Inversion Method of Consistency Measure Estimation Expert Opinions

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Abstract

The problem of collective choice is the problem of combining several individual experts' opinions about the order of preference of objects (alternatives) being compared into a single "group" preference. The complexity of collective choice consists in the necessity of processing the ratings of the compared alternatives set by different experts in their own private scales. This article presents the author's original algorithm for processing expert preferences in the problem of collective choice, based on the notion of the total "error" of the experts and measuring their contribution to the collective measure of their consistency. The presentation of the material includes the necessary theoretical part consisting of basic definitions and rules, the statement of the problem and the method itself based on the majority rule, but in the group order of objects.

Keywords: collective choice, permutation, group, inconsistency, inversion, graph, rating, Schulze method, skating method, Pareto-optimal solutions.

1. Introduction

In practice, the efficiency of decision-making requires the development and application of specialized algorithmic and methodological support. If a group of experts participates in the decision support process, the so-called collective (group) choice problem arises. The existing algorithms for solving collective choice problems [1-3] can be divided into three classes.

A representative of the first class is the Schulze method [4] (based on the proof of the Arrow theorem) with the selection of Pareto-optimal solutions (Schwartz exception) from the first ranking to the last, with the selection recalculating the criteria for the next step. The disadvantage of the method is a rather complicated algorithm of constant recalculation, which significantly complicates the practical use of the method.

A typical representative of the second class is the skating-system well-proven in ballroom dance competitions [5]. It is simple in computational calculations and is based on the so called understandable majority principle. Unfortunately, in many ways, this simplicity can lead to unstable decisions, and therefore, the impossibility to distribute the final places among the competitors in one round, or recognize a draw between competitors [6, 7].

The third class consists of regression models, type nonlinear factor analysis and other methods of information compression [8, 9], in which the desired solution is constructed in the form of the problem of minimization of accumulated errors. The difference between the methods of the third class is that they are not focused on the choice of the leader in the ratings, but are determined by the optimum, which is influenced by the entire volume of data.

The mentioned methods of solving the problems of collective choice in general are inherent to the problem of coordinating the experts' evaluations when comparing the evaluated objects.

In 1951 C. Arrow formulated [10] the theorem "On the impossibility of collective choice within the framework of the ordinality method", mathematically generalizing the Condorcet paradox [11]. The theorem states that within the framework of this approach there is no method for combining individual preferences for three or more alternatives, which would satisfy some quite fair conditions (the axioms of choice) and would always give a logically consistent result.

When ambiguous expert opinions are superimposed on the uncertainty of the objects themselves, some hierarchy is assumed in solving the choice problem. This is the case, for example, in the method of hierarchy analysis [12], when each of M of experts has his/her own, different from the others, opinion concerning the weights of the objects under consideration N objects through the coefficients of the preference matrix ($S_{ij}^{m} = \frac{w_j^m}{w_i^m}\,(i = 1, \ldots, N;\ j = 1, \ldots, N\,; i \neq j; m = 1, \ldots, M)$).

Usually, weights are averaged and work with a generalized matrix S_{ij} this usually leads, as a rule, to a violation of the basic axioms of the "right" choice (universality, completeness, monotone, lack of a dictator, independence) proposed by W. Pareto [13, 14], R. Koch [15], C. Plott [16] and others. The rejection of one or another averaging procedure complicates the choice problem and leads, for example, to the need to solve the problem of "merging multidimensional scales" [17].

Earlier [18], the authors argued that to obtain consistent decisions, experts need to reach consensus, at least within the accuracy of determining private ratings in the full order of objects, and then seek agreement in the weighting coefficients between the neighboring nearest objects, setting a single scale. In this article we consider a method belonging to the third class of algorithms in decision theory, aimed at finding the optimum of the consistency measure, the restoration of the full collective order in preferences based on private ratings of experts.

2. Basic definitions and rules

Let us introduce several basic definitions.

Definition 1. Arbitrary mutually one-valued mapping $g\colon X \leftrightarrow g(X)$ of multiple first N natural numbers $X = \langle 1, 2, 3, \ldots, N\rangle$ is called a permutation $N$of row (permutation):

$$\begin{array}{llccccc} & X= & \{ & x_1 = 1 & x_2 = 2 & \ldots & x_N = N & \} \\ g: & \updownarrow & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ & g(X)= & \{ & g_1 = g(x_1) & g_2 = g(x_2) & \ldots & g_N = g(x_N) & \} \end{array}$$

The set $G = \{g\}$ forms a group of dimensionalities $N!$.

