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A QUASI SUJA DISTRIBUTION

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Abstract

A two-parameter quasi Suja distribution which contains Suja distribution as particular case has been proposed for extreme right skewed data. Its statistical properties including moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, Renyi entropy measures, and stress-strength reliability have been derived and studied. The estimation of parameters using method of moments and maximum likelihood has been discussed. A simulation study has been presented to know the performance of maximum likelihood estimation. The goodness of fit of the proposed distribution has been presented.

Keywords: Suja distribution, Statistical Properties, parameters estimation, Goodness of fit.

I. Introduction

The search for a suitable distribution for modeling of lifetime data is very challenging because the lifetime data are stochastic in nature. The analysis and modeling of lifetime data are essential in almost every fields of knowledge including engineering, medical science, demography, social sciences, physical sciences finance, insurance, demography, social sciences, physical sciences, literature etc and during recent decades several researchers in statistics and mathematics tried to introduce lifetime distributions. Recently, Sharma *et al* [1] studied comparative study of several one parameter lifetime distributions and observed that there are some datasets which are extreme skewed to the right where these distributions were not giving good fit. In the search for a new lifetime distribution which can be used to model data from various fields of knowledge, Shanker [2] proposed a one parameter distribution Suja distribution which is defined by its probability density function (pdf) and cumulative distribution function (cdf) given by

$$f(x;\theta)=\frac{\theta^5}{\theta^4+24}\left(1+x^4\right)e^{-\theta x};x>0,\theta>0 \tag{1.1}$$

$$F(x;\theta)=1-\left[1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right]e^{-\theta x};x>0,\theta>0 \tag{1.2}$$

Shanker [2] studied its statistical properties, estimation of parameter using method of moment and method of maximum likelihood and applications to some real lifetime data and observed that Suja distribution gives much closer fit than several one parameter lifetime distributions. Recently, Al-Omari and Alsmairan [3] obtained length-biased Suja distribution and studied its statistical properties and applications. Al-Omari et al [4] proposed power lenth-biased Suja distribution and discussed its properties and applications. Alsmairan and Al-Omari [5] derived weighted Suja distribution and discussed its statistical properties and applications to ball bearings data in safety engineering. Todoka et al [6] have studied on the cdf of various modifications of Suja distribution and discussed their applications in the field of analysis of computer- virus propagation and debugging theory.

The main objective of this paper is to propose a two-parameter quasi Suja distribution which contains Suja distribution as particular case. Its statistical properties including moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, Renyi entropy measures, and stress-strength reliability have been derived and studied. The estimation of parameters using method of moments and maximum likelihood methods has been discussed. A simulation study has been presented to know the performance of maximum likelihood estimation. Applications and goodness of fit of the proposed distribution have been discussed.

II. A Quasi Suja Distribution

The pdf and the cdf of quasi Suja distribution QSD are expressed as

$$f(x;\theta,\alpha)=\frac{\theta^4}{\alpha\theta^3+24}\left(\alpha+\theta x^4\right)e^{-\theta x};x>0,\theta>0,\alpha>0 \tag{2.1}$$

$$F(x;\theta,\alpha)=1-\left[1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\alpha\theta^3+24}\right]e^{-\theta x};x>0,\theta>0,\alpha>0 \tag{2.2}$$

The survival function of QSD is given by

$$S(x;\theta,\alpha)=\left[\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x+\left(\alpha\theta^3+24\right)}{\alpha\theta^3+24}\right]e^{-\theta x};x>0,\theta>0,\alpha>0\cdot$$

At $\alpha=\theta$, the pdf and the cdf of QSD reduces to the corresponding pdf and cdf of Suja distribution. Like Suja distribution, QSD is also a convex combination of exponential distribution with parameter θ and gamma distribution with parameters $(5,\theta)$ with mixing proportion $p=\frac{\alpha\theta^3}{\alpha\theta^3+24}$.

The nature of pdf and cdf of QSD for varying values of parameters are shown in the following figures 1 and 2 respectively. From the pdf plots of the QSD, it is clear that it is extreme skewed to the right.

