Estimation of the Change Point in the Mean Control Chart for Autocorrelated Processes

Rupali Kapase, Vikas Ghute*

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Department of Statistics
Punyashlok Ahilyadevi Holkar
Solapur University Solapur (MS)-413255, India
rakapase@sus.ac.in, vbghute_stats@rediffmail.com

* Corresponding Author

Abstract

Control charts are the most popular monitoring tools used to monitor changes in a process and distinguish between assignable and chance causes of variations. The time that a control chart gives an out-of-control signal is not the real time of change. The actual time of change is called the change point. Knowing the real time of change will help and simplify finding the assignable causes of the signal which may be the result of the shift in the process parameters. In this paper, we propose a maximum likelihood estimator of the process change point when a Shewhart $\bar{X}$ chart with autocorrelated observations signals a change in the process mean. The performance of the proposed change point estimator when used with $\bar{X}$ chart with AR(1) *process is investigated using simulation study. The results show that the performance of the proposed estimator has good properties in the aspect of expected length and coverage probability. We illustrate the use of proposed change point estimator through an example.*

Keywords: Change point; maximum likelihood estimator; control chart; average run length; autocorrelation; autoregressive process.

1. Introduction

Control chart is an important statistical process tool used to improve the quality of the manufacturing process which is widely used to monitor the process by distinguishing the assignable and chance causes of variations. It also helps us to detect changes in a process with issuing an out-of-control signal. Once control chart gives an out-of-control signal, that is, the assignable cause is present in the process, it is necessary to process practitioner that to identify and remove the sources responsible for the special cause of variability.

For the purpose of process improvement, process practitioner can take corrective action to return the process in the state of statistical control. The time at which the control chart gives an out-of-control signal, is not the real time of change and it shows the change with delay which depends on the size of the shift in process parameter. The actual time of change in the process is called change point. Finding the actual change point has been in great importance for many industries. The change point estimator when used with a control chart monitoring scheme helps engineers to find actual time of step change. Knowing the actual time of change, leads to easier detection of change cause by limiting the scope of time within which searching for the change cause is done. Change point estimation also reduces the dependency of searching to process engineer's expertise and knowledge which leads to cost reduction. Therefore, change point estimation improves the overall change detection ability of the monitoring system.

Estimating the exact time of a step change in the process parameter of univariate and multivariate control charts has been discussed by various researchers. For univariate processes, Samuel et al. [1, 2] present a technique to estimate the time when a change in the process mean and variance of an independent normal process take place. Samuel et al. [3] estimated the change point in a normal process mean in SPC applications and compared the performance of maximum likelihood estimator with build-in change point estimators of CUSUM and EWMA. Park and Park [4] estimated the change point in the process for mean and variance when $\bar{X}$ and S control chart used simultaneously. Khoo [5] determined the permanent shift in the process mean with CUSUM control chart. Kapase and Ghute [6] estimated the time of a step change in the process mean with Tukey's control chart and individual X control chart and compared the both control charts in detecting the occurrence of the special cause in the process. Kapase and Ghute [7] discussed the maximum likelihood estimator of process change to identify the time of permanent shift in the normal mean with EWMA and MA control charts. Amiri and Allahari [8] presented a literature review on change point estimation methods for a control chart post signal diagnostics. For multivariate processes, Nedumaran et al. [9] have developed a change point MLE for sudden step change in mean of multivariate normal process when the process is monitored by χ^2 control chart. Dogu and Deveci-Kocakoc [10] proposed change point model for generalized variance control chart. Dogu and Deveci-Kocakoc [11] developed a change point estimation procedure for jointly monitoring the mean and covariance of multivariate normal process. Dogu [12] discussed multivariate joint change point estimation procedure for simultaneous monitoring of both location and dispersion under the assumption that the process is being monitored with a multivariate single control chart. Atashgar [13] reviewed the literature on the mean change point of univariate and multivariate processes.

The literature on the use of change point estimation in control charts assumes the independence of observations. In practice, we may come across processes dealing with autocorrelated observations. The processes such as those found in chemical manufacturing, refinery operations, smelting operations, wood product manufacturing, waste water processing and operation of nuclear reactors have been shown to have autocorrelated observations (Timmer et al. [14]). The autocorrelation can have a significant effect on statistical performance of the control charts for monitoring process parameters. In literature, little attention has been given to change point estimation in control charts when the observations are autocorrelated. Timmer and Pignatiello [15] developed three MLE change point estimators for the location, variance and autoregressive parameters for the use when the control charts are applied to autocorrelated data modeled by AR(1) process. Each of these estimators is then applied after a signal from a control chart indicates presence of special cause source of variability. It was shown that for all the three parameters the performance of the MLE change point estimator was superior to that of the other competing control charts.

Shewhart $\bar{X}$ control chart is most commonly used in practice for monitoring the process mean. It is based on the assumption that the sampled process is a normal whose observations are independent and identically distributed. In practice, we may come across processes dealing with autocorrelated observations. The purpose of this paper is to propose a change point estimator based on maximum likelihood approach for estimating the time of step change in the $\bar{X}$ chart when the observations are autocorrelated. We assume that the process with autocorrelated observations can be modeled by first order autoregressive(AR(1)) model. We derive the maximum likelihood estimator for the time of step change in autocorrelated process mean. The proposed change point estimator is used when $\bar{X}$ chart with AR(1) process to monitor process mean issues a signal. We analyze the performance of our proposed estimator when it applied after $\bar{X}$ chart with AR(1) process signals that a assignable cause is present. An example of the use of proposed estimator is also given.