Definition 2. An inverse permutation to g is defined as $(g^{-1}(j) = k) \Leftrightarrow (g(k) = j)\ \forall j, k$.

An example of all permutations for $N = 4$ is given in Table 1. In principle, for any N each permutation can be assigned an index in the lexicographic order (LG-order) of values $g(X)$ (columns 1, 2 in Table 1).

Definition 3. The first permutation by index which is equal to the unit permutation in the group $^{1}g = \langle 1234\rangle = E$ we will call "true" or natural order. The last permutation with ordinates in reverse order: ${}^{24}g = \langle 4321 \rangle = \bar{E}$- the complete inversion, which is at the last determinable level $N(N-1)/2$.

Definition 4. For a permutation of $g = \langle g_1\ g_2\ g_{3\ldots} g_N \rangle$ a pair of indices $(g_i,\ g_j)$ is called an inversion [19] if $(i < j) \& (g_j > g_i)$.

Definition 5: A table of permutation inversions g is a sequence of numbers $\{b_1\ b_2\ \ldots\ b_N\}$ where b_j - is the number of elements greater than j and to the left of j. In other words, b_j - is the number of inversions with the second term equal to j.

Table 1. *Full table of permutations for* $N = 4$

Index g	Numeric code g in the LG order	Root synonym g	Synonym g, based on in versions, [other synonyms of minimum word length]	Error level, word length	Reverse re-installation	Inversion Tables, their sum, Σ				
1	2	3	4	5	6	*7*	*8*	*9*	*10*	*11*
1	< 1234>	E	E, [-]	**0**	< 1234>	0	0	0	0	**0**
2	< 1243>	c	c, [-]	**1**	< 1243>	0	0	1	0	**1**
3	< 1324>	b	b, [-]	**1**	< 1324>	0	1	0	0	**1**
4	< 1342>	cb	cb, [-]	**2**	< 1423>	0	2	0	0	**2**
5	< 1423>	bc	bc, [-].	**2**	< 1342>	0	1	1	0	**2**
6	< 1432>	bcb	cbc, [-]	**3**	< 1432>	0	2	1	0	**3**
7	< 2134>	a	a, [-]	**1**	< 2134>	1	0	0	0	**1**
8	< 2143>	ac	ac, [ca]	**2**	< 2143>	1	0	1	0	**2**
9	< 2314>	ba	ba, [-].	**2**	< 3124>	2	0	0	0	**2**
10	< 2341>	cba	cba, [-]	**3**	< 4123>	3	0	0	0	**3**
11	< 2413>	bac	bac, [bca].	**3**	< 3142>	2	0	1	0	**3**
12	< 2431>	bcba	cbac, [cbca]	**4**	< 4132>	3	0	1	0	**4**
13	< 3124>	ab	ab, [-]	**2**	< 2314>	1	1	0	0	**2**
14	< 3142>	acb	acb, [cab].	**3**	< 2413>	1	2	0	0	**3**
15	< 3214>	aba	bab, [-]	**3**	< 3214>	2	1	0	0	**3**
16	< 3241>	acba	cbab, [caba].	**4**	< 2413>	3	1	0	0	**4**
17	< 3412>	bacb	bacb, [bcab]	**4**	< 3412>	2	2	0	0	**4**
18	< 3421>	bacba	cbacb, [cbcab, bcbab, bcaba]	**5**	< 4312>	3	2	0	0	**5**
19	< 4123>	abc	abc, [-]	**3**	< 2341>	1	1	1	0	**3**
20	< 4132>	abcb	acbc, [cabc].	**4**	< 2431>	1	2	1	0	**4**
21	< 4213>	abac	babc, [abca].	**4**	< 3241>	2	1	1	0	**4**
22	< 4231>	abcba	cbabc, [acbca, cabca, cabac]	**5**	< 4231>	3	1	1	0	**5**
23	< 4312>	abacb	bacbc, [babcb, bcabc, abcab]	**5**	< 3421>	2	2	1	0	**5**
24	< 4321>	abacba	cbacbc, [cbcabc, bcbabc, bcabac, bcabca, bacbac, bac bca, babcba, abcaba, abcbab, acbcab, acbacb, cabcab, cabacb]	**6**	< 4321>	3	2	1	0	**6**

Definition 6. For any g there exists a set of $AT(g)$ (English, Adjacent Trans position) - "adjacent, neighboring" permutations, the number of which is exactly $(N-1)$. All the edges $E \times AT(E)$ consist of the forming elements of the group G. The elements of the multiplicity of formants $E \times AT(E)$ can be regarded as symbols s of some alphabet A (Table 2).

