + 2\mu\bar{P}_{i,j,-1,2}(s) \qquad i > 1, i > j \ge 0 \tag{11}$$

$$(s + \lambda + 2\mu)\bar{P}_{i,j,2}(s) = \lambda \bar{P}_{i-1,j,1}(s) + \lambda\left(1 - \delta_{i-1,j}\right)\bar{P}_{i-1,j,2}(s) + (i-j-1)\theta\bar{P}_{i,j,1}(s) \qquad i > 2, i > j \ge 0 \tag{12}$$

3.3. Solution of the Problem

Solving equations (7) to (12) recursively, the following results are obtained

$$\bar{P}_{0,0,0}(s) = \frac{1}{s+\lambda} \tag{13}$$

$$\bar{P}_{1,1,0}(s) = \frac{\lambda\mu}{(s+\lambda)^2\,(s+\lambda+\mu)} \tag{14}$$

$$\bar{P}_{i,i,0}(s) = \frac{1}{s+\lambda}\mu \sum_{\gamma=1}^{K}\left(\sum_{l=\gamma}^{K} b_l\right)\bar{P}_{i,i-\gamma,1}(s) \qquad i > 1 \tag{15}$$

$$\bar{P}_{i,2,0}(s) = \frac{\mu}{(s+\lambda+\mu+(i-2)\theta)}\left[b_1\bar{P}_{i,1,1}(s) + b_2\bar{P}_{i,0,1}(s)\right] \quad i > 2 \quad (16)$$

$$\bar{P}_{1,0,1}(s) = \left(\frac{1}{s+\lambda}\right)\left(\frac{\lambda}{s+\lambda+\mu}\right) \quad (17)$$

$$\bar{P}_{2,1,1}(s) = \frac{\lambda}{(s+\lambda+\mu)}\bar{P}_{1,1,0}(s) + 2\mu\frac{\lambda}{(s+\lambda+2\mu)(s+\lambda+\mu)}\bar{P}_{1,0,1} \quad (18)$$

$$\bar{P}_{i,1,1}(s) = \frac{2\mu}{(s+\lambda+\mu+(i-2)\theta)}\frac{\lambda^{i-1}}{(s+\lambda+2\mu)^{i-1}}\bar{P}_{1,0,1}(s) \quad i > 2 \quad (19)$$

$$\bar{P}_{i,i-1,1}(s) = \frac{\lambda}{(s+\lambda+\mu)}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{(s+\lambda+\mu)}\bar{P}_{i,i-1,0}(s) + \frac{2\mu}{(s+\lambda+\mu)}\bar{P}_{i,i-2,2}(s) \quad i > 2 \quad (20)$$

$$\bar{P}_{i,0,2}(s) = \frac{\lambda^{i-1}}{(s+\lambda+2\mu)^{i-1}}\bar{P}_{1,0,1}(s) \quad i > 1 \quad (21)$$

$$\bar{P}_{i,j,2}(s) = \left(\sum_{k=1}^{i-j}\left(\frac{\lambda}{s+\lambda+2\mu}\right)^{i-j-k}\eta'_k(s)\bar{P}_{j+k,j,1}(s)\right) \quad i \geq j+2, j \geq 1 \quad (22)$$

$$\text{where } \eta'_k(s) = \begin{cases} 1 & for\ k = 1 \\ \left(1 + \frac{(k-1)\theta}{s+\lambda+2\mu}\right) & for\ k = 2\ to\ i-j-1 \\ \frac{(k-1)\theta}{s+\lambda+2\mu} & for\ k = i-j \end{cases}$$

$$\bar{P}_{i,j,1}(s) = \frac{\lambda}{(s+\lambda+\mu+(i-j-1)\theta)}\bar{P}_{i-1,j,0}(s) + \frac{(i-j)\theta}{(s+\lambda+\mu+(i-j-1)\theta)}\bar{P}_{i,j,0}(s)$$
$$\frac{2\mu}{(s+\lambda+\mu+(i-j-1)\theta)}\left(\sum_{k=0}^{i-j}\left(\frac{\lambda}{s+\lambda+2\mu}\right)^{i-j-k}\eta'_k(s)\bar{P}_{j+k,j-1,1}(s)\right)$$
$$i \geq j+2, j \geq 2 \quad (23)$$