The paper is organized as follows. In section 2, first order autoregressive AR(1) model is discussed. Section 3 provides the details of $\bar{X}$ chart with AR(1) process. Section 4 gives the details of the change point model. The performance assessment and other performance measurements are provided in section 5. An illustrative example is given in Section 6. In Section 7 conclusions are given.

2. AR (1) Model of Autocorrelation

In this section, the first order autoregressive AR(1) model is presented. The AR(1) process is a time series model that fits many naturally occurring processes with autocorrelated observations. Assume that quality characteristic X follows $N(\mu_0, \sigma_0^2)$ when the process is in-control, where μ_0 and σ_0^2 are known as in-control mean and variance respectively. After each sampling interval i, n observations of quality characteristic X are collected. We also assume that the consecutive observations $\{X_{i,1}, X_{i,2}, \ldots, X_{i,n}\}$ of the quality characteristic fit to a first order autoregressive AR(1) model as

$$X_{i,j} - \mu_0 = \phi\left(X_{i,j-1} - \mu_0\right) + \epsilon_j \ , i = 1,2, \ldots \ \ . j = 1,2, \ldots, n \tag{1}$$

where $\epsilon_j, j = 1,2, \ldots, n$ are independently and identically distributed $N(0, \sigma_\epsilon^2)$ random error variables and ϕ is autocorrelation parameter. It is well known that the mean and variance of $X_{i,j}$ for such a model are $\frac{\mu}{1-\phi}$ and $\frac{\sigma_\epsilon^2}{1-\phi^2}$ respectively.

We consider the situation where the monitoring parameter μ can take one of the two values, either in-control value $\mu_0$or an out-of-control value μ_1. Initially the monitoring parameter has the value μ_0. The change point denoted by τ is the time at which the monitoring parameter changes its in-control value to its out-of-control value. Thus, the AR(1) model for the change point problem with monitoring parameter μ is given by

$$X_t = \mu_t + \phi X_{t-1} + \epsilon_t \ , \tag{2}$$

where $\mu_t = \begin{cases} \mu_0 \ , & if\ t < \tau \\ \mu_1 \ , & if\ t \geq \tau \end{cases}$

3. The $\bar{X}$ Chart for AR(1) Process

In this section, we assume that at time $i = 1,2, \ldots$ the consecutive observations $X_{i,1}, X_{i,2}, \ldots, X_{i,n}$ of quality characteristic X denote a sequence of samples each of size n taken on quality characteristic X generated by AR(1) model. The plotting statistic of the $\bar{X}$ chart is the sample mean $\bar{X}_i$ given as

$$\bar{X}_i = \frac{X_{i,1} + X_{i,2} + \cdots + X_{i,n}}{n} \tag{3}$$

Alwan and Roberts [16] give the standard deviation of the sample mean $\bar{X}_i$ is for AR(1) process as

$$\sigma(\bar{X}_i) = \frac{\sigma_0}{\sqrt{n}\psi} \quad (4)$$

where, $\psi^{-1} = \sqrt{1 + \frac{2}{n}\left\{\frac{\phi^{n+1} - n\phi^2 + (n-1)\phi}{(\phi-1)^2}\right\}}$

The control limits LCL and UCL of two-sided control chart for monitoring the AR(1) quality characteristic are

$$UCL, LCL = \mu_0 \pm k\frac{\sigma_0}{\sqrt{n}\psi} \quad (5)$$

where $k > 0$ is a positive real valued constant defined to satisfy some desired in-control ARL denoted as $ARL_{\bar{X}}(0)$. In the $\bar{X}$ chart, if value of the plotting statistic $\bar{X}_i$ falls above the UCL or below the LCL then there is an indication that the process is out-of-control, that is, the process mean has been changed.

After the occurrence of assignable cause, process mean shifts from μ_0 to $\mu_1 = \mu_0 + \delta\sigma_0$ where δ is the standard shift size expresses as $\delta = \frac{(\mu_1 - \mu_0)}{\sigma_0}$. If $\delta = 0$, process is considered to be in-control otherwise the process is out of control.

The in-control ARL values of the $\bar{X}$ chart for the autocorrelated data can be calculated as

$$ARL_{\bar{X}}(0) = \frac{1}{\alpha} \quad (6)$$

$ARL_{\bar{X}}(0)$ is usually specified by the user based on the requirement of the false alarm rate and α is obtained. Using α, the value of the control limits coefficient k is obtained by

$$k = -\phi^{-1}(\alpha/2) \quad (7)$$

When the process is out-of-control,

$$ARL_{\bar{X}}(\delta) = \frac{1}{1-\beta} \quad (8)$$

where α and β are type I and type II error probabilities respectively. For $\bar{X}$ chart the probability of type II error is calculated by

$$\beta = \Phi\left(k - \delta\sqrt{n}\,\psi\right) - \Phi\left(-k - \delta\sqrt{n}\,\psi\right) \quad (9)$$

where $\Phi(.)$ is the cumulative distribution function of standard normal $N(0,1)$ distribution.