Table 2. *Nodes of the 1st error level consist of one symbol of the alphabet A*

In order, adjacent to the "truth" E	Error level	Alphabet A	The node of the 1st error level is a word of one symbol A
	0		$E = E \times E$
1	1	a	$a = E \times a = \langle 2, 1, 3, 4, \ldots, N-1, N \rangle$
2	1	b	$b = E \times b = \langle 1, 3, 2, 4, \ldots, N-1, N \rangle$
3	1	c	$c = E \times c = \langle 1, 2, 4, 3, \ldots, N-1, N \rangle$
…	1	…	…
$N-1$	1	z	$z = E \times z = \langle 1, 2, 4, 3, \ldots, N, N-1 \rangle$

z - conditional symbol $(N-1)$ of the formant. For $N = 4$, $E \times AT(E) = \{a, b, c\}$, consequently: $z = c$.

Definition 7. A weighted graph of a group $V(G, G \times G)$ consists of nodes G and the weight of an edge $(g_1 \times g_2)$ is equal to s, when $\left(g_2 \in AT(g_1)\right) \& (g_2 = g_1 s)$.

The structure of the graph $V(G, G \times G)$ is determined dynamically by the error levels. At the upper (zero) level there is only a single permutation E. At the second and further levels there are only nodes formed by joining only one symbol of the alphabet A.

The parity property of permutations is noteworthy: $aa = bb = cc = \cdots = zz = E,$ because of which the graph of the group can be treated as an undirected graph.

Definition 8. Each g can be interpreted as a path, or some sequence of directed segments of the graph $V$and vice versa.

The way from $v \in G$ в $v' \in G$ passes through the edges connecting neighboring permutations and is equal to $v' = v s_1 \cdot \ldots \cdot s_T$ where $s_1 \ldots s_T$ - symbols of the alphabet A, T - is the length of the word.

Definition 9. The set of all finite words S over a finite alphabet A is countable. Hence, each nonzero word can be assigned an index q.

$$S = \bigcup_{q=1}^{\infty} S^q, \; S^q = \prod_{t=1}^{T^q} s_t^q,$$

where T^q - is the number of symbols in the word S^q.

For each word $S^q = \left\{S_1^q \cdot \ldots \cdot S_{T^q}^q\right\}$ there is one inverse word S^{q^*}: $S^{q^*} = \left\{s_1^{q^*} = s_{T^q}^{q^*}; \ldots; s_{T^q}^{q^*} = s_1^q\right\}$.

Definition 10. The words S^q и S^k - are synonymous if $s_1^q \cdot \ldots \cdot s_{T^q}^q \cdot s_1^{k^*} \cdot \ldots \cdot s_{T^{k^*}}^{k^*} = E$.

Definition 11. Among identical synonyms we can distinguish a finite set of minimal words in length T_{min} from the node $g = v'$ in "truth" E $(v^{-1} = E)$: $g = v^{-1} v' = s_1 \ldots s_T$ - (**synonym**(g)).

Definition 12. Among the words from (**synonym**(g)) there is a "root synonym" with a minimal form of LG-order based on the order of elements in A.

A root synonym is a word derived from a numerical code by the "bubble" sort algorithm [20] when moving toward the "truth" E. A different way of obtaining root synonyms is presented in Table 3. It proceeds from the method of sequentially destroying inversions followed by transforming a sequence of synonyms from the current synonym to the root synonym using group-forming equations (see Definition 13).

Each permutation g has exactly one root synonym of length $T(g)$, coinciding with dynamically determined error rate and total number of inversions in the table of inversions - Σ (Table 3). In our case for <4321> it is a word of 6 symbols "**abacba**".

Table 3: *Algorithms of building synonyms by different methods*

Method of sequential inversion reduction ("first" optimal in word length synonym)									Bubble sort algorithm (optimal word-length "root" synonym)								
Error level	Code	1	2	3	4		Σ		Error level	Code	1	2	3	4		Σ	
6	4321	**3**	2	1	0	**c**	***6***		6	4321	3	2	**1**	0	**a**	***6***	
5	4312	**2**	2	1	0	**b**	***5***		5	3421	3	**2**	0	0	**b**	***5***	
4	4132	**1**	2	1	0	**a**	***4***		4	3241	**3**	1	0	0	**a**	***4***	
3	1432	0	**2**	1	0	**c**	***3***		3	2341	**3**	0	0	0	**c**	***3***	
2	1423	0	**1**	1	0	**b**	***2***		2	2314	**2**	0	0	0	**b**	***2***	
1	1243	0	0	**1**	0	**c**	***1***		1	2134	**1**	0	0	0	**a**	***1***	
0	1234	0	0	0	0		***0***	**cbacbc**	0	1234	0	0	0	0		***0***	**abacba**

When the numerical code coincides, a network structure of the graph V is formed when the neighboring nodes are at a distance of one on the inversion level.

