ANALYSIS OF THE CONTROLLED SECURITY MODEL WITH UNCOUNTABLE NUMBER OF LINEAR LIMITATIONS ON MANAGEMENT STRATEGIES

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Abstract

The work analyzes the security model described by the controlled semi-Markov process with catastrophes. Management optimization is associated with determining the frequency of restoration work of the subsystem (security subsystem) which acts up attempts of malicious persons to disrupt the normal operation of the main system. The optimization criterion is the mathematical expectation of the time before the catastrophe (the moment of the first successful attempt to disrupt the normal operation of the main system). In the context of new linear limitations on management strategies, the structure of the optimal strategy was examined.

Keywords: controlled semi-Markov process, safety, management, strategy, randomization, optimization problem.

1. Introduction. Description Of Structural Elements Of Security Model Under Examination

Before we set a mathematical problem, let us describe a real physical (practical) situation, the mathematical model of which will be the object of examination of the present work.

Let us suppose that some system S is operating, performing important, responsible and necessary work. Let us suppose that there is another system S_0 (opponent, malicious person) which tries to disrupt the high-quality performance of the work of the original system S. Finally, there is a security system S_1 which must act up attempts to disrupt the uninterrupted operation of the original system and ensure its safe operation.

Regarding the "functioning" of the system S_0 we assume that the moments of attempts (attacks) to disrupt the operation of the main system S form a discrete set on the time axis. Regarding the operation of the system S_1 we assume that it can break and restore (the exact description of this process will be given below). Therefore, periods of good operation interchange with periods when the security system cannot perform its protection functions. For this reason one of the main hypothesis is formulated as follows: *if the moment of the attempt to disrupt the operation of the system S falls during the period of good operation, then this attempt is acted up, if this moment falls during the period of broken operation and restoration, then an undesired event (catastrophe) occurs*. If it is possible to reduce the periods of broken operation and restoration of the security system, that is, to manage the operation process, then by linking the moment of catastrophe with this control process, it is possible to set a mathematical task of finding the optimal management strategy. This will be done below.

2. Mathematical Hypotheses

1. With regard to system S, there is no need to introduce any mathematical hypotheses, since management is related only to the security system S_0, and the moment of catastrophe depends on the state of the system S_1 and the characteristics of the system S_0.
2. Regarding the system S_0, we assume that the flow of attempts (attacks) in time is described by the stationary Poisson process with the parameter λ. In the Poisson process the intervals η between adjoined attacks are independent collectively and distributed according to exponential law with the parameter λ

$$P\{\eta < x\} = \begin{cases} 0, & x \leq 0, \\ 1 - e^{-\lambda x}, & x > 0. \end{cases} \quad (1)$$

The Poisson process is the Markov process. For the arbitrary moment t, the waiting time until the next attempt (attack) is also distributed according to the exponential law (1) with the same parameter. This property will be used when deriving basic relations.

3. The system S_1 breaks and the uptime ξ is distributed according to the law

$$F(x) = P\{\xi < x\} = \begin{cases} 0, & x \leq 0, \\ 1 - e^{-\mu x}, & x > 0, \end{cases} \quad \bar{F}(x) = 1 - F(x) = P\{\xi \geq x\} = e^{-\mu x}, \quad x > 0.$$

4. Let us suppose that the breakdown that appeared during the operation of the system is not detected (does not appear) by itself, and if we mark random time of the independent manifestation of the breakdown as ζ, then its distribution is equal to $(x) = P\{\zeta < x\} = 0, x < \infty$, that is, the random time of the independent manifestation of the breakdown with the probability of one is equal to infinity.

Further, we will describe the maintenance process (control process). At the beginning moment
$t_0 = 0$, the operation of the security system begins and a planned preventive update (prevention) of the system is assigned through the time $\upsilon \geq 0$ distributed according to the law $G(u) = P\{\upsilon < u\}, \quad G(0) = 0$.