4. Change Point Estimator for Process Mean

We will assume that the parameter μ of the time-series model is initially in-control with a known value of μ_0. However, after an unknown point in time a change in the process parameter occurs from μ_0 to $\mu_1 = \mu_0 + \delta\,\sigma_0/\sqrt{n}$, $\delta \neq 0$, where n is the subgroup size and δ is the unknown magnitude of the change. We also assume that once this step change in the process parameter occurs, the process remains at the new level of μ_1 until assignable cause has been identified and eliminated.

We will consider $\bar{X}$ chart signals at subgroup T that the process is no longer in a state of statistical control. This is the point at which quality engineer must initiate a search for assignable cause of variation. Let τ denote the last subgroup from the in-control process. Thus, $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_\tau$ are the subgroup averages from the in-control process, while $\bar{X}_{\tau+1}, \bar{X}_{\tau+2}, \dots, \bar{X}_T$ are the subgroup averages when the process parameter changed.The change point estimator for μ is found by formulating a likelihood function and solving for an estimate of τ, the process change point. The detailed derivation of the change point estimator for μ is derived as follows:

We will assume that the autocorrelated observations are observed from the AR(1) model.

We develop a technique for detecting the change point for the parameter μ.

$$f(\bar{x}) = \frac{\sqrt{n}\,\psi}{\sigma_0\sqrt{2\pi}}\, exp\left\{\frac{-n\,\psi^2}{2\sigma_0^2}(\bar{X}_i - \mu_0/(1-\phi))^2\right\}$$

The likelihood function for the change point problem for μ is defined to be

$$L = \left[\frac{\sqrt{n}\psi}{\sigma_0\sqrt{2\pi}}\right]^T exp\left\{\frac{-n\,\psi^2}{2\sigma_0^2}\sum_{i=1}^{T}(\bar{X}_i - \mu_0/(1-\phi))^2\right\}$$

The log likelihood function (apart from a constant) is

$$\begin{aligned}
\log L &= \frac{-n\,\psi^2}{2\sigma_0^2}\sum_{i=1}^{T}(\bar{X}_i - \mu_0/(1-\phi))^2 \\
&= \frac{-n\,\psi^2}{2\sigma_0^2}\left[(\bar{X}_1 - \mu_0/(1-\phi))^2 + \sum_{i=2}^{T}(\bar{X}_i - \mu_0/(1-\phi))^2\right] \\
&= \frac{-n}{2\sigma_0^2}\left[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \psi^2\sum_{i=2}^{T}(\bar{X}_i - \mu_0/(1-\phi))^2\right] \\
&= \frac{-n}{2\sigma_0^2}\left[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \sum_{i=2}^{T}(\bar{X}_i - \mu_0 - \phi\bar{X}_{i-1})^2\right] \\
&= \frac{-n}{2\sigma_0^2}\left[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \sum_{i=2}^{\tau}(\bar{X}_i - \mu_0 - \phi\bar{X}_{i-1})^2 + \sum_{i=\tau+1}^{T}(\bar{X}_i - \mu_1 - \phi\bar{X}_{i-1})^2\right] \\
\log L &= \frac{-n}{2\sigma_0^2}\left[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \sum_{i=2}^{\tau}(\bar{X}_i - \mu_0 - \phi\bar{X}_{i-1})^2\right] + \frac{-n}{2\sigma_0^2}\sum_{i=\tau+1}^{T}(\bar{X}_i - \mu_1 - \phi\bar{X}_{i-1})^2
\end{aligned} \tag{10}$$

We note that Eq. (10) is a function of two unknown parameters τ and μ_1. For a fixed value of τ, it is easy to show that the maximum likelihood estimator of μ_1 is

$$\hat{\mu}_1(\tau) = \frac{1}{T-\tau}\sum_{i=\tau+1}^{T}(\bar{X}_i - \phi\bar{X}_{i-1}) \tag{11}$$

Substituting this in Eq. (10) we get

$$\begin{aligned}
log\, L(\tau|\bar{x}) &= \frac{-n}{2\sigma_0^2}\left[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \sum_{i=2}^{\tau}(\bar{X}_i - \mu_0 - \phi\bar{X}_{i-1})^2\right] + \frac{-n}{2\sigma_0^2}\sum_{i=\tau+1}^{T}(\bar{X}_i - \hat{\mu}_1(\tau) - \phi\bar{X}_{i-1})^2 \\
&= \frac{-n}{2\sigma_0^2}\left[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \sum_{i=2}^{\tau}(\bar{X}_i - \mu_0 - \phi\bar{X}_{i-1})^2 + \sum_{i=\tau+1}^{T}(\bar{X}_i - \hat{\mu}_1(\tau) - \phi\bar{X}_{i-1})^2\right] \\
&= \frac{-n}{2\sigma_0^2}\left[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \sum_{i=2}^{\tau}\left((\bar{X}_i - \phi\bar{X}_{i-1}) - \mu_0\right)^2 + \sum_{i=\tau+1}^{T}\left((\bar{X}_i - \phi\bar{X}_{i-1}) - \hat{\mu}_1(\tau)\right)^2\right] \\
&= \frac{-n}{2\sigma_0^2}\left[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \sum_{i=2}^{\tau}\{(\bar{X}_i - \phi\bar{X}_{i-1})^2 - 2\mu_0(\bar{X}_i - \phi\bar{X}_{i-1}) + \mu_0^2\}\right. \\
&\left. + \sum_{i=\tau+1}^{T}\{(\bar{X}_i - \phi\bar{X}_{i-1})^2 - 2\hat{\mu}_1(\tau)(\bar{X}_i - \phi\bar{X}_{i-1}) + \hat{\mu}_1^2(t)\}\right]
\end{aligned}$$