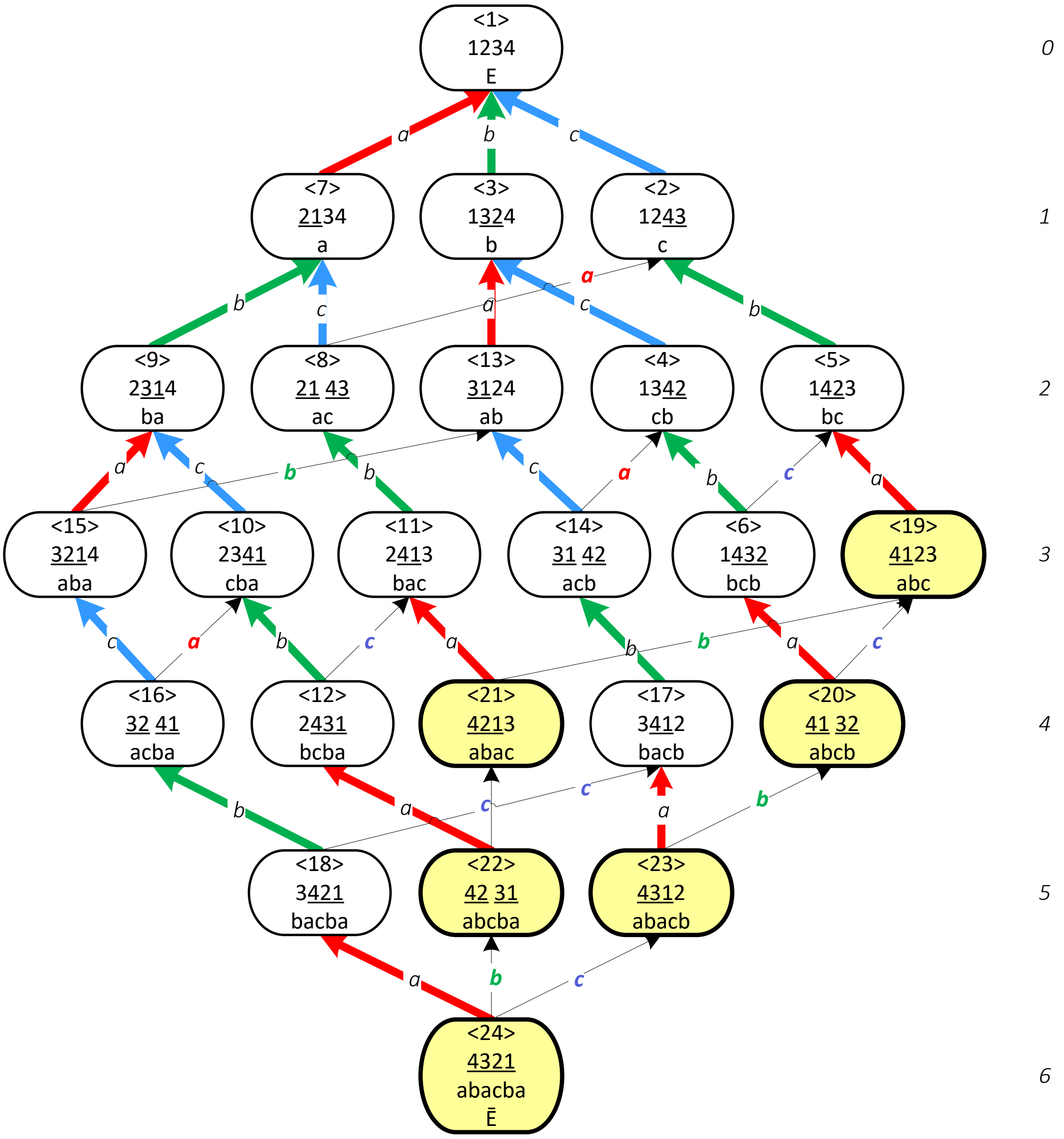


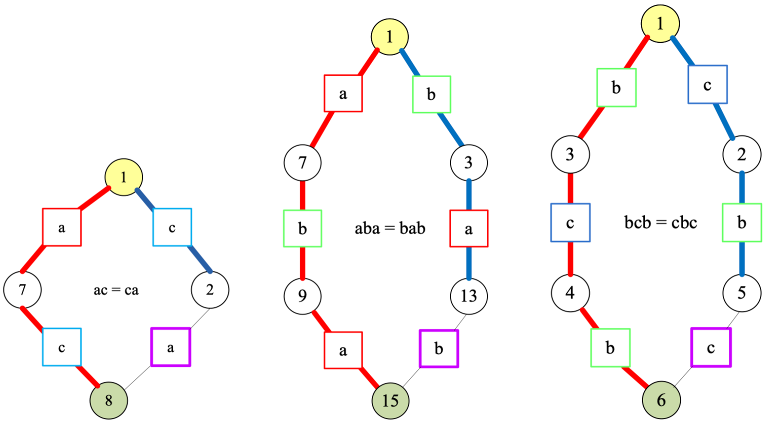
Figure 1: *Structure of the graph for* $N = 4$

Fig. 1 shows the structure of such a graph V for $N = 4$. The thick and thin edges represent the symbols s, in layers participating in generation of new neighboring nodes. The content of filling nodes is a representation of the permutation index g (column 1 in Table 1), its numerical code (column 2) and the content of the root word symbol for permutation g (column 3). The thick lines of the graph V in Fig. 1 correspond to its representation as a dictionary of root synonyms of permutation in the form of a spanning tree graph V. Their direction coincides with whether there is an inversion (up) or not (down) at the specified place. The six end nodes of the tree are represented by elements with yellow filling, such as {<19>; < 4123>; abc}. The underlined inversion, for <19> only (a) brings the next node closer to the "truth" E, reducing the number of errors by exactly one.

The dictionary of root synonyms is built according to the following principle: only those relations between permutations that are older in LG-order remain in cycles. For example, the cycle $<1><7><8><2><1>$ (a) we break by the connection $<2><8>$ since the connection $<1>$ $<7>$ (Fig. 2a) is smaller than connection $<1><2>$ (c).

A similar operation must be repeated for the lower sections of cycles <6>, <14>, <15>, <12>, <16>,

<20>, <21>, <18>, <22>, <23> and twice on <24> to break 12 more cycles and form the tree. For $N = 4$ analyzing the equalities describing the right and left branches of the cycles, we come to the necessity and sufficiency of six equalities: $= E,\ bb = E,\ cc = E,\ ac = ca\ (Fig. 2a),\ aba = bab,\ bcb = cbc,$ (Fig. 2b), which are necessary to construct synonyms of words in permutations (Table 1, columns 3, 4).