$$where\ \eta'_k(s) = \begin{cases} 1 & for\ k = 0 \\ \left(1 + \frac{k\theta}{s+\lambda+2\mu}\right) & for\ k = 1\ to\ i-j-1 \\ \frac{k\theta}{s+\lambda+2\mu} & for\ k = i-j \end{cases}$$

$$\bar{P}_{i,j,0}(s) = \frac{1}{(s+\lambda+(i-j)\theta)}\left(\mu\sum_{\gamma=1}^{K} b_\gamma\right)\bar{P}_{i,j-\gamma,1}(s) \quad i > j \geq 3 \quad (24)$$

Using the Inverse Laplace transformation

$\frac{Q(p)}{P(p)}$ is $\sum_{k=1}^{n}\sum_{l=1}^{m_k}\frac{t^{m_k-l}e^{a_k t}}{(m_k-l)!(l-1)!}\times\frac{d^{l-1}}{dp^{l-1}}\left(\frac{Q(p)}{P(p)}\right)(p-a_k)^{m_k}$ $\forall\, p = a_k$, $a_i \neq a_k$ for $i \neq k$.

where

$P(p) = (p-a_1)^{m_1}(p-a_2)^{m_2}\dots\dots\dots(p-a_n)^{m_n}$

$Q(p)$ is a polynomial of degree $< m_1 + m_2 + m_3 + \dots\dots\dots m_n - 1$.

If L^{-1}\{f(s)\} = F(t) and L^{-1}\{g(s)\} = G(t), then

L^{-1}\{f(s) g(s)\} = $\int_0^t F(u)G(t-u)du$ = F * G, F * G is called the convolution of F and G.

and

The Laplace inverse of $\bar{N}^{a,b,c}_{n_1,n_2,n_3}(s) = \frac{1}{(s+a)^{n_1}(s+b)^{n_2}(s+c)^{n_3}}$ is

$$N^{a,b,c}_{n_1,n_2,n_3}(t) = \sum_{l=1}^{n_3}\sum_{m=1}^{l}\frac{e^{-at}\ t^{n_3-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_2+g_2)\right)}{(n_3-l)!(m-1)!\,(b-a)^{n_2+m-1}(c-a)^{n_1+l-m}}$$

$$+\sum_{l=1}^{n_2}\sum_{m=1}^{l}\frac{e^{-bt}\ t^{n_2-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_2-l)!\,(m-1)!\ (a-b)^{n_3+m-1}(c-b)^{n_1+l-m}}$$

$$+\sum_{l=1}^{n_1}\sum_{m=1}^{l}\frac{e^{-ct}\ t^{n_1-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(n_2+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_1-l)!\,(m-1)!\ (a-c)^{n_3+m-1}(b-c)^{n_2+l-m}}$$

in equations (13) to (24), the following probabilities are

$$P_{0,0,0}(t) = e^{-\lambda t} \tag{25}$$

$$P_{1,1,0}(t) = \lambda\mu(te^{-\lambda t})e^{-(\lambda+\mu)t} \tag{26}$$

$$P_{i,i,0}(t) = \left\{\mu\sum_{\gamma=1}^{K}\left(\sum_{l=\gamma}^{K} b_l\right)e^{-\lambda t}\right\} * P_{1,i-\gamma,1}(t) \qquad i>1 \tag{27}$$

$$P_{i,2,0}(t) = \mu b_1 e^{-(\lambda+\mu+(i-2)\theta)t} * P_{i,1,1}(t) + \mu b_2 e^{-(\lambda+\mu+(i-2)\theta)t} * P_{i,0,1}(t) \qquad i>2 \tag{28}$$

$$P_{1,0,1}(t) = \lambda e^{-\lambda t}\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right) \tag{29}$$

$$P_{2,1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{1,1,0}(t) + 2\lambda\mu e^{-(\lambda+\mu)t}\left(\frac{1}{2\mu}-\frac{e^{-2\mu t}}{2\mu}\right) * P_{1,0,1}(t) \tag{30}$$