$$= \frac{-n}{2\sigma_0^2}\Bigg[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \left\{\sum_{i=2}^{\tau}(\bar{X}_i - \phi\bar{X}_{i-1})^2 - 2\mu_0\sum_{i=2}^{\tau}(\bar{X}_i - \phi\bar{X}_{i-1}) + (\tau-1)\mu_0^2\right\}$$
$$+ \left\{\sum_{i=\tau+1}^{T}(\bar{X}_i - \phi\bar{X}_{i-1})^2 - 2\hat{\mu}_1(\tau)\sum_{i=\tau+1}^{T}(\bar{X}_i - \phi\bar{X}_{i-1}) + (T-\tau)\hat{\mu}_1^2(t)\right\}\Bigg]$$
$$= \frac{-n}{2\sigma_0^2}\Bigg[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \Bigg\{\sum_{i=2}^{T}(\bar{X}_i - \phi\bar{X}_{i-1})^2 - 2\mu_0\sum_{i=2}^{\tau}(\bar{X}_i - \phi\bar{X}_{i-1})$$
$$- 2\hat{\mu}_1(\tau)\sum_{i=\tau+1}^{T}(\bar{X}_i - \phi\bar{X}_{i-1}) + (\tau-1)\mu_0^2 + (T-\tau)\hat{\mu}_1^2(t)\Bigg\}\Bigg]$$
$$= \frac{-n}{2\sigma_0^2}\Bigg[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \Bigg\{\sum_{i=2}^{T}(\bar{X}_i - \phi\bar{X}_{i-1})^2 - 2\mu_0\sum_{i=2}^{\tau}(\bar{X}_i - \phi\bar{X}_{i-1}) - 2\mu_0\sum_{i=\tau+1}^{T}(\bar{X}_i - \phi\bar{X}_{i-1})$$
$$+ 2\mu_0\sum_{i=\tau+1}^{T}(\bar{X}_i - \phi\bar{X}_{i-1}) - 2\hat{\mu}_1(\tau)\sum_{i=\tau+1}^{T}(\bar{X}_i - \phi\bar{X}_{i-1}) + (\tau-1)\mu_0^2 + T\mu_0^2 - T\mu_0^2 + (T-\tau)\hat{\mu}_1^2(t)\Bigg\}\Bigg]$$
$$= \frac{-n}{2\sigma_0^2}\Bigg[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \Bigg\{\sum_{i=2}^{T}(\bar{X}_i - \phi\bar{X}_{i-1})^2 - 2\mu_0\sum_{i=2}^{T}(\bar{X}_i - \phi\bar{X}_{i-1}) - 2\mu_0(T-\tau)\hat{\mu}_1(\tau)$$
$$- 2\hat{\mu}_1(\tau)(T-\tau)\hat{\mu}_1(\tau) + \tau\mu_0^2 - \mu_0^2 + T\mu_0^2 - T\mu_0^2 + (T-\tau)\hat{\mu}_1^2(\tau)\Bigg\}\Bigg]$$
$$= \frac{-n}{2\sigma_0^2}\Bigg[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \Bigg\{\sum_{i=2}^{T}(\bar{X}_i - \phi\bar{X}_{i-1})^2 - 2\mu_0\sum_{i=2}^{T}(\bar{X}_i - \phi\bar{X}_{i-1}) + 2\mu_0(T-\tau)\hat{\mu}_1(\tau)$$
$$- 2(T-\tau)\hat{\mu}_1^2(\tau) - (T-\tau)\mu_0^2 + (T-1)\mu_0^2 + (T-\tau)\hat{\mu}_1^2(\tau)\Bigg\}\Bigg]$$
$$= \frac{-n}{2\sigma_0^2}\Bigg[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \Bigg\{\sum_{i=2}^{T}(\bar{X}_i - \phi\bar{X}_{i-1})^2 - 2\mu_0\sum_{i=2}^{T}(\bar{X}_i - \phi\bar{X}_{i-1})$$
$$+ 2\mu_0(T-\tau)\hat{\mu}_1(\tau) - (T-\tau)\mu_0^2 + (T-1)\mu_0^2 - (T-\tau)\hat{\mu}_1^2(\tau)\Bigg\}\Bigg]$$
$$= \frac{-n}{2\sigma_0^2}\Bigg[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \Bigg\{\sum_{i=2}^{T}(\bar{X}_i - \phi\bar{X}_{i-1})^2 - 2\mu_0\sum_{i=2}^{T}(\bar{X}_i - \phi\bar{X}_{i-1}) + (T-1)\mu_0^2$$
$$+ (T-\tau)[\hat{\mu}_1^2(\tau) - 2\mu_0\hat{\mu}_1(\tau)] + \mu_0^2\Bigg\}\Bigg]$$
$$= \frac{-n}{2\sigma_0^2}\Bigg[\psi^2(\bar{X}_1 - \mu_0/(1-\phi))^2 + \Bigg\{\sum_{i=2}^{T}(\bar{X}_i - \phi\bar{X}_{i-1})^2 - 2\mu_0\sum_{i=2}^{T}(\bar{X}_i - \phi\bar{X}_{i-1}) + (T-1)\mu_0^2$$
$$+ (T-\tau)[\hat{\mu}_1(\tau) - \mu_0]^2\Bigg\}\Bigg]$$