a) b)

Figure 2: *Illustration of breaking network cycles*

For $N \geq 5$ such enumeration of relations is difficult, so it is reasonable to introduce the notion of canonical formative equations (CFE).

Definition 13. The canonical formative equations are a necessary and sufficient list of equations that fully specify the rules for constructing root and other synonyms through the alphabet A.

Thus, in Table 3 it is possible to perform conversions with the help of CFE: "cbacbc" = "cbabcb" = "cabacb" = "acbcab" = "abcbab" = "abcaba" = "abacba".

Table 4 shows regularities that take into account the parity laws of permutations and lozenge closures at the 2 error level.

The sign "*" in Table 4 marks the necessary inverse permutations of the 2nd level of inversions. For example, if the left-hand side of the equations "ac" sets the equivalent right-hand side of "ca" then you must replace "ca" with "ac", etc. By adding third-level inversion relations versions: $aba = bab$; $bcb = cbc$; $cdc = dcd$; $ded = ede$; …; $xyx = yxy$; $yzy = zyz$, we define a complete set of CFE.

Table 4: *Automatic construction of CFE at 1-2 error levels*

		The second argument								
		a	b	c	d	e	...	x	y	z
The first argument	a	*E*		**ac***	ad*	ae*	...	ax*	ay*	az*
	b		*E*		bd*	be*	...	bx*	by*	bz*
	c	**ca**		*E*		ce*	...	cx*	cy*	cz*
	d	da	db		*E*		...	dx*	dy*	dz*
	e	ea	eb	ec		*E*	...	ex*	ey*	ez*
	...	...	...	...	...	...	...	...	...	...
	x	xa	xb	xc	xd	xe	...	*E*		xz*
	y	ya	yb	yc	yd	ye	...		*E*	
	z	za	zb	zc	zd	ze	...	zx		*E*