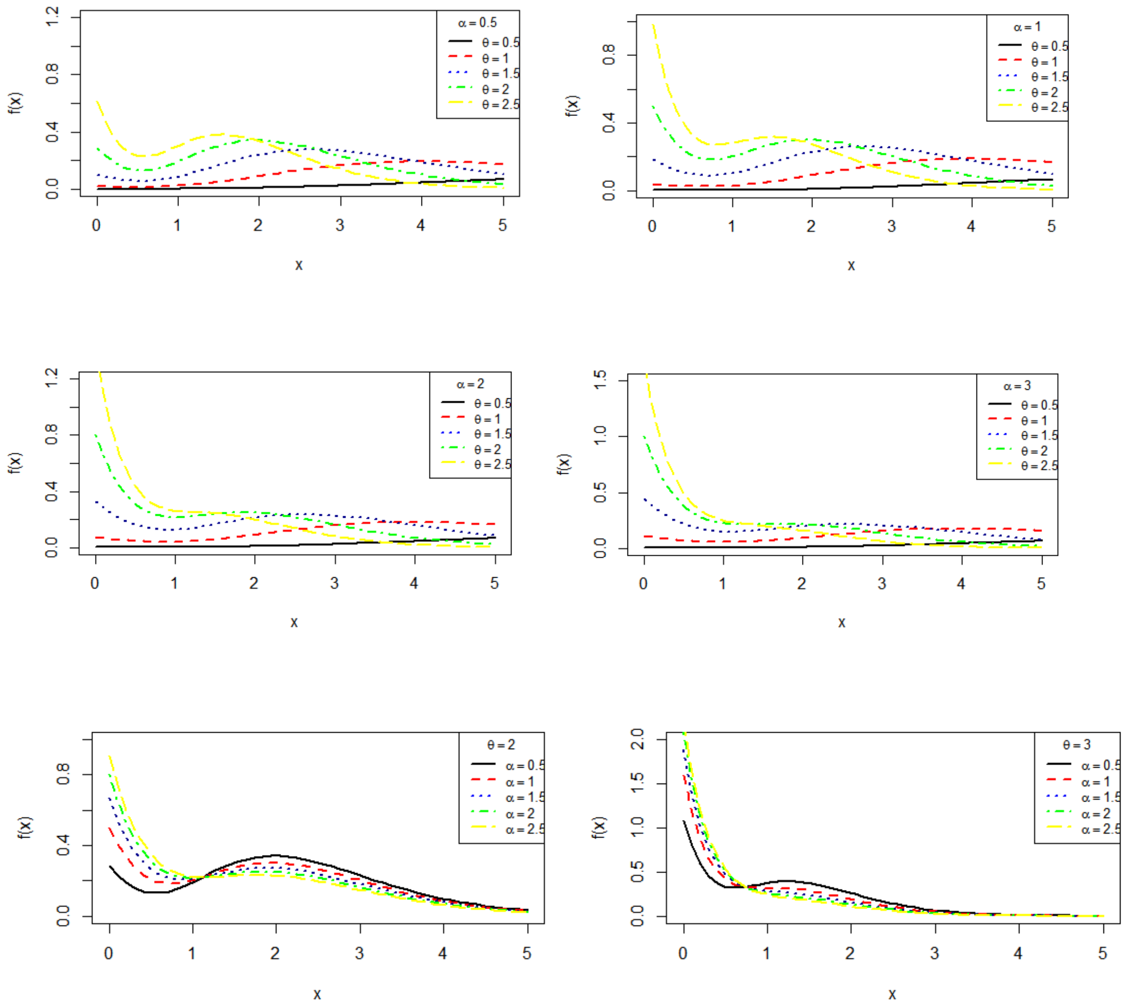

Figure1: *pdf plots of QSD for varying values of parameters*

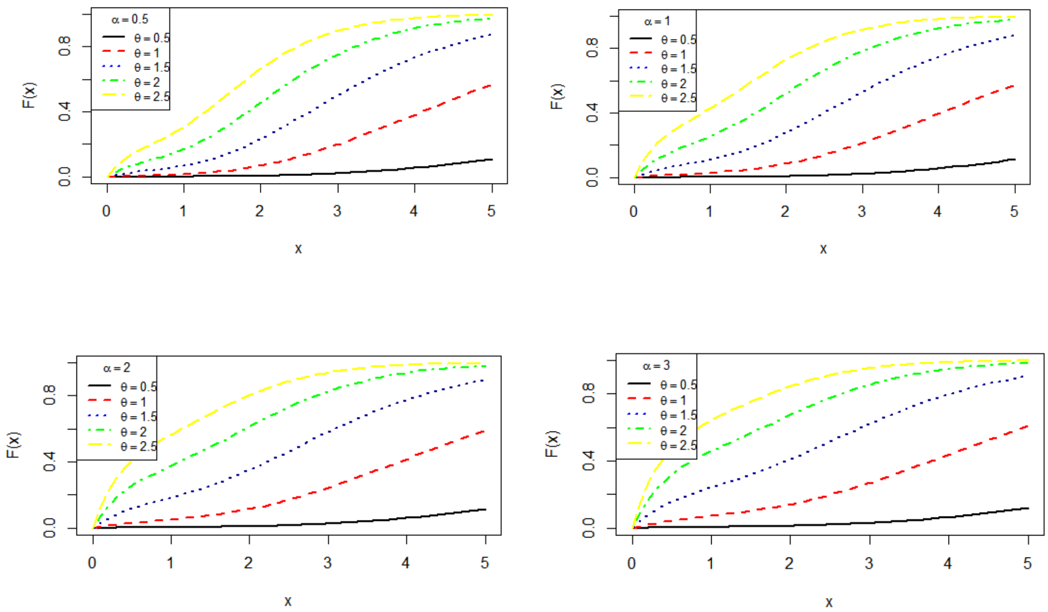

Figure 2: *cdf plots of QSD for varying values of parameters*

III. Measures based on Moments

The r th moment about origin μ_r' of QSD can be obtained as

$$\mu_r' = \frac{r!\left\{\alpha\theta^3 + (r+1)(r+2)(r+3)(r+4)\right\}}{\theta^r\left(\alpha\theta^3+24\right)};r=1,2,3,... \quad (3.1)$$

Taking $r=1,2,3$ and 4, the first four raw moments of QSD can be expressed as

$$\mu_1' = \frac{\alpha\theta^3+120}{\theta\left(\alpha\theta^3+24\right)},\ \mu_2' = \frac{2\left(\alpha\theta^3+360\right)}{\theta^2\left(\alpha\theta^3+24\right)},\ \mu_3' = \frac{6\left(\alpha\theta^3+840\right)}{\theta^3\left(\alpha\theta^3+24\right)} \text{ and } \mu_4' = \frac{24\left(\alpha\theta^3+1680\right)}{\theta^4\left(\alpha\theta^3+24\right)}$$

Now the relationship between central moments and raw moments gives the central moments as

$$\mu_2 = \frac{\alpha^2\theta^6+528\alpha\theta^3+2880}{\theta^2\left(\alpha\theta^3+24\right)^2}$$

$$\mu_3 = \frac{2\left(\alpha^3\theta^9+1512\alpha^2\theta^6+1728\alpha\theta^3+69120\right)}{\theta^3\left(\alpha\theta^3+24\right)^3}$$

$$\mu_4 = \frac{9\left(\alpha^4\theta^{12}+2656\alpha^3\theta^9+58752\alpha^2\theta^6+1234944\alpha\theta^3+3870720\right)}{\theta^4\left(\alpha\theta^3+24\right)^4}$$

The descriptive measures based on moments of QRD such as coefficient of variation (C.V), coefficient of skewness, $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis $\left(\beta_2\right)$ and index of dispersion $\left(\gamma\right)$ of QSD are obtained as

$$C.V. = \frac{\sqrt{\mu_2}}{\mu_1'} = \frac{\sqrt{\alpha^2\theta^6+528\alpha\theta^3+2880}}{\alpha\theta^3+120}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\left(\mu_2\right)^{3/2}} = \frac{2\left(\alpha^3\theta^9+1512\alpha^2\theta^6+1728\alpha\theta^3+69120\right)}{\left(\alpha^2\theta^6+528\alpha\theta^3+2880\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{9\left(\alpha^4\theta^{12}+2656\alpha^3\theta^9+58752\alpha^2\theta^6+1234944\alpha\theta^3+3870720\right)}{\left(\alpha^2\theta^6+528\alpha\theta^3+2880\right)^2}$$

$$\gamma = \frac{\mu_2}{\mu_1'} = \frac{\alpha^2\theta^6+528\alpha\theta^3+2880}{\theta\left(\alpha\theta^3+24\right)\left(\alpha\theta^3+120\right)}$$

The coefficient of variation, skweness, kurtosis and index of dispersion for varying values of parameters are shown in the following figures 3, 4, 5, and 6 respectively