If the security system is not broken by the appointed moment $\upsilon \geq 0$, that is the event $\{\upsilon < \xi\}$ occurred, then at the moment of $\upsilon \geq 0$ a planned preventive update of the system will begin which according to the hypothesis updates the system completely. Let us mark the duration of this planned preventive (prevention) update γ_1, and the distribution function of this time length we mark $\bar{F}_1(x) = P\{\gamma_1 \geq x\}$.

If the system breaks by the assigned moment $\upsilon \geq 0$ (the event $\{\upsilon \geq \xi\}$ occurred), but its breakdown was not detected by itself (the event $\{\upsilon < \xi + \zeta\}$ occurred), then the planned emergency system update will begin at the assigned moment $\upsilon \geq 0$. The duration of this restoration work we will mark γ_2, and the distribution law we will mark $F_2(x) = P\{\gamma_2 < x\}, \bar{F}_2(x) = P\{\gamma_2 \geq x\}$.

In the future for mathematical expectations entered above the restoration times $\gamma_i, \quad i = 1,2$ we will use the symbols

$$T_i = \int_0^\infty \bar{F}_i(x)dx, \quad i = 1,2. \quad (2)$$

After the possible restoration works, when the security system is fully updated according to the hypothesis, the moment of the next preventive restoration work is rescheduled regardless of the past run of the process and the entire maintenance process is repeated again.

It is easy to see that the evolution of the security system S_1 is described with the semi-Markov process $\xi(t)$, in which Markov moments are the moments of the beginning and end of restoration work. When determining a set of states let us note that at the start of restoration work it is known which restoration work begins. The duration of restoration work depends only on its type. Therefore, at this moment, the past run of the process and the future behavior of the process are independent if it is known which restoration work begins.

At the end of restoration work, the system by definition is a new one and the moment of eventual preventive prevention is rescheduled regardless of the past. Therefore, even at this moment of the end of restoration, the past run of the process and the future behavior of the process are independent.

Let us consider $\xi(t) = 1$, if at the nearest Markov moment, preceding moment t, the planned preventive prevention of the system begins. Let us consider $\xi(t) = 2$, if at the nearest Markov moment, preceding t, the planned emergency restoration of the system begins. Finally, we assume $\xi(t) = 0$ if the closest Markov moment, preceding t, is the moment the security system is updated. It should be noted here that part of the time in this state the security system spends in a broken state (hidden breakdown), and the breakdown does not appear itself, and that is why the model provides for periodic determination of the state of the security system.

5. In accordance with the basic hypothesis of the occurrence of catastrophes, formulated above, it is now possible to describe the effect of this process on the evolution of the security system S_1.

Let us enter into consideration the absorption state, in which the process $\xi(t)$ passes at the moment of the first unreflected attack. Such an event will occur when the attack appears on the restoration period (states 1 or 2) or on the latent breakdown period (state 0). At this moment t we consider that $\xi(t) = 3$.

Therefore, the process $\xi(t)$ characterizing the evolution of the security system and taking values from the finite set $E = \{0,1,2,3\}$ is described. Figure 1 shows the transition graph of the process $\xi(t)$.

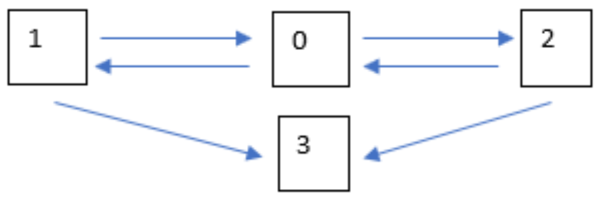

Figure 1: *Transition graph of the process* $\xi(t)$

3. Determination Of Process Characteristics. Setting The Mathematical Task

The random process $\xi(t)$ described above is a homogeneous controlled semi-Markov process with a finite set of states $E = \{0,1,2,3\}$, management is realized only in state 0 and is contained in choosing the period through which the restoration work should begin. Therefore, trajectories are step functions and equality $U = 0, +\infty)$ is true for the management (decision) space U in state 0. The management strategy is determined by the distribution of $G(u) = P\{v < u\}$, $G(0) = 0$, that is, it belongs to a set of Markov homogeneous randomize strategies.

More details on the construction, characteristics and properties of homogeneous controlled semi-Markov processes with a finite set of states can be found in [1].