3. Problem statement

Let us consider N comparison objects $O_1, \dots, O_k, \dots, O_N$ which indices are the first N members of the natural series $E_{POI} = \langle 1, \dots, k, \dots, N \rangle$ - correspond to the order of presentation of the objects for the expertise. In the examination of objects participate M experts $E_1, \dots, E_m, \dots, E_M$. Each of the experts E_m has his own idea of the order of objects $g_m = \langle g_{m,1}, \dots, g_{m,n}, \dots, g_{m,N} \rangle$ which indexes increase with decreasing of some quality of objects from the expert's point of view. The value $g_{m,1}$ corresponds to the index of object O_{k_1}, taking part in examination with maximal quality according to expert's opinion E_m, a $g_{m,N}$ - the worst-quality object with the index O_{k_N}:

$$\mathrm{G} = (g_{m,n})_{\substack{m=\overline{1,M} \\ n=\overline{1,N}}} = \begin{pmatrix} g_{1,1} & \cdots & g_{1,N} \\ \cdots & \ddots & \cdots \\ g_{M,1} & \cdots & g_{M,N} \end{pmatrix}.$$

Thereby g_m - it is a permutation of object ratings (POR), the argument of which is the order of $E_{POR} = \langle 1, \dots, n, \dots, N \rangle$.

Places $p_m = \langle p_{m,1}, \dots, p_{m,k}, \dots, p_{m,N} \rangle$ by values inverse to POR g_m $(p_m = g_m^{-1})$ are permutations of object indices (POI) with argument E_{POI}:

$$\mathrm{P} = (p_{m,n})_{\substack{m=\overline{1,M} \\ n=\overline{1,N}}} = \begin{pmatrix} p_{1,1} = g_{1,1}^{-1} & \cdots & p_{1,N} = g_{1,N}^{-1} \\ \cdots & \ddots & \cdots \\ p_{M,1} = g_{M,N}^{-1} & \cdots & p_{M,N} = g_{M,N}^{-1} \end{pmatrix}.$$

It is necessary to find the compression of all private POR rankings g_m $(m = 1, \dots, M)$ in the form of a POR $g_m^* = \langle g_1^*, \dots, g_N^* \rangle$ which would reduce the total inconsistency of expert evaluations $g_{m,n} \to g_m^*$ (based on the equality of all participants in the examination), measured in the inversions of the transitions from $g_{m,n}$ к g_m^*, that is

$$K^* = \min K(g) = \min_{g_m} \left(\sum_{m=1}^{M} K_m(\langle g_1, \dots, g_N \rangle) \right),$$

where $K_m(\langle g_1, \dots, g_N \rangle)$ - is the sum of inversions in the evaluations of the m expert, K^*- is the limiting measure of inconsistency of experts' opinions chapter.

Finding an optimum in permutations of object rankings is equivalent to finding an object index over p^*: $p^* = \langle p_1^*, \dots, p_N^* \rangle$, since $K(g_m^*) = K(p_m^*)$where $p^*=(g^*)^{-1}$ (the lengths of the reciprocal paths $(E \to g)$ are the same as the forward paths $(p = g^{-1} \to E)$ at any g) (Table 5).

Table 5: *Solution search table $P(g)$ with table of inversions $B(g, P)$*

POR	Arg E	1	…	n	…	N	POI	Arg E	1	…	k	…	N	Criterion inconsistencies
POR	Func g	g_1	…	g_n	…	g_N	POI	Func p	g_1^{-1}	…	g_k^{-1}	…	g_N^{-1}	
POI	Arg E	1	…	k	…	N	POI	Arg E	1	…	k	…	N	
1	$p_1(g)$	p_{1,g_1}	…	p_{1,g_k}	…	p_{1,g_N}		$B_1(p_1(g))$	$B_{1,1}$	…	$B_{1,k}$	…	$B_{1,N}$	$K_1(g) = \sum_{k=1}^{N} B_{1,k}$
…	…	…	…	…	…	…			…	…	…	…	…	…
m	$p_m(g)$	p_{m,g_1}	…	p_{m,g_k}	…	p_{m,g_N}		$B_m(p_m(g))$	$B_{1,1}$	…	$B_{m,k}$	…	$B_{m,N}$	$K_m(g) = \sum_{k=1}^{N} B_{m,k}$
…	…	…	…	…	…	…			…	…	…	…	…	…
M	$p_M(g)$	p_{M,g_1}	…	p_{M,g_k}	…	p_{M,g_N}		$B_M(p_M(g))$	$B_{M,1}$	…	$B_{M,k}$	…	$B_{M,N}$	$K_M(g) = \sum_{k=1}^{N} B_{M,k}$
														$K(g) = \sum_{m=1}^{M} K_m(g)$

4. Method description

This problem belongs to the class of integer programming problems (on the structure as a graph - $V(G, G \times G)$ POR graph, arranged by error levels). Methods for solving such problems are well developed [21, 22], but none of them guarantees that, starting with some permutation, we will certainly get into a global minimum, which may not be the only one. At the very least, what can be guaranteed is a complete search of all POR. This option is possible for $N \leq 10$. For each g counts $K(P_m, g)$ and the sum of $K(g)$, and the current state of the set of global minima is "memorized".