The maximum likelihood estimate of τ is the value of t that maximizes the likelihood function. So, $\hat{\tau} = \arg\max_{1<t<T}[(\mathrm{T}-\mathrm{t})(\hat{\mu}_1(\tau) - \mu_0)^2]$.

5. Performance Evaluation

In this section, we study the performance of our proposed estimator using Monte Carlo simulation. The average change point estimate and the precision of the change point estimate are used as the performance measures for the use of proposed estimator. A simulation study is conducted to examine the performance of proposed change point estimator used to $\bar{X}$ chart when process data are observed from the AR(1) model. In order to assess the performance, the observations are generated from AR(1) process. The in-control average run length (ARL) for the control chart is considered as 370. The change in the process is simulated at the point $\tau = 100$. Assuming that in-control AR(1) process has a $N(\mu_0, \sigma_0^2)$ distribution; the observations from 1 to 100 subgroups of size $n = 4$ with autocorrelation level ϕ are generated from $N(0,1)$ distribution. Starting from the subgroup 101, observations are randomly generated from a $N(\delta, 1)$ distribution until AR(1) $\bar{X}$ chart produces out-of-control signal and this is not a false alarm. At this point $\hat{\tau}$ is computed. The procedure is repeated 10000 times for each of the magnitude that are studied namely for $\delta =$ 0.5,1.0,1.5,2.0 and 3.0 with different levels of autocorrelation $\phi = -0.8$ to 0.8 in steps of size 0.2.

In Table 1, the expected length of each simulation run for various magnitudes of change in process mean is presented. The expected length $E(T)$ is the expected time at which the $\bar{X}$ chart is expected to issue a signal of a change in the process mean. Since the change had actually occurred following subgroup $\tau = 100$, $E(T) = 100 + ARL$, where ARL is the average run length of control chart for the out-of-control process. The values of $\bar{\hat{\tau}}$, the average change point estimate from 10000 simulation runs for various sizes of change in process mean for different levels of autocorrelation along with its corresponding standard error estimates are presented in Table 1. As the actual change point for the simulations is $\tau = 100$, the average estimated time of process change $\bar{\hat{\tau}}$ is expected to be close to 100.

Table 1: *Average change point estimates for μ and their standard errors.*

	ϕ								
	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
					$\delta = 0.5$				
$E(T)$	352.39	347.40	319.12	290.70	254.64	214.49	172.54	136.42	113.53
$\bar{\hat{\tau}}$	99.76	100.52	101.64	103.70	104.05	104.58	104.33	103.10	99.81
$s.e.(\bar{\hat{\tau}})$	0.085	0.1471	0.1947	0.222	0.081	0.229	0.206	0.167	0.127
					$\delta = 1.0$				
$E(T)$	225.05	211.69	187.16	165.51	143.27	127.01	114.78	107.89	104.52
$\bar{\hat{\tau}}$	100.10	99.99	100.04	100.21	100.30	100.22	100.01	99.34	98.87
$s.e.(\bar{\hat{\tau}})$	0.0226	0.043	0.056	0.066	0.071	0.077	0.075	0.074	0.078
					$\delta = 1.5$				
$E(T)$	163.70	150.88	134.82	123.75	114.85	109.02	105.50	103.73	102.80
$\bar{\hat{\tau}}$	100.10	100.00	99.75	99.93	99.82	99.66	99.34	98.99	99.13
$s.e.(\bar{\hat{\tau}})$	0.0161	0.0256	0.035	0.0388	0.051	0.054	0.0606	0.067	0.064
					$\delta = 2.0$				
$E(T)$	132.99	123.42	115.32	110.09	106.31	104.20	103.05	102.43	102.10
$\bar{\hat{\tau}}$	100.07	99.98	99.94	99.84	99.73	99.51	99.31	99.27	99.56
$s.e.(\bar{\hat{\tau}})$	0.015	0.0211	0.0249	0.0371	0.0378	0.0444	0.0504	0.049	0.0391
					$\delta = 3.0$				
$E(T)$	105.67	104.82	103.33	102.46	102.00	101.73	101.58	101.50	101.40
$\bar{\hat{\tau}}$	99.86	99.93	99.72	99.72	99.60	99.56	99.43	99.59	99.72
$s.e.(\bar{\hat{\tau}})$	0.0332	0.0216	0.038	0.030	0.0374	0.0344	0.043	0.029	0.026

From Table 1, we observe that for autocorrelation level $\phi = 0.4$ and magnitude of change in mean $\delta = 0.5$, the control chart issue a signal at time 172.54 on average. In this case, the average estimated time of process change is 104.33, which is fairly close to the actual change point 100. Also for, $\phi = 0.4$ and $\delta = 1.0$, the average estimated time of signal is 100.01. For the cases with $\phi = 0.4$ and $\delta = 1.5, 2.0$ and 3.0, the corresponding estimated times of signal are 99.34, 99.31 and 99.43 respectively. Similar pattern of estimated time of signal is observed for all considered levels of autocorrelations and magnitudes of change in process mean. Thus, we conclude that, on average, the proposed maximum likelihood estimator of the time of process change is fairly close to the actual time of change regardless of the magnitude of change in the process mean and the level of autocorrelation.