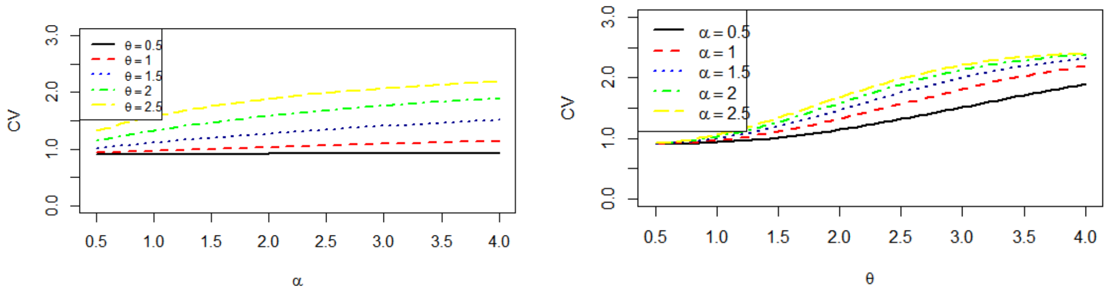

Figure3: *Plots of Coefficient of variation (C.V) of QSD for varying values of parameters*

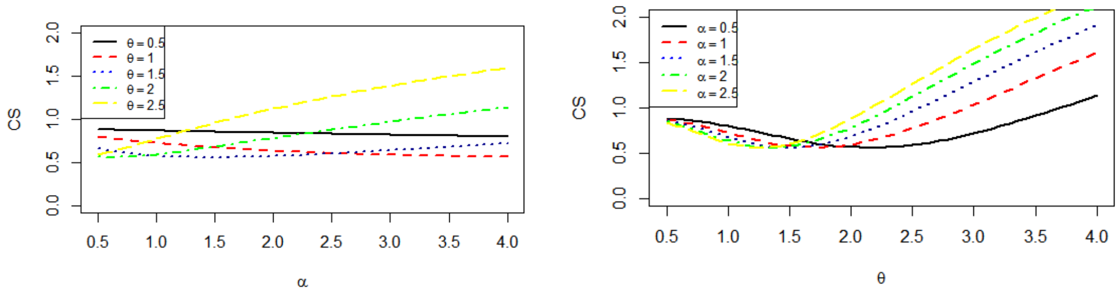

Fig. 4: *Plots of Coefficient of skweness of QSD for varying values of parameters*

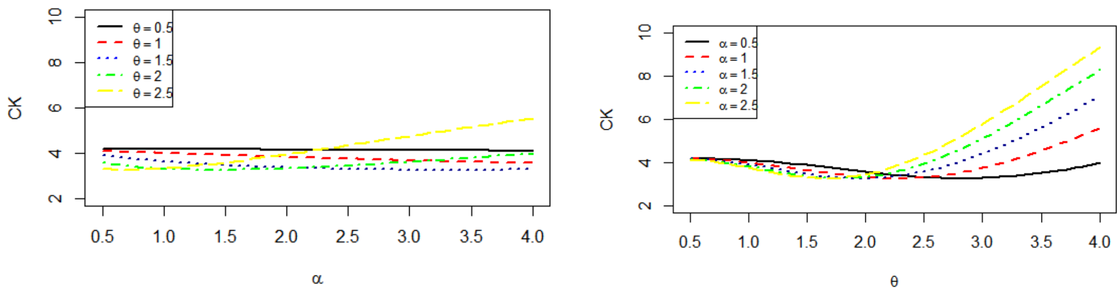

Figure 5: *Plots of Coefficient of kurtosis of QSD for varying values of parameters*

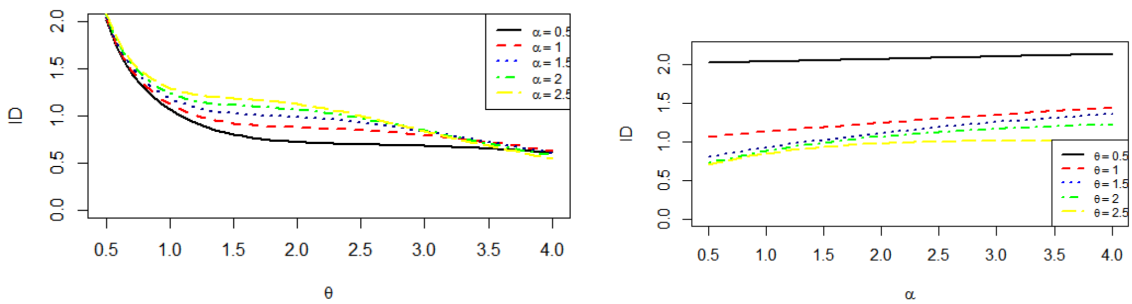

Figure 6:*Plots of Index of dispersion of QSD for varying values of parameters*

IV. Reliability Measures

Let X be a random variable having pdf $f(x)$ and cdf $F(x)$. The hazard rate function $h(x)$ (also known as the failure rate function) and the mean residual life function $m(x)$ of X are respectively defined as

$$h(x)=\lim_{\Delta x\to 0}\frac{P\left(X<x+\Delta x\,|\,X>x\right)}{\Delta x}=\frac{f(x)}{1-F(x)} \quad \text{and}$$

$$m(x)=E\left[X-x\,|\,X>x\right]=\frac{1}{1-F(x)}\int_x^{\infty}\left[1-F(t)\right]dt=\frac{1}{S(x)}\int_x^{\infty}t\,f(t)\,dt-x\,.$$

Now using the pdf and cdf of QSD, the hazard rate function, $h(x)$ and the mean residual life function, $m(x)$ of the QSD are thus obtained as

$$h(x)=\frac{\theta^4\left(\alpha+\theta x^4\right)}{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x+\left(\alpha\,\theta^3+24\right)}$$

and $$m(x)=\frac{\theta^4x^4+8\theta^3x^3+36\theta^2x^2+96\theta x+\left(\alpha\,\theta^3+120\right)}{\theta\left[\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x+\left(\alpha\,\theta^3+24\right)\right]}.$$

Obviously, we have $h(0)=\dfrac{\alpha\,\theta^4}{\alpha\,\theta^3+24}=f(0)$ and $m(0)=\dfrac{\alpha\theta^3+120}{\theta\left(\alpha\theta^3+24\right)}=\mu_1'$. The hazard rate function and the mean residual life function of QSD for varying values of parameters are shown in figures 7 and 8 respectively.