A homogeneous controlled semi-Markov process $X(t) = (\xi(t), u(t))$ is defined as a two-dimensional step process, the first component of which $\xi(t)$ describes the evolution of the controlled component, and the second one $u(t)$ describes the control process. One of the main characteristics of the homogeneous controlled semi-Markov process is the semi-Markov kernel $Q_{ij}(t, u)$, $i, j \in E$, $t \in 0, +\infty)$, $u \in U$ which is defined as the conditional probability that the following condition of the first component will be the condition j and transition to this state will happen until moment t under as long as the previous condition is i and the decision u is made.

Construction of a semi-Markov kernel.

From the description of the system functioning it follows that if $i = 1,2$, the following equalities are true

$$Q_{i0}(t) = Q_{i0}(t,u) = P\{\gamma_i < t, \eta > \gamma_i\} = \int_0^t e^{-\lambda x} dF_i(x),$$
$$Q_{ij}(t,u) = 0, \quad j = 1,2,$$
$$Q_{i3}(t) = Q_{i3}(t,u) = P\{\eta < t, \eta < \gamma_i\} = \int_0^t \lambda e^{-\lambda x} \bar{F}_i(x) dx \tag{3}$$

The last equalities in (3) are true, since the Markov moments of the beginning and end of restoration work interchange. In addition, to go to zero state, it is necessary and enough that the carried out restoration work ends before moment t and there are no attacks during the restoration work. To go to state 3, it is necessary and enough that the attack takes place before t, and the carried out restoration work before that moment does not end.

It should be noted that these functions are independent of management, since in states $i = 1,2$, management is not realized.

If $i = 0$, the following equalities are true:

$$Q_{00}(t,u) = 0, \quad Q_{01}(t,u) = \begin{cases} 0, & u > t, \\ e^{-\mu u}, & u < t, \end{cases}$$
$$Q_{02}(t,u) = \begin{cases} 0, & u > t, \\ \frac{\mu}{\mu-\lambda}\left(e^{-\lambda u} - e^{-\mu u}\right), & u < t, \end{cases} \quad Q_{03}(t,u) = \begin{cases} 1 - \frac{\mu}{\mu-\lambda} e^{-\lambda t} + \frac{\lambda}{\mu-\lambda} e^{-\mu t}, & u > t, \\ 1 - \frac{\mu}{\mu-\lambda} e^{-\lambda u} + \frac{\lambda}{\mu-\lambda} e^{-\mu u}, & u < t, \end{cases} \tag{4}$$

With the limit transition if $t \to \infty$, we get the transition probabilities of the states of the imbedded Markov chain $p_{ij} = lim_{t\to\infty} \int_0^\infty Q_{ij}(t,u) dG(u)$, $\quad i,j \in E = \{0,1,2,3\}$.

For the model under examination, we have from equalities (3) and (4)

$$p_{00} = 0, \quad p_{01} = \int_0^\infty e^{-\mu u} dG(u), \quad p_{02} = \int_0^\infty \frac{\mu}{\mu - \lambda}\left(e^{-\lambda u} - e^{-\mu u}\right) dG(u)$$
$$p_{03} = \int_0^\infty \left(1 - \frac{\mu}{\mu - \lambda} e^{-\lambda u} + \frac{\lambda}{\mu - \lambda} e^{-\mu u}\right) dG(u)$$
$$p_{i0} = \int_0^\infty e^{-\lambda x} dF_i(x), p_{ij} = 0, \quad j = 1,2, p_{i3} = \int_0^\infty \lambda e^{-\lambda x} \bar{F}_i(x) dx, i = 1,2. \tag{5}$$