$$P_{i,1,1}(t) = \left[2\mu\lambda^{i-1}e^{-(\lambda+\mu+(i-2)\theta)t}\left\{\frac{1}{(2\mu)^{i-1}} - e^{-2\mu t}\sum_{r=0}^{i-2}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-r}}\right\}\right] * P_{1,0,1}(t) \qquad i>2 \tag{31}$$

$$P_{i,i-1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu)t} * P_{i,i-1,0}(t) + 2\mu e^{-(\lambda+\mu)t} * P_{i,i-2,2}(t) \qquad i>2 \tag{32}$$

$$P_{i,0,2}(t) = \left(\lambda^{i-1}\frac{t^{i-2}}{(i-2)!}e^{-(\lambda+2\mu)t}\right) * P_{1,0,1}(t) \qquad i>1 \tag{33}$$

$$P_{i,j,2}(t) = \left(\lambda^{i-j-1}\frac{t^{i-j-2}}{(i-j-2)!}e^{-(\lambda+2\mu)t}\right) * P_{j+1,j,1}(t) + \sum_{k=2}^{i-j-1}\left(\lambda^{i-j-k}\frac{t^{i-j-k-1}}{(i-j-k-1)!}e^{-(\lambda+2\mu)t}\right) * P_{j+k,j,1}(t) + \sum_{k=2}^{i-j-1}\left(\lambda^{i-j-k}(\mathrm{k}-1)\theta\frac{t^{i-j-k}}{(i-j-k)!}e^{-(\lambda+2\mu)t}\right) * P_{j+k,j,1}(t) + \left((\mathrm{i}-\mathrm{j}-1)\theta e^{-(\lambda+2\mu)t}\right) * P_{i,j,1}(t) \qquad i\ge j+2, j\ge 1 \tag{34}$$

$$P_{i,j,1}(t) = \lambda e^{-(\lambda+\mu+(i-j-1)\theta)t} * P_{i-1,j,0}(t) + (i-j)\theta e^{-(\lambda+\mu+(i-j-1)\theta)t} * P_{i,j,0}(t) + 2\mu\lambda^{i-j}e^{-(\lambda+\mu+(i-j-1)\theta)t}\left\{\frac{1}{(2\mu)^{i-j}} - e^{-2\mu t}\sum_{r=1}^{i-j-1}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-r}}\right\} * P_{j,j-1,1}(t) + 2\mu e^{-(\lambda+\mu+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\left\{\frac{1}{(2\mu)^{i-j-k}} - e^{-2\mu t}\sum_{r=0}^{i-j-k-1}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-k-r}}\right\} * P_{j+k,j-1,1}(t) + 2\mu e^{-(\lambda+\mu+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}(\mathrm{k}\theta)\left\{\frac{1}{(2\mu)^{i-j-k+1}} - e^{-2\mu t}\sum_{r=0}^{i-j-k}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-k+1-r}}\right\} * P_{j+k,j-1,1}(t) + 2\mu(i-j)\theta e^{-(\lambda+\mu+(i-j-1)\theta)t}\left(\frac{1}{2\mu}-\frac{e^{-2\mu t}}{2\mu}\right) * P_{i,j-1,1}(t)$$

$i \geq j+2, j \geq 2$ (35)

$P_{i,j,0}(t) = \left(\mu \sum_{\gamma=1}^{K} b_\gamma e^{-(\lambda+\mu+(i-j)\theta)t}\right) * P_{i,j-\gamma,1}(t)$ $\quad i > j \geq 3$ (36)

4. Measures of Effectiveness

4.1. The Laplace transform of the probability $P_{i.}(t)$ that exactly i units arrive by time t is :

$$\bar{P}_{i.}(s) = \sum_{j=0}^{i} \bar{P}_{i,j}(s) = \frac{\lambda^i}{(s+\lambda)^{i+1}};i > 0 \quad (37)$$

And its Inverse Laplace transform is

$$P_{i.}(t) = \frac{e^{-\lambda t}(\lambda t)^i}{i!} \quad (38)$$

The basic assumption on primary arrivals is that it forms a Poisson process and above analysis of abstract solution also verifies the same.

