# A new continuous probability model based on a trigonometric function: Theory and applications

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**Abstract**

*In this manuscript, we highlight a new probability distribution based on a trigonometric function, obtained by specializing the Sine-G family of distributions with exponentiated exponential distribution. The proposed distribution is quite flexible in terms of density and hazard rate functions. Several mathematical properties of the proposed distribution are also explored. For applicability of proposed distribution, two real data sets are scrutinized and it is sensed that proposed distribution leads to a better fit than all other models taken under consideration.*

**Keywords:** Sine-G; Exopentiated exponential distribution; Hazard rate function; Order statistic; Simulation study; Maximum likelihood estimation.

## 1. Introduction

In distribution theory proposing new family of distributions by incorporating an extra parameter is most common among researchers. The purpose of incorporating a new parameter is to increasing the data fitting strength of the proposed probability models. Although researchers are quite successful in doing so, however they are little concerned about the over parameterization

and complexities that arise due to addition of new parameters. To know more about such families one can go through [11],[9],[6],[3],[8], [18], [17] and [1]. So keeping in view the limitations of having extra parameters, some researchers have come up with family of distributions that not only can be used to model the complex data structures, but also are void of extra parameters. Among them lets recall [10],[12],[13],[15] and [16]. After getting motivated by aforementioned work on probability models having parsimony in parameters we have introduced a new model based on Sine-G (SG) family of distribution proposed by [15]. The introduced model, namely Sine-G exponentiated exponential (SGEE) distribution has been obtained by taking the baseline distribution as exponentiated exponential distribution. The SGEE distribution has same number of parameters as baseline distribution and has greater flexibility than some well-known two parameteric probability distributions including the baseline distribution.
The rest of the manuscript is presented as, In Section 2 a brief introduction about Sine-G family of distributions is given. In Section 3, a member of the SG family namely, SGEE distribution is examined in detail and its general properties are studied including, quantile, moments, moment generating functon, order statistic etc. The maximum likelihood estimation and simulation study are discussed in Section 4. In Section 5, to check the applicability of SGEE two data sets have been scrutinized. Finally, the paper ends in Section 6 with concluding remarks.

## 2. Sine-G (SG) Family Of Disrtibutions

Let $G(y)$ be the baseline commulative distribution function (CDF) of any random variable Y. Then the CDF, $F(y)$ of the sine-G family of distributions proposed by [15] is given by

$$F(y) = \sin\left(\frac{\pi}{2}G(y)\right) \quad ; y \in \mathbb{R},$$

The corresponding probability density function( PDF) is given by

$$f(y) = \frac{\pi}{2}g(y)\cos\left(\frac{\pi}{2}G(y)\right) \quad ; y \in \mathbb{R},$$

The survival function $S(y)$ for SG is given by

$$\begin{aligned} S(y) =& 1 - \sin\left(\frac{\pi}{2}G(y)\right) \quad ; y \in \mathbb{R} \\ =& \left(\sin\left(\frac{\pi}{4}G(y)\right) - \cos\left(\frac{\pi}{4}G(y)\right)\right)^2 \end{aligned}$$

The hazard rate function $\lambda(y)$ is given by

$$\lambda(y) = \frac{\frac{\pi}{2}g(y)\cos\left(\frac{\pi}{2}G(y)\right)}{\left(\sin\left(\frac{\pi}{4}G(y)\right) - \cos\left(\frac{\pi}{4}G(y)\right)\right)^2} \tag{1}$$

## 3. Sine-G Exponentiated Exponential (SGEE) Distribution And Its Properties

Suppose the random variable Y has exponentiated exponential distribution with CDF $G(y) = (1 - e^{-\theta y})^{\alpha}$; $y, \alpha, \theta > 0$ then the CDF of the SGEE distribution is given by

$$F(y) = \sin\left(\frac{\pi}{2}(1 - e^{-\theta y})^{\alpha}\right) \quad ; \quad \alpha, \theta, y > 0$$