A subset of $g \in G$ for which $K(g) = K^*$, we call the set of global minima - G^K. Since M is odd, it, like the set of local minima, consists of isolated solutions (permutations).

Let us consider a pair of $(l,\ l+1)$ columns in $P(g)$. $l = 1, \dots, N-1$ corresponds to the symbol s_l of the alphabet A (Table 6).

Table 6: *Neighboring Pair Table* $P(g)$

Expert	POR g	g_l	g_{l+1}
		l	$l+1$
1	$P_1(g)$	$P_{1,g_l}(E)$	$P_{1,g_{l+1}}(E)$
…		…	…
m	$P_m(g)$	$P_{m,g_l}(E)$	$P_{m,g_{l+1}}(E)$
…		…	…
M	$P_M(g)$	$P_{M,g_l}(E)$	$P_{M,g_{l+1}}(E)$

<u>Rule 1</u>. If $P_m\, g_l(E) < P_m\, g_{l+1}(E)$, then the sum of inversions $K(g_m)$ is increased by 1, and if $P_m\, g_l(E) > P_m\, g_{l+1}(E)$, the sum of inversions decreases by 1.

<u>Rule 2</u>. The decrease and increase of the sum depend on the number of rows in which the second condition M^2 (Rule1) dominates the first condition M^1. The ratio is $M^1 + M^2 = M$. Then the sum $K(g)$ from the influence of s_l will decrease by exactly $M^2 - M^1$ units ($if\ M^2 > M/2$) or increase by $M^1 - M^2$ units ($if\ M^2 < M/2$).

<u>Rule 3</u>. "Cutoff condition." The POR g belongs to the set of local minima G^P if for all $j = 1, \dots,$ $N-1$ the sum of errors only increases with rotation of neighboring columns by the symbol s_j. That is $g \in G^P$ has neighboring nodes of the graph V, exceeding by sum the found local optimum g by at least one.

The search G^P makes sense with large N, but with small N it is also effective, since the decrease (increase) of some selected pair does not depend on the place where the pair is located, but only on the contents of the resulting inversions.

Depending on the number of compared objects (N) two variants of the range are possible.

<u>Variant 1</u>. "Direct calculation". At small N ($N \leq 6$) it is possible to create a "directory" in LG-order. Then to build $G^P$and G^K it is necessary to exclude from it the POR where the "cut-off condition" is not satisfied. Calculate for all $g \in G^P$ value $K(g)$ and choose the optimal one.

Let us explain the above on the example for $N = 4$ (Table 7).

Table 7: *Initial data* $P(E)$ *and optimality criterion calculation* $K(E)$

	POI E	1	2	3	4	Inversion Tables				$K_m(E)$
		O_1	O_2	O_3	O_4	1→1	2→2	3→3	4→4	
E_1	P_1	1	4	2	3	0	1	1	0	2
E_2	P_2	2	3	1	4	2	0	0	0	2
E_3	P_3	3	2	1	4	2	1	0	0	3
E_4	P_4	4	2	3	1	3	1	1	0	4
E_5	P_5	1	4	3	2	0	2	1	0	3
E_6	P_6	2	4	1	3	2	0	1	0	3
E_7	P_7	2	1	4	3	1	0	1	0	2
						Optimality criterion $K(E)$:				20

Let us create a matrix of full pairwise comparisons of columns for the POI $P(E)$ for $(i = \overline{1, N};\ j = \overline{1, N};\ i \neq j)$ (Table 8).

Table 8: *Results of counting inversions on a pair of columns*

j / i	1	2	3	4
1	x	3	4	1
2	4	x	5	4
3	3	2	x	3
4	6	3	4	x

A fragment of the calculation for $(i = 1;\ j = 2, 3, 4)$ is given in Table 9.