4.2. The probability that exactly j customers have been served by time t. $P_{.j}(t)$ in terms of $P_{i,j}(t)$ is given by:

$$P_{.j}(t) = \sum_{i=j}^{\infty} P_{i,j}(t)$$

4.3. From the abstract solution of our model, we verified that the sum of all possible probabilities is one i.e. taking summation over i and j on equations (15)-(31) and adding, we get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\{\bar{P}_{i,j,0}(s) + \bar{P}_{i,j,1}(s) + \bar{P}_{i,j,2}(s)\} = \frac{1}{s}$$

Taking inverse Laplace transformation, we get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\{P_{i,j,0}(t) + P_{i,j,1}(t) + +P_{i,j,2}(t)\} = 1,$$

which is a verification of our results.

4.4. Converting two-state model into single state model:

To convert two-dimensional state model into a single state model probability $Q_{n,k}(t)$ is defined as under:

$Q_{n,k}(t)$ = Probability that there are n customers in the orbit at time t and the servers are free or busy according as $k = 0,1,2$.

The probability of exactly n customers in the system at time t in terms of $P_{i,j,0}(t)$ and $P_{i,j,k}(t)$:

When the server is free, it is defined by probability $Q_{n,0}(t)$

$$Q_{n,0}(t) = \sum_{j=0}^{\infty} P_{j+n,j,0}(t)$$

In this case, the number of customers in the orbit is equal to n which is obtained by using:

n = (number of arrivals – number of departures)

When k servers are busy, it is defined by probability $Q_{n,k}(t)$

$$Q_{n,k}(t) = \sum_{j=0}^{\infty} P_{j+n+k,j,k}(t) \quad (k = 1,2)$$

where k defines the number of servers.

In this case, the number of customers in the orbit is equal to n which is obtained by using:

n = (number of arrivals – number of departures – k)

Using the above definitions from the equations (1) to (6) the set of equations in statistical equilibrium are:

$$\lambda Q_{0,0} = \mu\left[\sum_{\gamma=1}^{K}\left(\sum_{l=\gamma}^{K} b_l\right)\right]Q_{\gamma,1} \tag{39}$$

$$(\lambda + n\theta)Q_{n,0} = \mu\left(\sum_{l=\gamma}^{K} b_l\right)Q_{n+\gamma,1} \qquad n > 0 \tag{40}$$

$$(\lambda + n\theta + \mu)Q_{n,1} = \lambda Q_{n,} + (n+1)\theta Q_{n+1,0} + 2\mu Q_{n,2,0} \qquad n \geq 0 \tag{41}$$

$$(\lambda + 2\mu)Q_{n,2} = \lambda Q_{n,1} + (n+1)\theta Q_{n+1,1} + \lambda Q_{n-1,2} \qquad n \geq 0 \tag{42}$$

4.5. Special Case:

1. When the units are served singly and considering$K = 1,\ b_1 = 1,\ b_2 = b_3 = b_4 = \cdots = b_K = 0$ in equations (25) to (36), then the probabilities coincide with the results of Singla and Kalra [12].

$$P_{0,0,0}(t) = e^{-\lambda t} \tag{43}$$

$$P_{1,1,0}(t) = \lambda\mu(te^{-\lambda t})e^{-(\lambda+\mu)t} \tag{44}$$

$$P_{i,i,0}(t) = \lambda\mu e^{-\lambda t}\left(\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right) * P_{i-1,i-1,0}(t) + \mu\theta e^{-\lambda t}\left(\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right) * P_{i,i-1,0}(t) + 2\mu^2 e^{-\lambda t}\left(\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right) * P_{i,i-2,2}(t) \qquad i > 1 \tag{45}$$

$$P_{i,2,0}(t) = 2\mu^2 e^{-(\lambda+(i-2)\theta)t}\left(\frac{1}{(\mu+(i-2)\theta)} - \frac{e^{-(\mu+(i-2)\theta)t}}{(\mu+(i-2)\theta)}\right) * P_{i,0,2}(t) \qquad i \geq 3 \tag{46}$$

$$P_{1,0,1}(t) = \lambda e^{-\lambda t}\left(\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right) \tag{47}$$