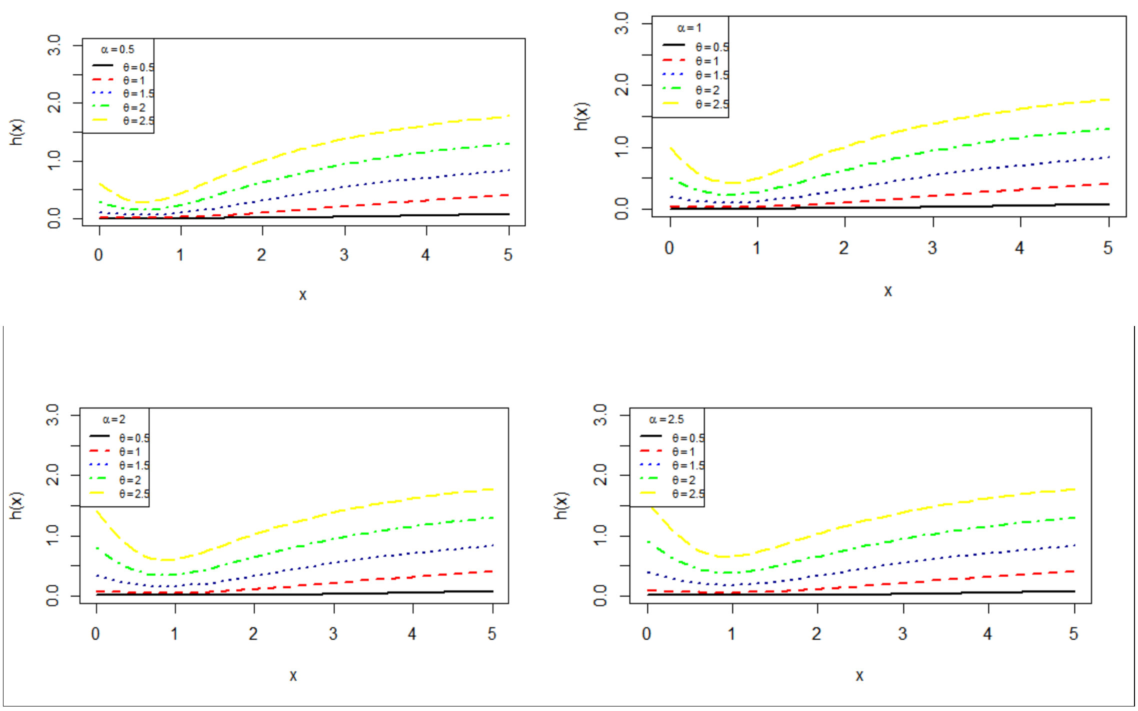

Figure7: *Plots of Hazard function of QSD for varying values of parameters*

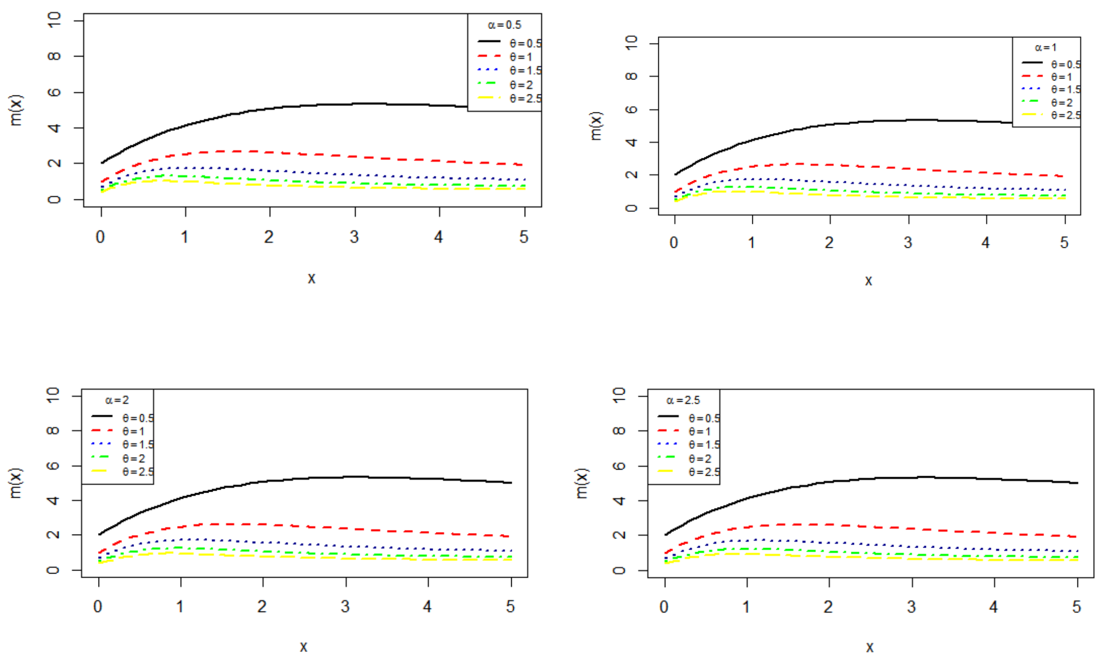

Figure8: *Plots of Mean residual life function of QSD for varying values of parameters*

V. Mean Deviations

The amount of scatter in a population is measured to some extent by the totality of deviations usually from mean and median. These are known as the mean deviation about the mean and the mean deviation about the median and are defined as

$\delta_1(X)=\int_0^\infty |x-\mu| f(x)\,dx$ and $\delta_2(X)=\int_0^\infty |x-M| f(x)\,dx$, respectively, where $\mu=E(X)$ and $M=\text{Median}\,(X)$. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the relationships

$$\delta_1(X)=\int_0^\mu (\mu-x)f(x)\,dx+\int_\mu^\infty (x-\mu)f(x)\,dx$$

$$=\mu F(\mu)-\int_0^\mu x\,f(x)\,dx-\mu\left[1-F(\mu)\right]+\int_\mu^\infty x\,f(x)\,dx$$

$$=2\mu F(\mu)-2\mu+2\int_\mu^\infty x\,f(x)\,dx$$

$$=2\mu F(\mu)-2\int_0^\mu x\,f(x)\,dx \tag{5.1}$$

and

$$\delta_2(X)=\int_0^M (M-x)f(x)\,dx+\int_M^\infty (x-M)f(x)\,dx$$

$$=M\,F(M)-\int_0^M x\,f(x)\,dx-M\left[1-F(M)\right]+\int_M^\infty x\,f(x)\,dx$$

$$= -\mu + 2\int_M^\infty x\, f(x)\, dx$$

$$= \mu - 2\int_0^M x\, f(x)\, dx \tag{5.2}$$

Using the pdf of QSD and the mean of QSD, we get

$$\int_0^\mu x\, f(x;\theta,\alpha)\, dx = \mu - \frac{\left\{\theta^5\mu^5 + 5\theta^4\mu^4 + 20\theta^3\mu^3 + 60\theta^2\mu^2 + \left(\alpha\theta^3+120\right)\theta\mu + \left(\alpha\theta^3+120\right)\right\} e^{-\theta\mu}}{\theta\left(\alpha\theta^3+24\right)} \tag{5.3}$$

$$\int_0^M x\, f(x;\theta,\alpha)\, dx = \mu - \frac{\left\{\theta^5 M^5 + 5\theta^4 M^4 + 20\theta^3 M^3 + 60\theta^2 M^2 + \left(\alpha\theta^3+120\right)\theta M + \left(\alpha\theta^3+120\right)\right\} e^{-\theta M}}{\theta\left(\alpha\theta^3+24\right)} \tag{5.4}$$