The precision of the change point estimator for the parameter μ is evaluated by examining the probability that $\hat{\tau}$ is within m observations of the actual change point. The estimated probability that the average change point estimate is within m subgroups of actual change point is shown in Table 2 for several values of m based on 10000 simulations with the autocorrelation levels $\phi = 0.2, 0.4$ and 0.6.

Table 2: *Precision of estimator for* μ *when* $n = 4$ *and* $\tau = 100$.

$\phi = 0.2$					
δ	0.5	1.0	1.5	2.0	3.0
$P(\lvert\hat{\tau}-\tau\rvert = 0)$	0.0894	0.2721	0.4725	0.6226	0.8173
$P(\lvert\hat{\tau}-\tau\rvert \le 1)$	0.1842	0.4728	0.7058	0.8367	0.9352
$P(\lvert\hat{\tau}-\tau\rvert \le 2)$	0.2665	0.5697	0.8124	0.9146	0.9631
$P(\lvert\hat{\tau}-\tau\rvert \le 3)$	0.3192	0.6837	0.8756	0.9469	0.9737
$P(\lvert\hat{\tau}-\tau\rvert \le 4)$	0.3707	0.7448	0.9134	0.9634	0.9798
$P(\lvert\hat{\tau}-\tau\rvert \le 5)$	0.4174	0.7925	0.9360	0.9724	0.9834
$P(\lvert\hat{\tau}-\tau\rvert \le 6)$	0.4599	0.8323	0.9535	0.9773	0.9851
$P(\lvert\hat{\tau}-\tau\rvert \le 7)$	0.4971	0.8594	0.9639	0.9805	0.9872
$P(\lvert\hat{\tau}-\tau\rvert \le 8)$	0.5326	0.8838	0.9709	0.9824	0.9888
$P(\lvert\hat{\tau}-\tau\rvert \le 9)$	0.5619	0.9031	0.9761	0.9844	0.9903
$P(\lvert\hat{\tau}-\tau\rvert \le 10)$	0.5856	0.9169	0.9781	0.9863	0.9915
$P(\lvert\hat{\tau}-\tau\rvert \le 11)$	0.6102	0.9276	0.9812	0.9879	0.9920
$\phi = 0.4$					
$P(\lvert\hat{\tau}-\tau\rvert = 0)$	0.1204	0.2302	0.5147	0.6456	0.8239
$P(\lvert\hat{\tau}-\tau\rvert \le 1)$	0.2095	0.5148	0.7374	0.8504	0.9310
$P(\lvert\hat{\tau}-\tau\rvert \le 2)$	0.2815	0.6447	0.8380	0.9170	0.9584
$P(\lvert\hat{\tau}-\tau\rvert \le 3)$	0.3475	0.7250	0.8917	0.9436	0.9692
$P(\lvert\hat{\tau}-\tau\rvert \le 4)$	0.3968	0.7827	0.9258	0.9585	0.9755
$P(\lvert\hat{\tau}-\tau\rvert \le 5)$	0.4432	0.8280	0.9443	0.9671	0.9798
$P(\lvert\hat{\tau}-\tau\rvert \le 6)$	0.4822	0.8596	0.9572	0.9720	0.9827
$P(\lvert\hat{\tau}-\tau\rvert \le 7)$	0.5192	0.8869	0.9638	0.9753	0.9854
$P(\lvert\hat{\tau}-\tau\rvert \le 8)$	0.5507	0.9298	0.9679	0.9786	0.9870
$P(\lvert\hat{\tau}-\tau\rvert \le 9)$	0.5805	0.9240	0.9718	0.9811	0.9885
$P(\lvert\hat{\tau}-\tau\rvert \le 10)$	0.6038	0.9368	0.9746	0.9832	0.9898
$P(\lvert\hat{\tau}-\tau\rvert \le 11)$	0.6293	0.9466	0.9778	0.9849	0.9910
$\phi = 0.6$					
$P(\lvert\hat{\tau}-\tau\rvert = 0)$	0.1596	0.3972	0.5823	0.7016	0.8516
$P(\lvert\hat{\tau}-\tau\rvert \le 1)$	0.2654	0.6030	0.7961	0.8782	0.9411
$P(\lvert\hat{\tau}-\tau\rvert \le 2)$	0.3411	0.7145	0.8749	0.9269	0.9623
$P(\lvert\hat{\tau}-\tau\rvert \le 3)$	0.4057	0.7894	0.9141	0.9489	0.9724
$P(\lvert\hat{\tau}-\tau\rvert \le 4)$	0.4595	0.8379	0.9344	0.9602	0.9800