Table 9: *Checking the "cutoff condition" (sum of column inversions* $(i = 1)$

1	→	2		1	→	3		1	→	4
1		4		1		2		1		3
2		3		2	1	1		2		4
3	1	2		3	1	1		3		4
4	1	2		4	1	3		4	1	1
1		4		1		3		1		2
2		4		2	1	1		2		3
2	1	1		2		4		2		3
	3				4				1	

Table 8 shows that the "cutoff condition" is not satisfied by 6 pairs of columns: 1→3, 2→1, 2→3, 2→4, 4→1, 4→3. As you can see, for the $N = 4$ directory g (Table 1) will contain 24 POIs. At the level of inversions (a) from 24 indexes values 12 elements with indexes (3-4, 7-12, 19-20, 23-24) will be discarded (due to "wrong pairs"), at the level (b) - 7 POI with indexes (1, 6, 15-17, 21-22), at the level (c) - 4 POI with indexes (2, 5, 13, 18). As a result, G^P and G^K consist of one POI with the index $^{14}g = g^* = \langle 3\ 1\ 4\ 2\rangle$ for which we will further calculate the value of the optimal criterion (Table 10).

Table 10. *POR optimum* g^* *and calculation of the optimality criterion* $K(g^*)$

	POR g^*	3	1	4	2	Inversion Tables				$K_m(g^*)$
		O_3	O_1	O_4	O_2	1→1	2→2	3→3	4→4	
E_1	P_1	2	1	3	4	1	0	0	0	1
E_2	P_2	1	2	4	3	0	0	1	0	1
E_3	P_3	1	3	4	2	0	2	0	0	2
E_4	P_4	3	4	1	2	2	2	0	0	4
E_5	P_5	3	1	2	4	1	1	0	0	2
E_6	P_6	**1**	**2**	**3**	**4**	0	0	0	0	**0**
E_7	P_7	4	2	3	1	3	1	1	0	5

Optimality criterion $K(g^*)$: 15

As can be seen from Table 10, expert E6 "guessed" the optimal solution $K_6(g^*) = 0$. Experts E1 and E2 made only one error each, E3 and E5 made two errors each, and E4 and E7 made too many errors. The next step is to use the POR g^* to reconstruct the optimal POI $p^* = (g^*)^{-1}$. Consequently, the required places $p^*(E) = \langle 2, 4, 1, 3\rangle$.

Variant 2. "Iterations." In general, you can use a cutoff rule directly starting with some starting POI, e.g. from $^{0}\boldsymbol{g} = E_{POI}$. The complete absence of cutoff guarantees that the local minimum is found in the $^{3}\boldsymbol{g} = \boldsymbol{bac}$ (Table 11).

Table 11.

1	2	3	4	5	6	7	8	9	10
	g	$^{0}\boldsymbol{g}$ =E	$K_m(^{0}\boldsymbol{g})$	$^{1}\boldsymbol{g} = \boldsymbol{b}$	$K_m(^{1}\boldsymbol{g})$	$^{2}\boldsymbol{g} = \boldsymbol{ba}$	$K_m(^{2}\boldsymbol{g})$	$^{3}\boldsymbol{g} = \boldsymbol{bac}$	$K_m(^{3}\boldsymbol{g})$
		⟨1234⟩		⟨1324⟩		⟨3124⟩		**⟨3142⟩**	
E_1	P_1	1423	2	1243	1	2143	2	2134	1
E_2	P_2	2314	2	2134	1	1234	0	1243	1
E_3	P_3	3214	3	3124	2	1324	1	1342	2
E_4	P_4	4231	5	4321	6	3421	5	3412	4
E_5	P_5	1432	3	1342	2	3142	3	3124	2
E_6	P_6	2413	3	2143	2	1243	1	1234	0
E_7	P_7	2143	2	2413	3	4213	4	4231	5
		a=3; **b=5**; c=3	20	**a=4**; b=2; **c=4**	17	a=3; b=3; **c=4**	16	a=3; b=1; c=3	15

The presence at the end of iteration (a=4; b=2; c=4) of ambiguity of choice makes us return to the beginning of this stage and consider another alternative $^{2+}\boldsymbol{g} = \boldsymbol{bc}$ c $K(^{2+}\boldsymbol{g})$=16 and conclude that the search is terminated because $^{3+}\boldsymbol{g} = \boldsymbol{bca}$ c $K(^{3+}\boldsymbol{g})$=15 is a copy of ⟨3142⟩ by CFE.

5. Concluding remarks

It is beyond the scope of this article to compare the proposed method with other methods of information compression (e.g., factor analysis, the averaging method, or the Schulze method), which will be discussed later.

The further development of this method implies its application in ranking determinations that allow equality of evaluations of compared objects when determining the weights of compared objects (similar to pairwise comparisons in the method of hierarchy analysis [12] and solving problems of heterogeneous scales merging [17, 18]).

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