Using expressions from (5.1), (5.2), (5.3), and (5.4), the mean deviation about mean, $\delta_1(X)$ and the mean deviation about median, $\delta_2(X)$ of QSD are obtained as

$$\delta_1(X) = \frac{2\left\{\theta^4\mu^4 + 8\theta^3\mu^3 + 36\theta^2\mu^2 + 96\theta\mu + \left(\alpha\theta^3+120\right)\right\} e^{-\theta\mu}}{\theta\left(\alpha\theta^3+24\right)} \tag{5.5}$$

$$\delta_2(X) = \frac{2\left\{\theta^5 M^5 + 5\theta^4 M^4 + 20\theta^3 M^3 + 60\theta^2 M^2 + \left(\alpha\theta^3+120\right)\theta M + \left(\alpha\theta^3+120\right)\right\} e^{-\theta M}}{\theta\left(\alpha\theta^3+24\right)} - \mu \tag{5.6}$$

VI. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves (Bonferroni [7]) and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu}\int_0^q x\, f(x)\, dx = \frac{1}{p\mu}\left[\int_0^\infty x\, f(x)\, dx - \int_q^\infty x\, f(x)\, dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^\infty x\, f(x)\, dx\right] \tag{6.1}$$

and $$L(p) = \frac{1}{\mu}\int_0^q x\, f(x)\, dx = \frac{1}{\mu}\left[\int_0^\infty x\, f(x)\, dx - \int_q^\infty x\, f(x)\, dx\right] = \frac{1}{\mu}\left[\mu - \int_q^\infty x\, f(x)\, dx\right] \tag{6.2}$$

respectively or equivalently

$$B(p) = \frac{1}{p\mu}\int_0^p F^{-1}(x)\, dx \tag{6.3}$$

and $$L(p) = \frac{1}{\mu}\int_0^p F^{-1}(x)\, dx \tag{6.4}$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$.

The Bonferroni and Gini indices are thus defined as

$$B = 1 - \int_0^1 B(p)\, dp \tag{6.5}$$

$$\text{and } G = 1 - 2\int_0^1 L(p)\,dp \tag{6.6}$$

respectively.

Using the pdf of QSD, we have

$$\int_q^\infty x f(x;\theta,\alpha)\,dx = \frac{\left\{\theta^5 q^5 + 5\theta^4 q^4 + 20\theta^3 q^3 + 60\theta^2 q^2 + \left(\alpha\theta^3 + 120\right)\theta q + \left(\alpha\theta^3 + 120\right)\right\} e^{-\theta q}}{\theta\left(\alpha\theta^3 + 24\right)} \tag{6.7}$$

Now using equation (6.7) in (6.1) and (6.2), we get

$$B(p) = \frac{1}{p}\left[1 - \frac{\left\{\theta^5 q^5 + 5\theta^4 q^4 + 20\theta^3 q^3 + 60\theta^2 q^2 + \left(\alpha\theta^3 + 120\right)\theta q + \left(\alpha\theta^3 + 120\right)\right\} e^{-\theta q}}{\left(\alpha\theta^3 + 120\right)}\right] \tag{6.8}$$

$$L(p) = 1 - \frac{\left\{\theta^5 q^5 + 5\theta^4 q^4 + 20\theta^3 q^3 + 60\theta^2 q^2 + \left(\alpha\theta^3 + 120\right)\theta q + \left(\alpha\theta^3 + 120\right)\right\} e^{-\theta q}}{\left(\alpha\theta^3 + 120\right)} \tag{6.9}$$

Now using equations (6.8) and (6.9) in (6.5) and (6.6), the Bonferroni and Gini indices are obtained as

$$B = 1 - \frac{\left\{\theta^5 q^5 + 5\theta^4 q^4 + 20\theta^3 q^3 + 60\theta^2 q^2 + \left(\alpha\theta^3 + 120\right)\theta q + \left(\alpha\theta^3 + 120\right)\right\} e^{-\theta q}}{\left(\alpha\theta^3 + 120\right)} \tag{6.10}$$

$$G = \frac{2\left\{\theta^5 q^5 + 5\theta^4 q^4 + 20\theta^3 q^3 + 60\theta^2 q^2 + \left(\alpha\theta^3 + 120\right)\theta q + \left(\alpha\theta^3 + 120\right)\right\} e^{-\theta q}}{\left(\alpha\theta^3 + 120\right)} - 1 \tag{6.11}$$

VII. Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n from QSD. Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ denote the corresponding order statistics. The pdf and the cdf of the k th order statistic, say $Y = X_{(k)}$ are given by

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(y)\left\{1 - F(y)\right\}^{n-k} f(y)$$

$$= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l F^{k+l-1}(y) f(y)$$

and

$$F_Y(y) = \sum_{j=k}^{n} \binom{n}{j} F^j(y)\left\{1 - F(y)\right\}^{n-j} = \sum_{j=k}^{n} \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} (-1)^l F^{j+l}(y)$$

respectively, for $k = 1, 2, 3, ..., n$.

Thus, the pdf and the cdf of the k th order statistics of QSD are obtained as

$$f_Y(y)=\frac{n!\theta^4\left(\alpha+\theta x^4\right)e^{-\theta x}}{\left(\alpha\theta^3+24\right)(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l$$
$$\times\left[1-\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x+\left(\alpha\theta^3+24\right)}{\alpha\theta^3+24}e^{-\theta x}\right]^{k+l-1}$$

and

$$F_Y(y)=\sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l\left[1-\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x+\left(\alpha\theta^3+24\right)}{\alpha\theta^3+24}e^{-\theta x}\right]^{j+l}$$

VIII. Stochastic Orderings

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $\left(X\le_{st}Y\right)$ if $F_X(x)\ge F_Y(x)$ for all x

(ii) hazard rate order $\left(X\le_{hr}Y\right)$ if $h_X(x)\ge h_Y(x)$ for all x

(iii) mean residual life order $\left(X\le_{mrl}Y\right)$ if $m_X(x)\le m_Y(x)$ for all x

(iv) likelihood ratio order $\left(X\le_{lr}Y\right)$ if $\dfrac{f_X(x)}{f_Y(x)}$ decreases in x. The following results due to Shaked and Shanthikumar [8] are well known for establishing stochastic ordering of distributions

$$\begin{array}{c}X\le_{lr}Y\Rightarrow X\le_{hr}Y\Rightarrow X\le_{mrl}Y\\ \Downarrow\\ X\le_{st}Y\end{array}\qquad(8.1)$$