It is not difficult to test the obvious equality if *i=0,1,2*

$$lim_{t\to\infty} \sum_{j\in E} Q_{ij}(t) = \sum_{j\in E} p_{ij} = 1.$$

Now we will formulate a mathematical task. When describing the security model, it was noted that attacks are reflected, when the moments of attacks fall on the periods of good operation of the security system, and the moment of the first unreflected attack is determined as a catastrophe moment. The security system will not be able to defeat the attack if it is in restoration states $i = 1,2$. In addition, and in the state $i = 0$ part of the period, the security system can be in the state of latent breakdown and skip the attack. Therefore, the increase in the safety of operation of the main system S or the increase in the efficiency of operation of the security system S_1 will be associated with the numerical characteristic of a random moment of catastrophe - the mathematical expectation of the time before the catastrophe. This expectation depends on the initial characteristics, in particular, on the distribution function $G(u)$, which determines the periodicity of planned restoration work. In the future, we will say that the distribution $G(u)$ determines the management strategy. It should be noted here that in the model under examination, the management is realized only in state $i = 0$ and depends only on the state at the moment of decision-making. Therefore, in this case, the strategy has the property of Markov, homogeneity (there is no dependence on calendar time). In addition, the model considers a class of randomize strategies, since a random experiment is used when choosing a solution (the implementation of a random coefficient having a distribution $G(u)$ determines the solution). The set of distribution definition $G(u)$ we will call the management space $R^+ = (0, \infty)$.

Thus, we come to the following mathematical tasks:

- to calculate the dependence of the mathematical expectation of the time before the catastrophe $M(\tau/\xi(0) = 0) = M_0(G)$ from the distribution $G(u)$ determining the management strategies (here τ is the time before the catastrophe and it is assumed that the security system is good at zero moment);

- to find the maximum functional $M_0(G)$ by the set of *permissible* management strategies and determine the strategy on which this extremum is achieved.

Further, let us be clear the concept of permissible strategies (distributions) introduced above.

Let us mark the functional $M(\tau/\xi(0)=i)=M_i(G)=M_i$ and Ω_1, $G\in\Omega_1$, the set of distributions for which these functionals exist. Next, we will introduce two distribution functions.

$$0\le G_1(t)\le G_2(t)\le 1, t\ge 0, G_1(t)=G_2(t)=0, t\le 0,$$

and define a set of distributions

$$\Omega_2=\{0\le G_1(t)\le G(t)\le G_2(t)\le 1\} \tag{6}$$

Then the set of permissible distributions by the equality

$$\Omega=\Omega_1\cap\Omega_2 \tag{7}$$

Let us note that limitations (6) can be represented as limitations of the type of inequalities on linear functionals

$$G_1(t)\le\int_0^{+\infty}u(x,t)\,dG(x)\le G_2(t), u(x,t)=\begin{cases}1, x<t,\\0, x\ge t;\end{cases} t\in[0,+\infty).$$

Now, in the accepted notations, it is possible to formulate a mathematical task: *to determine the maximum $M(\tau/\xi(0)=i)=M_i(G)=M_i$ by the set of permissible distributions (7) and the distribution $G^{(0)}$ - the strategy on which this maximum is achieved, $\max_{G\in\Omega}M_0(G)=M_0(G^{(0)})$.*

4. Definition Of Target Functional Structure

Let us calculate the dependency of mathematical expectations of the time before the catastrophe. For conditional mathematical expectations $M_i=M(\tau/\xi(0)=i)$, $i=0,1,2$, we write out a system of algebraic equations using the formula of full mathematical expectation

$$M_0=\sum_{i=1}^{3}M_{0i}p_{0i}$$
$$M_i=M_{i0}p_{i0}+M_{i3}p_{i3},\quad i=1,2, \tag{8}$$

where M_{ij} means a conditional mathematical expectation of the time before the catastrophe if $\xi(0)=i$ and this component made the first transition to the state j. From the above description, there are the following equalities

$$M_{0i}=\frac{\int_0^\infty\int_0^\infty t d_tQ_{i1}(t,u)dG(u)}{p_{0i}}+M_i, i=1,2,$$
$$M_{i3}=\frac{\int_0^\infty t d_tQ_{i3}(t)}{p_{i3}},\quad i=0,1,2,\quad M_{i0}=\frac{\int_0^\infty t d_tQ_{i0}(t)}{p_{i0}}+M_0, i=1,2. \tag{9}$$