$$P_{2,1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{1,1,0}(t) + 2\lambda\mu e^{-(\lambda+\mu)t}\left(\frac{1}{2\mu} - \frac{e^{-2\mu t}}{2\mu}\right) * P_{1,0,1}(t) \tag{48}$$

$$P_{i,1,1}(t) = \left[2\mu\lambda^{i-1}e^{-(\lambda+\mu+(i-2)\theta)t}\left\{\frac{1}{(2\mu)^{i-1}} - e^{-2\mu t}\sum_{r=0}^{i-2}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-r}}\right\}\right] * P_{1,0,1}(t) \qquad i > 2 \tag{49}$$

$$P_{i,i-1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu)t} * P_{i,i-1,0}(t) + 2\mu e^{-(\lambda+\mu)t} * P_{i,i-2,2}(t) \qquad i > 2 \tag{50}$$

$$P_{i,0,2}(t) = \left(\lambda^{i-1}\frac{t^{i-2}}{(i-2)!}e^{-(\lambda+2\mu)t}\right) * P_{1,0,1}(t) \qquad i > 1 \tag{51}$$

$$P_{i,j,2}(t) = \left(\lambda^{i-j-1}\frac{t^{i-j-2}}{(i-j-2)!}e^{-(\lambda+2\mu)t}\right) * P_{j+1,j,1}(t) + \sum_{k=2}^{i-j-1}\left(\lambda^{i-j-k}\frac{t^{i-j-k-1}}{(i-j-k-1)!}e^{-(\lambda+2\mu)t}\right) * P_{j+k,j,1}(t) + \sum_{k=2}^{i-j-1}\left(\lambda^{i-j-k}(\mathrm{k}-1)\theta\frac{t^{i-j-k}}{(i-j-k)!}e^{-(\lambda+2\mu)t}\right) * P_{j+k,j,1}(t) + \left((\mathrm{i}-\mathrm{j}-1)\theta e^{-(\lambda+2\mu)t}\right) * P_{i,j,1}(t) \qquad i \geq j+2, j \geq 1 \tag{52}$$

$$P_{i,j,1}(t) = \lambda e^{-(\lambda+\mu+(i-j-1)\theta)t} * P_{i-1,j,0}(t) + (i-j)\theta e^{-(\lambda+\mu+(i-j-1)\theta)t} * P_{i,j,0}(t) + 2\mu\lambda^{i-j}e^{-(\lambda+\mu+(i-j-1)\theta)t}\left\{\frac{1}{(2\mu)^{i-j}} - e^{-2\mu t}\sum_{r=1}^{i-j-1}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-r}}\right\} * P_{j,j-1,1}(t) + 2\mu e^{-(\lambda+\mu+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\left\{\frac{1}{(2\mu)^{i-j-k}} - e^{-2\mu t}\sum_{r=0}^{i-j-k-1}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-k-r}}\right\} *$$

$$P_{j+k,j-1,1}(t) +$$

$$2\mu e^{-(\lambda+\mu+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\,(\mathrm{k}\theta)\left\{\frac{1}{(2\mu)^{i-j-k+1}}-e^{-2\mu t}\sum_{r=0}^{i-j-k}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-k+1-r}}\right\}*$$

$$P_{j+k,j-1,1}(t) + 2\mu(i-j)\theta e^{-(\lambda+\mu+(i-j-1)\theta)t}\left(\frac{1}{2\mu}-\frac{e^{-2\mu t}}{2\mu}\right)*P_{i,j-1,1}(t)$$

$$i \geq j+2, j \geq 2 \qquad (53)$$

$$P_{i,j,0}(t) = \lambda\mu e^{-(\lambda+(i-j)\theta)t}\left(\frac{1}{\mu+(i-j)\theta}-\frac{e^{-(\mu+(i-j)\theta)t}}{\mu+(i-j)\theta}\right)*P_{i-1,j-1,0}(t) +$$

$$\mu(i-j+1)\theta e^{-(\lambda+(i-j)\theta)t}\left(\frac{1}{\mu+(i-j)\theta}-\frac{e^{-(\mu+(i-j)\theta)t}}{\mu+(i-j)\theta}\right)*P_{i,j-1,0}(t)$$