QSD is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem: Let $X\sim$ QSD$\left(\theta_1,\alpha_1\right)$ and $Y\sim$ QSD$\left(\theta_2,\alpha_1\right)$. If $\theta_1\ge\theta_2$ and $\alpha_1=\alpha_2$, or $\alpha_1\ge\alpha_2$ and $\theta_1=\theta_2$ then $X\le_{lr}Y$ and hence $X\le_{hr}Y$, $X\le_{mrl}Y$ and $X\le_{st}Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)}=\frac{\theta_1^4\left(\alpha_2\theta_2^3+24\right)}{\theta_2^4\left(\alpha_1\theta_1^3+24\right)}\left(\frac{\alpha_1+\theta_1x^4}{\alpha_2+\theta_2x^4}\right)e^{-(\theta_1-\theta_2)x};x>0$$

Now
$$\ln\frac{f_X(x)}{f_Y(x)}=\ln\left[\frac{\theta_1^4\left(\alpha_2\theta_2^3+24\right)}{\theta_2^4\left(\alpha_1\theta_1^3+24\right)}\right]+\ln\left(\frac{\alpha_1+\theta_1x^4}{\alpha_2+\theta_2x^4}\right)-\left(\theta_1-\theta_2\right)x$$

. This gives
$$\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)}=\frac{4\left(\alpha_2\theta_1-\alpha_1\theta_2\right)x^3}{\left(\alpha_1+\theta_1x^4\right)\left(\alpha_2+\theta_2x^4\right)}-\left(\theta_1-\theta_2\right)$$

Thus for $\theta_1\ge\theta_2$ and $\alpha_1=\alpha_2$, or $\alpha_1\ge\alpha_2$ and $\theta_1=\theta_2$, $\dfrac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)}<0$. This means that $X\le_{lr}Y$ and hence $X\le_{hr}Y$, $X\le_{mrl}Y$ and $X\le_{st}Y$.

IX. Renyi Entropy Measure

An entropy of a random variable X is a measure of variation of uncertainty. A popular entropy measure is Renyi entropy [9]. If X is a continuous random variable having pdf $f(.)$, then Renyi entropy is defined as

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)\,dx\right\}, \text{ where } \gamma>0 \text{ and } \gamma\neq 1.$$

Thus, the Renyi entropy of QSD can be obtained as

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left[\int_0^{\infty}\frac{\theta^{4\gamma}}{\left(\alpha\theta^3+24\right)^{\gamma}}\left(\alpha+\theta x^4\right)^{\gamma}e^{-\theta\gamma x}dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\int_0^{\infty}\frac{\theta^{4\gamma}\alpha^{\gamma}}{\left(\alpha\theta^3+24\right)^{\gamma}}\left(1+\frac{\theta}{\alpha}x^4\right)^{\gamma}e^{-\theta\gamma x}dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\int_0^{\infty}\frac{\theta^{4\gamma}\alpha^{\gamma}}{\left(\alpha\theta^3+24\right)^{\gamma}}\sum_{j=0}^{\infty}\binom{\gamma}{j}\left(\frac{\theta}{\alpha}x^4\right)^{j}e^{-\theta\gamma x}dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\theta^{4\gamma}\alpha^{\gamma}\theta^{j}}{\left(\alpha\theta^3+24\right)^{\gamma}\alpha^{j}}\int_0^{\infty}e^{-\theta\gamma x}x^{4j}dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\theta^{4\gamma+j}\alpha^{\gamma-j}}{\left(\alpha\theta^3+24\right)^{\gamma}}\int_0^{\infty}e^{-\theta\gamma x}x^{4j+1-1}dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\theta^{4\gamma+j}\alpha^{\gamma-j}}{\left(\alpha\theta^3+24\right)^{\gamma}}\frac{\Gamma(4j+1)}{(\theta\gamma)^{4j+1}}\right]$$

$$=\frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\theta^{4\gamma-3j-1}\alpha^{\gamma-j}}{\left(\alpha\theta^3+24\right)^{\gamma}}\frac{\Gamma(4j+1)}{(\gamma)^{4j+1}}\right]$$

X. Stress-Strength Reliability

The stress- strength reliability describes the life of a component which has random strength X that is subjected to a random stress Y. When the stress applied to it exceeds the strength, the component fails instantly and the component will function satisfactorily till $X>Y$. Therefore, $R=P(Y<X)$ is a measure of component reliability and in statistical literature it is known as stress-strength parameter. It has wide applications in almost all areas of knowledge especially in engineering such as structures, deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels etc.

Let X and Y are independent strength and stress random variables having QSD with parameter (θ_1,α_1) and (θ_2,α_2), respectively. Then, the stress-strength reliability R of QSD can be obtained as

$$R=P(Y<X)=\int_0^{\infty}P(Y<X\mid X=x)f_X(x)\,dx=\int_0^{\infty}f(x;\theta_1,\alpha_1)\,F(x;\theta_2,\alpha_2)\,dx$$

$$= 1 - \frac{\theta_1^4 \left[\begin{array}{l} 40320\,\theta_1\theta_2^4 + 20160\,\theta_1\theta_2^3\left(\theta_1+\theta_2\right) + 8640\theta_1\theta_2^2\left(\theta_1+\theta_2\right)^2 + 2880\theta_1\theta_2\left(\theta_1+\theta_2\right)^3 \\ +24\left(\alpha_1\theta_2^4 + \alpha_2\theta_1\theta_2^3 + 24\theta_1\right)\left(\theta_1+\theta_2\right)^4 + 24\alpha_1\theta_2^3\left(\theta_1+\theta_2\right)^5 + 24\alpha_1\theta_2^2\left(\theta_1+\theta_2\right)^6 \\ +24\alpha_1\theta_2\left(\theta_1+\theta_2\right)^7 + \left(\alpha_1\alpha_2\theta_2^3 + 24\alpha_1\right)\left(\theta_1+\theta_2\right)^8 \end{array} \right]}{\left(\alpha_1\theta_1^3+24\right)\left(\alpha_2\theta_2^3+24\right)\left(\theta_1+\theta_2\right)^9}.$$

XI. Estimation of Parameters

In this section, the method of moments and the method of maximum likelihood for estimating parameters of QSD have been discussed.