$P(\|\hat{\tau}-\tau\| \leq 5)$	0.5070	0.8719	0.9464	0.9671	0.9838
$P(\|\hat{\tau}-\tau\| \leq 6)$	0.5524	0.8986	0.9548	0.9719	0.9857
$P(\|\hat{\tau}-\tau\| \leq 7)$	0.5923	0.9149	0.9610	0.9756	0.9877
$P(\|\hat{\tau}-\tau\| \leq 8)$	0.6302	0.9301	0.9658	0.9786	0.9893
$P(\|\hat{\tau}-\tau\| \leq 9)$	0.6610	0.9422	0.9692	0.9808	0.9911
$P(\|\hat{\tau}-\tau\| \leq 10)$	0.6851	0.9505	0.9711	0.9826	0.9919
$P(\|\hat{\tau}-\tau\| \leq 11)$	0.7112	0.9555	0.9734	0.9846	0.9926

Consider the case of small level of autocorrelation $\phi = 0.2$ and small change of magnitude $\delta = 0.5$, we observe that the proposed estimator exactly identified the time of change in 8.94% of a total of 10000 simulations carried out. Further there would be 18.42% chance that the estimated change point is within ± 1 subgroup of actual change point and 25.65% chance that the estimated change point is within ± 2 subgroups of the actual change point and 61.02% chance that the estimated change point is within ± 11. As m increases, the value of precision also increases. For the case of large level of autocorrelation $\phi = 0.6$ and large change of magnitude $\delta = 3.0$, we observe that the proposed estimator exactly identified the time of change in 85.16% of a total of 10000 simulations carried out. Further there would be a 94.11 % chance that the estimated change point is within ± 1 subgroup of actual change point and 96.23% chance that the estimated change point is within ± 2 subgroups of the actual change point and 99.26% chance that the estimated change point is within ± 11. Thus, we conclude that the proposed estimator exhibits a good performance in identifying the time of change.

6. An Example

A numerical example is provided to illustrate the use of the proposed change point estimator for $\bar{X}$ chart with AR(1) process. The dataset in this example is generated from normally distributed quality characteristic with $\mu_0 = 1, \sigma_0 = 1$ and $\phi = 0.2$. When the desired type I error is $\alpha = 0.0027$, then the upper and lower control limits for $\bar{X}$ chart using Eq. (5) are UCL=1.73891 and LCL=-1.73891. Under the assumption that the process is in-control, the first 10 samples each of size $n = 4$ are generated from AR(1) process with $N(\mu_0, \sigma_0^2)$ and the remaining samples each of size $n = 4$ are generated from AR(1)process with $N(\mu_1, \sigma_0^2)$ until $\bar{X}$ chart issued a signal, here μ_1 denotes the changed mean.

The sample observations for each subgroup i, X_{i1}, X_{i2}, X_{i3} and X_{i4} and calculated subgroup averages $\bar{X}_i$ are shown in Table 3.

Table 3: *Observations from normal process with* $\phi = 0.2$ and $n = 4$.

i	X_{i1}	X_{i2}	X_{i3}	X_{i4}	$\bar{X}_i$	t	$\hat{\mu}_1(t)$	C_t
1	-0.7056	-0.3292	-0.7509	-1.0821	-0.7170	0		
2	1.5940	0.5129	-1.5725	-0.1931	0.0853	1	0.4583	7.1425
3	0.5516	1.2266	-0.3674	-0.8348	0.1440	2	0.4653	7.1446
4	-0.1713	1.9423	0.03207	0.2694	0.5181	3	0.4759	7.2465
5	1.5390	-0.9642	-0.3528	-0.8888	-0.1667	4	0.4754	7.0072
6	-0.8516	-0.3973	0.4710	1.7833	0.2514	5	0.5003	7.5089
7	1.0925	0.7734	0.7591	1.9817	1.1517	6	0.5077	7.4759
8	-1.4947	0.4468	-0.4431	-1.0924	-0.6459	7	0.4865	6.6279
9	-0.7333	2.8370	-0.6741	0.6855	0.5288	8	0.5370	7.7860
10	-1.0108	0.2336	2.4691	-1.0286	0.1658	9	0.5323	7.3682
11	1.2306	2.0690	3.2730	-0.0510	1.6304	10	0.5512	7.5966
12	-0.2249	0.8871	1.4378	1.3899	0.8725	11	0.5077	6.1852