$$+ 2\mu^2\lambda^{i-j+1}\left[\sum_{l=1}^{i-j+1}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\;t^{(i-j+1)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{\mathrm{g_1=0}}^{\mathrm{l-m-1}}(1+\mathrm{g_1})\right)\left(\prod_{\mathrm{g_2=0}}^{\mathrm{m-2}}(1+\mathrm{g_2})\right)}{((i-j+1)-l)!\,(m-1)!\;(\mu)^m(2\mu-(i-j)\theta)^{1+l-m}}\right.$$

$$\left.-\;\frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)^{(i-j+1)}(\mu-(i-j)\theta)}+\frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)^{(i-j+1)}(\mu-(i-j)\theta)}\right]*P_{j-1,j-2,1}(t) + 2\mu^2\sum_{k=1}^{i-j}\lambda^{(i-j+1)-k}$$

$$\left[\sum_{l=1}^{(i-j+1)-k}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\;t^{((i-j+1)-k)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{\mathrm{g_1=0}}^{\mathrm{l-m-1}}(1+\mathrm{g_1})\right)\left(\prod_{\mathrm{g_2=0}}^{\mathrm{m-2}}(1+\mathrm{g_2})\right)}{(((i-j+1)-k)-l)!(m-1)!\,(\mu)^m(2\mu-(i-j)\theta)^{1+l-m}}-\right.$$

$$\left.\frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)^{(i-j+1)-k}(\mu-(i-j)\theta)}+\frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)^{(i-j+1)-k}(\mu-(i-j)\theta)}\right]*P_{j+k-1,j-2,1}(t) + 2\mu^2\sum_{k=1}^{i-j}\lambda^{(i-j+1)-k}(k\theta)$$

$$\left[\sum_{l=1}^{((i-j+1)-k)+1}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\;t^{(((i-j+1)-k)+1)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{\mathrm{g_1=0}}^{\mathrm{l-m-1}}(1+\mathrm{g_1})\right)\left(\prod_{\mathrm{g_2=0}}^{\mathrm{m-2}}(1+\mathrm{g_2})\right)}{((((i-j+1)-k)+1)-l)!(m-1)!\,(\mu)^m(2\mu-(i-j)\theta)^{1+l-m}}-\right.$$

$$\left.\frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)^{((i-j+1)-k)+1}(\mu-(i-j)\theta)}+\frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)^{((i-j+1)-k)+1}(\mu-(i-j)\theta)}\right]*P_{j+k-1,j-2,1}(t) + 2\mu^2\,(i-j+$$

$$1)\theta\left[\frac{e^{-(\lambda+(i-j)\theta)t}}{(\mu)(2\mu-(i-j)\theta)}-\frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)(\mu-(i-j)\theta)}+\frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)(\mu-(i-j)\theta)}\right]*P_{i,j-2,1}(t)$$

$$i > j \geq 3 \qquad (54)$$

2. Letting $K = 1,\ b_1 = 1,\ b_2 = b_3 = b_4 = \cdots = b_K = 0$ and $\mu = 1$ in (39) to (42), then the following equations are:

$$(\lambda + n\theta)Q_{n,0} = Q_{n,1} \qquad n \geq 0 \qquad (55)$$

$$(\lambda + n\theta + 1)Q_{n,1} = \lambda Q_{n,0} + (n+1)\theta Q_{n+1,0} + 2Q_{n,2} \qquad n \geq 0 \qquad (56)$$

$$(\lambda + 2)Q_{n,2} = \lambda Q_{n,1} + (n+1)\theta Q_{n+1,1} + \lambda Q_{n-1,2} \qquad n \geq 0 \qquad (57)$$

which coincide with the results (2.1) – (2.3) of Falin and Templeton [2].

6. Conclusion

In this study, a two retrial queueing system with bulk departures having two identical parallel servers is investigated. Bulk queueing systems are common in real-life situations such as elevators, loading and unloading cargoes, giant wheel, chemical manufacturing process, communication networks and tourism etc.

Transient probabilities of exact number of arrivals and departures are found by solving difference differential equations recursively when no, one or both servers are busy. Further, some particular cases of interest are discussed along with special cases. From two-dimensional state queueing model, factors are well understood and quantified.

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