Substituting (9) into equations (8), we obtain a system of algebraic equations

$$M_0=\int_0^\infty\int_0^\infty t d_t\left(\sum_{i=1}^{3}Q_{0i}(t,u)\right)dG(u)+M_1p_{01}+M_2p_{02}$$

$$M_i=\int_0^\infty t d_tQ_{i0}(t)+\int_0^\infty t d_tQ_{i3}(t)+M_0p_{i0},\ i=1,2.$$

Solution of this system

$$M_0=\frac{\int_0^\infty\int_0^\infty t d_t(\sum_{i=1}^{3}Q_{0i}(t,u))dG(u)+\sum_{i=1,2}p_{0i}\sum_{j=0,3}\int^\infty t d_tQ_{ij}(t)}{1-p_{10}p_{01}-p_{20}p_{02}}$$

determines the functional under examination - the mathematical expectation of the time before the catastrophe. If we take into account the equations (2) and (4), we get

$$\sum_{i=1}^{3}Q_{0i}(t,u)=\begin{cases}1-\dfrac{\mu}{\mu-\lambda}e^{-\lambda t}+\dfrac{\lambda}{\mu-\lambda}e^{-\mu t},\quad u>t,\\1,\quad u<t,\end{cases}$$

$$\sum_{j=0,3} Q_{ij}(t) = 1 - \bar{F}_i(t)e^{-\lambda t}, i = 1,2.$$

Thus, taking into account (5), we finally obtain that $M_0 = M_0(G)$ is bilinear functional

$$M_0(G) = \frac{\int_0^{+\infty} A(u)dG(u)}{\int_0^{+\infty} B(u)dG(u)}, \tag{10}$$

in which the subintegral functions are determined by the equalities

$$A(u) = \frac{(\mu+\lambda)}{\lambda\mu} - \frac{1}{(\mu-\lambda)\lambda\mu}\left(\mu^2 e^{-\lambda u} - \lambda^2 e^{-\mu u}\right) +$$
$$+ e^{-\mu u}\int_0^{\infty} \bar{F}_1(t)e^{-\lambda t}dt + \frac{\mu}{\mu-\lambda}\left(e^{-\lambda u} - e^{-\mu u}\right)\int_0^{\infty} \bar{F}_2(t)e^{-\lambda t}dt$$
$$B(u) = 1 - e^{-\mu u}\int_0^{\infty} e^{-\lambda x}dF_1(x) - \frac{\mu}{\mu-\lambda}\left(e^{-\lambda u} - e^{-\mu u}\right)\int_0^{\infty} e^{-\lambda x}dF_2(x) \tag{11}$$

5. Solution Of Optimization Task. Optimal Management Strategy Construction

The above research proves that the task of constructing the optimal management strategy is now formulated as follows: *to determine the maximum of the bilinear functional (10), in which the subintegral functions are determined by equalities (11), by the set of distribution functions satisfying conditions (6) and (7), and the distribution $G^{(0)}$ where this maximum is reached*

$$M_0\left(G^{(0)}\right) = max_{G\in\Omega} M_0(G).$$

The basis for solving the task will be two statements:

- A theorem on distribution structure where the maximum linear functional over the set of distributions (7) is achieved, the proof of which is given in [2];
- A theorem on coincidence of sets of distributions where the maximum of bilinear functional is reached and the maximum of specially selected linear functional is reached, the proof of which is given in [3].

Further we will give the definitions of these statements.

THEOREM [2]. If there is the maximum of the linear functional $\mathrm{L(G)} = \int_0^{+\infty}\mathrm{C(u)dG(u)}$ with respect to the set (7) and the subintegral function $\mathrm{C(u)}$ has the maximum at point $0 \le t_1 \le +\infty$ and in the area $0 < t < t_1$ the subintegral function is non-decreasing, and in the area $t_1 < t < +\infty$ the subintegral function is non-increasing, then the maximum of the functional is achieved on the distribution

$$G^{(0)}(u) = \begin{cases} G_1(u), & 0 \le u \le u_1, \\ G_2(u), & u_1 < u \le +\infty, \end{cases} \tag{12}$$