I. Method of Moments

Since QSD has two parameters to be estimated, the first two moments about origin are required to estimate its parameters. We have

$$\frac{\mu_2'}{\left(\mu_1'\right)^2} = \frac{2\left(\alpha\,\theta^3+360\right)\left(\alpha\,\theta^2+24\right)}{\left(\alpha\,\theta^3+120\right)^2} = k \text{ (Say)}$$

Taking $b=\alpha\,\theta^3$, above equation becomes

$$\frac{2\left(b+360\right)\left(b+24\right)}{\left(b+120\right)^2} = k$$

$$\frac{2\left(b^2+384b+8670\right)}{b^2+240b+14400} = k$$

$$\left(k-2\right)b^2 + \left(240k-768\right)b + \left(14400k-17340\right) = 0 \tag{11.1.1}$$

Now, for real root of b, the discriminant of the above equation should be greater than and equal to zero. That is

$$\left(240k-768\right)^2 - 4\left(k-2\right)\left(14400k-17340\right) \ge 0 \Rightarrow k \le 2.45.$$

This means that the method of moments estimate is applicable if $k = \frac{m_2'}{\left(\bar{x}\right)^2} \le 2.45$, where m_2' is the second moment about origin of the dataset. Now taking $b=\alpha\,\theta^3$ and equating the population mean to the sample mean, we get the moment estimate $\tilde{\theta}$ of θ as

$$\frac{\alpha\,\theta^3+120}{\theta\left(\alpha\,\theta^3+24\right)} = \frac{b+120}{\theta\left(b+24\right)} = \bar{x} \Rightarrow \tilde{\theta} = \frac{b+120}{\left(b+24\right)\bar{x}}.$$

Using the moment estimate of θ in $b=\alpha\,\theta^3$, we get the moment estimate $\tilde{\alpha}$ of α as

$$\tilde{\alpha} = \frac{b}{\left(\tilde{\theta}\right)^3} = \frac{b\left(b+124\right)^3\left(\bar{x}\right)^3}{\left(b+120\right)^3}$$

Thus the method of moment estimates $\left(\tilde{\theta},\tilde{\alpha}\right)$ of parameters $\left(\theta,\alpha\right)$ of QSD are given by

$\left(\tilde{\theta},\tilde{\alpha}\right)=\left(\dfrac{b+120}{(b+24)\bar{x}},\dfrac{b(b+124)^3(\bar{x})^3}{(b+120)^3}\right)$, where b is the value of the quadratic equation (11.1.1).

II. Method of Maximum likelihood

Let $\left(x_1,x_2,x_3,...,x_n\right)$ be a random sample of size n from QSD$\left(\theta,\alpha\right)$. Then the likelihood function of QSD is given by

$$L=\left(\frac{\theta^4}{\alpha\theta^3+24}\right)^n\prod_{i=1}^{n}\left(\alpha+\theta x_i^{\,4}\right)e^{-n\theta\bar{x}}, \text{ where } \bar{x} \text{ is the sample mean.}$$

The log-likelihood function is thus obtained as

$$\log L=n\left[4\log\theta-\log\left(\alpha\theta^3+24\right)\right]+\sum_{i=1}^{n}\log\left(\alpha+\theta x_i^{\,4}\right)-n\theta\bar{x}.$$

The maximum likelihood estimates $\left(\hat{\theta},\hat{\alpha}\right)$ of parameters $\left(\theta,\alpha\right)$ are the solution of the following log-likelihood equations

$$\frac{\partial\log L}{\partial\theta}=\frac{4n}{\theta}+\frac{3n\theta\alpha}{\alpha\theta^3+24}+\sum_{i=1}^{n}\frac{x_i^{\,4}}{\alpha+\theta x_i^{\,4}}-n\,\bar{x}=0$$

$$\frac{\partial\log L}{\partial\alpha}=\frac{-n\theta^3}{\alpha\theta^3+24}+\sum_{i=1}^{n}\frac{1}{\alpha+\theta x_i^{\,4}}=0$$

These two log-likelihood equations do not seem to be solved directly. We have to use Fisher's scoring method for solving these two log-likelihood equations. We have

$$\frac{\partial^2\log L}{\partial\theta^2}=\frac{-4n}{\theta^2}+\frac{3n\alpha\left(\alpha\theta^4-48\theta\right)}{\left(\alpha\theta^3+24\right)^2}-\sum_{i=1}^{n}\frac{x_i^{\,8}}{\left(\alpha+\theta x_i^{\,4}\right)^2}$$

$$\frac{\partial^2\log L}{\partial\alpha^2}=\frac{n\theta^6}{\left(\alpha\theta^3+24\right)^2}-\sum_{i=1}^{n}\frac{1}{\left(\alpha+\theta x_i^{\,4}\right)^2}$$

$$\frac{\partial^2\log L}{\partial\theta\,\partial\alpha}=\frac{-72n\theta^2}{\left(\alpha\theta^3+24\right)^2}-\sum_{i=1}^{n}\frac{x_i^{\,4}}{\left(\alpha+\theta x_i^{\,4}\right)^2}=\frac{\partial^2\log L}{\partial\alpha\,\partial\theta}.$$

The following equations can be solved for MLEs $\left(\hat{\theta},\hat{\alpha}\right)$ of $\left(\theta,\alpha\right)$ of QSD

$$\begin{bmatrix}\dfrac{\partial^2\ln L}{\partial\theta^2} & \dfrac{\partial^2\ln L}{\partial\theta\,\partial\alpha}\\ \dfrac{\partial^2\ln L}{\partial\theta\,\partial\alpha} & \dfrac{\partial^2\ln L}{\partial\alpha^2}\end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0}}\begin{bmatrix}\hat{\theta}-\theta_0\\ \hat{\alpha}-\alpha_0\end{bmatrix}=\begin{bmatrix}\dfrac{\partial\ln L}{\partial\theta}\\ \dfrac{\partial\ln L}{\partial\alpha}\end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0}}$$

where θ_0 and α_0 are the initial values of θ and α, respectively, as given by the method of moments. . These equations are solved iteratively till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

XII. A Simulation Study

In this section, a simulation study has been carried out using R-software. Acceptance and rejection method is used to generate random number, where sample size, $n = 40, 60, 80, 100$, value of $\theta = 0.1, 0.5, 1.0, 1.5 \,\&\, (\alpha = 1, \alpha = 2)$ have been used for calculating Bias error (BE) and MSE (Mean square error) of parameter θ and α which are presented in tables1 &2 respectively.

Table 1. *BE and MSE for* θ *and* α *at* α *=1*

Sample	θ	BE(θ)(MSE(θ))	BE(α) (MSE (α))
40	0.1	0.026829 (0.02879335)	0.051492(0.11741)
	0.5	0.0168297(0.0113295)	0.0291919(0.034087)
	1.0	0.0004329(0.0007498)	0.0041920(0.0007029)
	1.5	-0.0081702(0.0026701)	-0.0208079(0.0173187)
60	0.1	0.01656832(0.0164705)	0.0363314(0.0791985)
	0.5	0.0099016(0.00588256)	0.0196648(0.0232022)
	1.0	0.0015683(0.00014757)	0.00299813(0.000539)
	1.5	-0.0067650(0.0027459)	-0.0136685(0.0112097)
80	0.1	0.01079101(0.00931567)	0.0619808(0.30733037)
	0.5	0.00579101(0.00268286)	0.04948088(0.1958686)
	1.0	-0.00045898(0.00001685)	0.03698088(0.1094068)
	1.5	-0.006708987(0.0036008)	0.02448088(0.0479450)
100	0.1	0.008494785(0.007216137)	0.045858(0.21029559)
	0.5	0.004494785(0.002020309)	0.035858(0.1285795)
	1.0	-0.000505214(0.0002552)	0.025858(0.0668636)
	1.5	-0.005505214(0.00303073)	0.015858(0.0251476)