13	1.3571	1.4480	-0.8006	0.9300	0.7336	12	0.5060	5.8882
14	-0.9501	-0.9501	-0.2132	0.3868	-0.4317	13	0.5036	5.5785
15	1.5943	1.2325	0.8867	2.3642	1.5194	14	0.5551	6.4703
16	0.9931	-0.4139	2.9077	0.0924	0.8948	15	0.5025	5.0509
17	1.8382	0.5626	1.0340	0.9139	1.0872	16	0.4979	4.7099
18	1.2815	1.5461	-0.4358	1.4192	0.9528	17	0.4751	4.0629
19	2.7691	1.4962	0.1349	0.3950	1.1988	18	0.4598	3.5939
20	1.9336	-0.3313	1.8259	1.7318	1.2900	19	0.4255	2.8969
21	-0.0468	0.9701	0.8853	0.2945	0.5258	20	0.3839	2.2102
22	1.0439	1.1534	0.7997	-0.3568	0.6601	21	0.3922	2.1530
23	1.9355	0.7652	0.8776	-0.0657	0.8782	22	0.3796	1.8736
24	0.7291	-0.9438	1.0241	0.5090	0.3296	23	0.3491	1.4624
25	0.8384	0.8187	-0.2365	0.0117	0.3581	24	0.3668	1.4802
26	1.7822	-0.1980	-1.0260	0.9183	0.3691	25	0.3743	1.4009
27	-0.7695	1.2146	0.2993	0.1080	0.2131	26	0.3828	1.3190
28	-0.4857	0.6726	-0.4337	0.6117	0.0912	27	0.4133	1.3663
29	0.5524	-0.7997	-0.3407	1.6733	0.2713	28	0.4654	1.5159
30	-0.1526	-1.4796	-0.4572	-0.2571	-0.5866	29	0.5007	1.5045
31	0.3958	0.6523	-0.3440	2.0372	0.6853	30	0.7291	2.6578
32	-1.3273	-1.1706	2.2603	1.1004	0.2157	31	0.71069	2.0203
33	-0.6394	-0.2231	2.0017	-0.2361	0.2258	32	0.9214	2.5468
34	-1.9289	1.4110	2.4092	1.3562	0.8119	33	0.8605	2.2213
35	1.5843	2.1275	2.4985	1.6982	1.9771	34	1.8147	3.2933

It can be seen that a total of 35 subgroup averages were obtained before one of them exceeded the upper control limit. Thus, following this first signal from the $\bar{X}$ chart at time $T = 35$, our proposed change point estimator can be applied. The values of C_t statistics are then calculated. The largest value of C_t is 7.7860 which is associated with subgroup 9. Thus, we estimate that subgroup 9 was the first subgroup obtained from the changed process and consequently that subgroup 8 was the last subgroup from the in-control process.

7. Conclusion

In this paper, monitoring the process mean using Shewhart $\bar{X}$ chart in the presence of autocorrelated data under normal process is considered. We have derived the maximum likelihood estimator that is useful for identifying the change point of a step change in the mean of normal process when autocorrelation may exist in the process. We have investigated the performance of our change point estimator when it is used with $\bar{X}$ chart with AR(1) process. The results show that the performance of the proposed estimator has good properties in the aspect of expected length and coverage probability for autocorrelated data.

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References

[1] Samuel, T. R., Pignatiello Jr., J. J. and Calvin, J. A. (1998a). Identifying the time of a step-change with $\overline{X}$ control charts. *Quality Engineering,* 10(3): 521-527.

[2] Samuel, T. R., Pignatiello Jr., J. J. and Calvin, J. A. (1998b). Identifying the time of a step-change in a normal process variance. *Quality Engineering,* 10(3): 529-538.

[3] Samuel, T. R., Pignatiello Jr., J. J. and Calvin, J. A. (2001). Estimation of the change point of a normal process mean in SPC applications, *Journal of Quality Technology*, 33: 82-95.

[4] Park, J. and Park, S. (2004). Estimation of the change point in the $\overline{X}$ and S control charts. *Communications in Statistics-Simulation and Computation,* 33(4): 1115-1132.

[5] Khoo, M. B. C. (2004). Determining the time of a permanent shift in the process mean of CUSUM control charts. *Quality Engineering,* 17(1): 87-93.

[6] Kapase, R. A. and Ghute, V.B. (2018). Estimating the period of a step-change in single observation data. *International Journal of Agricultural and Statistical Sciences,* 14: 433-438.

[7] Kapase, R. A. and Ghute, V.B. (2021). Identifying the time of permanent shift in the normal process mean with memory type control charts. *Statistics and Applications*, (Accepted on 09 June 2021).

[8] Amiri, A. and Allahyari, S. (2012). Change point estimation methods for control chart postsignal diagnostics: A literature review. *Quality and Reliability Engineering International*, 28(7): 673-685.

[9] Nedumaran, G., Pignatiello Jr., J. J. and Calvin, J. A. (2000). Identifying the time of a step-change with χ^2control charts. Quality Engineering, 13(2): 153-159.

[10] Dogu, E. and Deveci-Kocakoc, I. (2011). Estimation of change pointin generalized variance control chart. *Communication in Statistics- Simulation and Computation,* 40(3): 345-363.

[11] Dogu, E. and Deveci-Kocakoc, I. (2013). A multivariate change point detection procedure for monitoring mean and covariance simultenously. *Communication in Statistics- Simulation and Computation.* 42: 1235-1255.

[12] Dogu, E. (2015). Identifying the time of step change with multivariate single control charts. *Journal of Statistical Computation and Simulation,* 85(8): 1529-1543.

[13] Atashgar, A. (2013). Identification of change point: an overview. *International Journal of Advanced Manufacturing Technology*, 64: 1663-1683.

[14] Timmer, D. H, Pignatiello Jr., J. J. and Longnecker, M.T. (1998). The development and evaluation of cusum-based control charts for an AR (1) process. *IIE Transaction in Quality and Reliability,* 30: 525-534.

[15] Timmer, D. H. and Pignatiello Jr., J. J. (2003). Change point estimates for the parameters of an AR(1) process. *Quality and Reliability Engineering International* , 19: 355-369.

[16] Alwan, L.C. and Roberts, H.V. (1988). Time-series modeling for statistical process control. *Journal of Business and Economic Statistics,* 6:87-95.