that is

$$\max_{G\in\Omega} L(G) = L(G^{(0)}) =$$
$$= \int_0^{tu} \mathrm{C}(u)dG_1(u) + \mathrm{C}(u_1)\left[\lim_{u\to u_1+0} G_2(u) - G_1(u_1)\right] + \int_{u_1+0}^{+\infty}\mathrm{C}(u)dG_2(u) = C. \tag{13}$$

and the following equality is correct

$$\max_{u\in[0,\infty)}\left\{\int_0^{u}\mathrm{C}(x)dG_1(x) + \mathrm{C}(t)\left[\lim_{x\to t+0} G_2(x) - G_1(u)\right] + \int_{u+0}^{+\infty}\mathrm{C}(x)dG_2(x)\right\} = C. \tag{14}$$

LEMMA [3]. If there is the maximum of bilinear functional $\mathrm{M_0(G_0) = max_{G\in\Omega} M_0\,(G) = C}$ (10) in some set distributions Ω then

$$\{\Phi: M_0(\Phi) = C = max_{G\in\Omega} M_0\,(G)\} = \{\Phi: max_{G\in\Omega} L\,(G) = L(\Phi)\},$$

where the subintegral function of the linear functional $L(G) = \int_0^{+\infty} C(u)dG(u)$ is defined by the equality $C(u) = \mathrm{A}(u) - C\mathrm{B}(u)$.

The lemma condition about existence of the maximum of functional (10) means that the set $\{\Phi: M_0(\Phi) = C = max_{G\in\Omega} M_0\,(G)\}$ is not empty.

The converse statement is also true.

STATEMENT. If there is such a constant C for which

$$\left\{\Phi: max_{G\in\Omega}\int_0^{\infty}\left(A(u) - CB(u)\right)dG(u) = \int_0^{\infty}\left(A(u) - CB(u)\right)d\Phi(u)\right\} \ne \emptyset,$$

then there is the maximum of bilinear functional (10) and this maximum is C.

From the above statements follows

COROLLARY [2]. If there is the maximum of bilinear functional $M_0(G_0) = \max_{G\in\Omega} M_0(G) = C$ (10) in set distributions Ω, determined by correlations (6) and (7), and function $C(u) = A(u) - CB(u)$ has the maximum in the point $0 \le t_1 \le +\infty$ and in the area $0 < t < t_1$ subintegral function is non-decreasing, and in the area $t_1 < t < +\infty$ the subintegral function is non-increasing, then the maximum of this bilinear functional is reached on distribution (12) and

$$\max_{G\in\Omega} M_0(G) = M_0(G^{(0)}) =$$
$$= \max_{u\in[0,\infty)} \frac{\int_0^u A(x)dG_1(x) + A(u)\left[\lim_{t\to u+0} G_2(t) - G_1(u)\right] + \int_{u+0}^{+\infty} A(x)dG_2(u)}{\int_0^u B(x)dG_1(x) + B(u)\left[\lim_{t\to u+0} G_2(t) - G_1(u)\right] + \int_{u+0}^{+\infty} B(x)dG_2(x)} = C. \quad (15)$$

If you use these statements to find the maximum of the bilinear functional $M_0(G)$ *(10)*, in which the subintegral functions are determined by equations (11), then the task will result in examining the function $C(u) = A(u) - CB(u)$.

To simplify transformations let us enter symbols

$$\beta_i = \int_0^\infty \bar{F}_i(t)e^{-\lambda t}dt\,, \alpha_i = \int_0^\infty e^{-\lambda t}dF_i(t)\,, i = 1,2,$$
$$\alpha_i = -\int_0^\infty e^{-\lambda t}d\bar{F}_i(t) = 1 - \lambda\beta_i \ge 0.$$