Table 2. *BE and MSE for* θ *and* α *at* α *=2*

Sample	θ	BE (MSE)	BE (MSE) for alpha
40	0.1	0.01796973(0.0129164485)	0.17092052(1.1685530)
	0.5	0.00796973 (0.0025406641)	0.14592052 (0.8517119)
	1.0	-0.00453027(0.0008209337)	0.12092052 (0.5848709)
	1.5	-0.01703027 (0.0116012033)	0.09592052(0.3680299)
60	0.1	0.016696288(0.0167259624)	0.031357314(0.058996870)
	0.5	0.010029622(0.0060355985)	0.014690648(0.012948908)
	1.0	0.001696288 (0.0001726436)	-0.001976019(0.000234279)
	1.5	-0.006637045 (0.002643022)	-0.018642686(0.020852983)
80	0.1	0.0107846385(0.0093046743)	0.05327416(0.22705089)
	0.5	0.0057846385(0.0026769634)	0.04077416(0.13300257)
	1.0	-0.000465361(0.0000173249)	0.02827416 (0.06395425)
	1.5	-0.006715361(0.0036076864)	0.01577416(0.01990593)
100	0.1	0.0086072384(0.007408455)	0.037134273(0.137895421)
	0.5	0.0046072384(0.002.122665)	0.027134273(0.073626875)
	1.0	-0.000392761(0.0001.542617)	0.017134273(0.029358330)
	1.5	-0.0053927616(0.002908188)	0.007134273(0.005089785)

It is obvious from tables 1 and 2 that as the sample size increases, both the BE and MSE decreases.

XIII. Applications

In this section, the goodness of fit of QSD has been discussed and compared with one parameter life time distributions including exponential distribution, Lindley distribution introduced by Lindley [10], Akash distribution proposed by Shanker [11], Suja distribution and two-parameter lifetime distributions including quasi Lindley distribution (QLD) of Shanker and Mishra [12] and Quasi Akash distribution of Shanker [13]. The pdf and the cdf of these distributions are presented in the table 3.

Table 3: ***pdf and the cdf of one parameter and two-parameter distributions***

Distributions	Pdf	Cdf
QLD	$f(x;\theta,\alpha)=\frac{\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}$	$F(x;\theta,\alpha)=1-\left(1+\frac{\theta x}{\alpha+1}\right)e^{-\theta x}$
QAD	$f(x;\theta,\alpha)=\frac{\theta^2}{\alpha\theta+2}(\alpha+\theta x^2)e^{-\theta x}$	$F(x;\theta,\alpha)=1-\left(1+\frac{\theta x(\theta x+2)}{\alpha\theta+2}\right)e^{-\theta x}$
LD	$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$	$F(x;\theta)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}$
AD	$f(x;\theta)=\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}$	$F(x;\theta)=1-\left(1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right)e^{-\theta x}$
Exponential	$f(x;\theta)=\theta e^{-\theta x}$	$F(x;\theta)=1-e^{-\theta x}$

The following dataset table 4 regarding the failure times of 50 electronic components which are extreme skewed to the right available in Murthy et al [14] has been considered to test the goodness of fit of the considered distributions.ML estimates of parameters of the considered distributions along with the values of $-2\log L$, AIC, K-S and p values are presented in table 5. The fitted plots of the considered distributions for dataset in table 4 are shown in figure 9.

Table 4: Failure times data of 50 electronic components

0.036	0.058	0.061	0.074	0.078	0.086	0.102	0.103	0.114	0.116
0.148	0.183	0.192	0.254	0.262	0.379	0.381	0.538	0.570	0.574
0.590	0.618	0.645	0.961	1.228	1.600	2.006	2.054	2.804	3.058
3.076	3.147	3.625	3.704	3.931	4.073	4.393	4.534	4.893	6.274
6.816	7.896	7.904	8.022	9.337	10.940	11.020	13.880	14.730	15.080

Table 5: *ML estimates of the parameters of the considered distributions along with values of* $-2\log L$, *AIC, K-S and p-value*

Distributions	**ML parameters**		$-2\log L$	AIC	**K-S**	**p-value**
	θ	α				
QSD	0.63822	179.0985	182.40	186.40	0.159	0.142
QLD	0.30731	35.0346	220.70	224.70	0.972	0.000
QAD	0.3680	44.9617	218.13	222.13	0.271	0.000
LD	0.59603	--------	197.71	199.71	0.283	0.000
SD	1.12234	--------	317.17	319.17	0.441	0.000
Exponential	0.29913	------	220.68	222.68	0.284	0.000

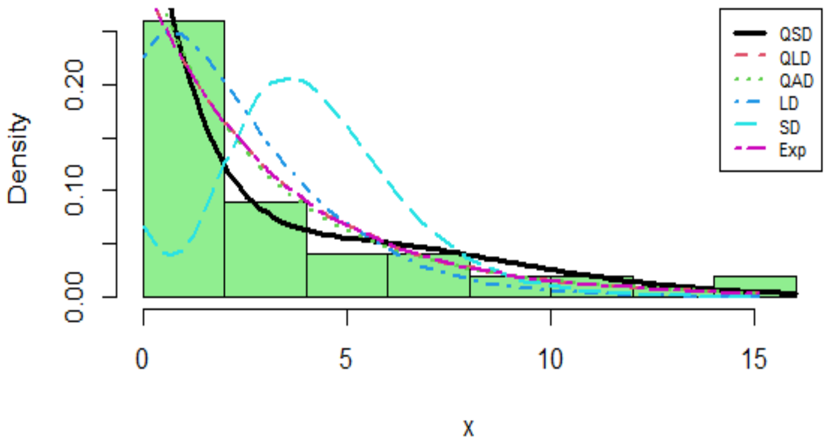

Figure 9 : *Fitted plot of considered distributions on dataset*

It is obvious from the goodness of fit in table 5 and the fitted plots in figure 9 that the QSD gives much closer fit as compared to other considered distributions for the failure time data of electronic components.

XIV. Conclusion

A two-parameter quasi Suja distribution (QSD) of which Suja distribution is a special case has been suggested. The proposed distribution is useful for extreme right skewed data. Its moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, Renyi entropy measures, and stress-strength reliability have been derived and studied. Method of moments and maximum likelihood estimation has been studied for estimating parameters. A simulation study has been presented. The goodness of fit of QSD has been presented with failure time data and the fit shows that QSD is the best distribution among the considered distributions.

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