Then, taking into account the equalities (11) and the accepted symbols for the function $C(u)$, we have

$$C(u) = A(u) - CB(u) =$$
$$= \frac{(\mu+\lambda)}{\lambda\mu} - \frac{1}{\mu\lambda(\mu-\lambda)}\left(\mu^2 e^{-\lambda u} - \lambda^2 e^{-\mu u}\right) + e^{-\mu u}\beta_1 + \frac{\mu}{\mu-\lambda}\left(e^{-\lambda u} - e^{-\mu u}\right)\beta_2 -$$
$$- C\left(1 - e^{-\mu u}(1-\lambda\beta_1) - \frac{\mu}{\mu-\lambda}\left(e^{-\lambda u} - e^{-\mu u}\right)(1-\lambda\beta_2)\right),$$

$$C(0) = \beta_1(1 - C\lambda) \le 0, C(\infty) = \frac{(\mu+\lambda)}{\lambda\mu} - C \quad (16)$$

The latter inequality follows from the obvious correlations

$$\beta_1 > 0, C \ge \frac{1}{\lambda}.$$

For the derivative of the function $C(u)$ after transformations, we have the equality

$$\frac{dC(u)}{du} == \frac{1}{(\mu-\lambda)}\left(\mu e^{-\lambda u} - \lambda e^{-\mu u}\right) - \mu e^{-\mu u}\beta_1 - \frac{\mu}{\mu-\lambda}\left(\lambda e^{-\lambda u} - \mu e^{-\mu u}\right)\beta_2 -$$
$$- C\left(\frac{\mu^2}{\mu-\lambda}(\beta_2-\beta_1)(1-\lambda C)(1-\lambda\beta_1) + \frac{\mu}{\mu-\lambda}\left(\lambda e^{-\lambda u} - \mu e^{-\mu u}\right)(1-\lambda\beta_2)\right) =$$
$$= e^{-\mu u}\left[\frac{1}{(\mu-\lambda)}(C\mu\lambda - 1) + \frac{\lambda}{\mu-\lambda}\beta_1(1-\lambda C) + \frac{\mu^2}{\mu-\lambda}(\beta_2-\beta_1)(1-\lambda C)\right] +$$
$$+ e^{-\lambda u}\left[\frac{\mu}{(\mu-\lambda)}(1-\lambda\beta_2)(1-\lambda C)\right].$$

Now the derivative under examination can be represented as the product of two functions

$$\frac{dC(u)}{du} = \Psi_1(u)\Psi_2(u)$$
$$\Psi_1(u) = e^{-\mu u}, \Psi_2(u) = C_0 + C_1,$$
$$C_0 = \left[\frac{1}{(\mu-\lambda)}(C\mu\lambda - 1) + \frac{\lambda}{\mu-\lambda}\beta_1(1-\lambda C) + \frac{\mu^2}{\mu-\lambda}(\beta_2-\beta_1)(1-\lambda C)\right]$$
$$C_1 = \frac{\mu}{(\mu-\lambda)}(1-\lambda\beta_2)(1-\lambda C)$$

The first function $\Psi_1(u)$ is positive and the second function $\Psi_2(u)$ is decreasing since

$$\frac{d\Psi_2(u)}{du} = e^{(\mu-\lambda)u}\mu(1-\lambda\beta_2)(1-\lambda C) \le 0.$$

Thus, it has been proved that the derivative of the function (16) changes the sign from plus to minus up to one time, that is, the function (16) is either monotonic in the area $R^+ = 0, \infty)$ or has the maximum on area boundaries and fully satisfies the conditions of the above statements.

Therefore, the maximum of the functional (10), in which the subintegral functions are determined by the equalities (11), over the set distributions satisfying the inequalities $\{G: 0 \leq G_1(u) \leq G(u) \leq G_2(u) \leq 1, 0 \leq u \leq +\infty\}$ (6), is achieved on the distribution

$$G^{(0)}(u) = \begin{cases} G_1(u), & 0 \leq u \leq u_1, \\ G_2(u), & u_1 < u \leq +\infty, \end{cases}$$

moreover, the parameter u_1 is defined as maximum point of function 15. We remind that functions A(u) and B(u) are defined by equalities (11).

6. Conclusion

The conducted researches allowed to bring the task of functional analysis - the search for the extremum of bilinear functional by the set distribution functions to the task of mathematical analysis - the search for the maximum of some function by the set of real numbers. This essentially simplifies the practical use of the results set forth.

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