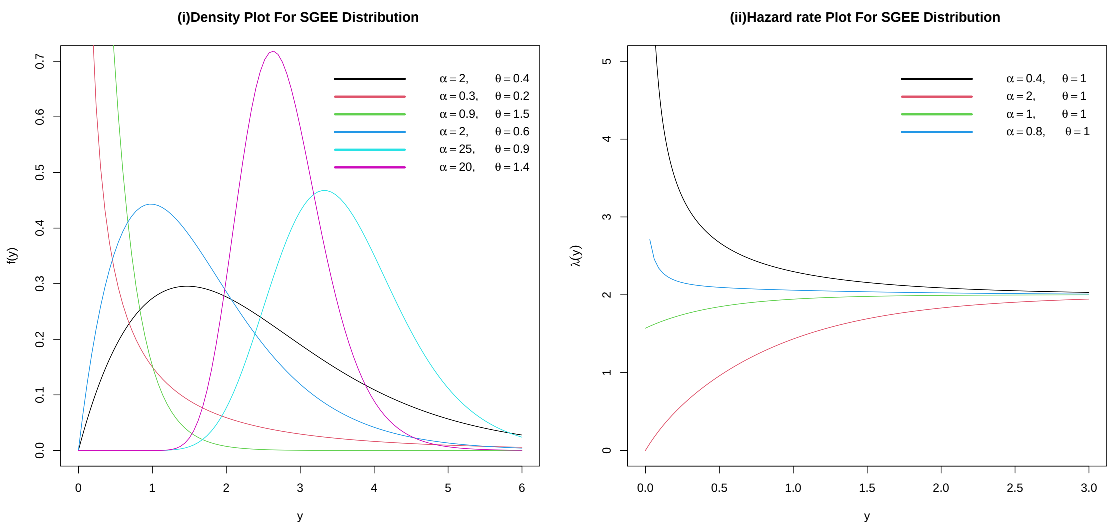


**Figure 1:** *Plots of the SGEE density and hazard function for different values of $\alpha$ and $\theta$.*

The corresponding PDF is

$$f(y) = \alpha\frac{\pi}{2}\theta e^{-\theta y}(1-e^{-\theta y})^{\alpha-1}\cos\left(\frac{\pi}{2}(1-e^{-\theta y})^{\alpha}\right) \quad ;\alpha,\theta,y>0. \tag{2}$$

The survival and hazard rate functions are, respectively, given by

$$S(y) = 1-\sin\left(\frac{\pi}{2}(1-e^{-\theta y})^{\alpha}\right) \quad ;\alpha,\theta,y>0 \tag{3}$$

and

$$\lambda(y) = \frac{f(y)}{S(y)}$$

$$\lambda(y) = \frac{\alpha\frac{\pi}{2}\theta e^{-\theta y}(1-e^{-\theta y})^{\alpha-1}\cos\left(\frac{\pi}{2}(1-e^{-\theta y})^{\alpha}\right)}{\left(\sin\left(\frac{\pi}{4}(1-e^{-\theta y})^{\alpha}\right)-\cos\left(\frac{\pi}{4}(1-e^{-\theta y})^{\alpha}\right)\right)^{2}};\ \ \alpha,\theta,y>0$$

Some important series expansions:

$$e^{x} = \sum_{j=0}^{\infty}\frac{x^{j}}{j!} \tag{4}$$

$$\cos x = \sum_{j=0}^{\infty}(-1)^{j}\frac{x^{2j}}{2j!} \tag{5}$$

$$\sin x = \sum_{j=0}^{\infty}(-1)^{j}\frac{x^{2j+1}}{2j+1!} \tag{6}$$

$$(1-x)^{n} = \sum_{j=0}^{n}(-1)^{j}\binom{n}{j}x^{j} \tag{7}$$

## 3.1. Quantile Function

The quantile function of SGEE is given by

$$Y = -\frac{1}{\theta}log\left[1-\left(\frac{2}{\pi}\sin^{-1}(1-U)\right)^{\frac{1}{\alpha}}\right]$$

where U $\sim (0,1)$ distribution. The $q^{th}$ quantile of SGEE distribution is given by

$$y_q = -\frac{1}{\theta} log \left[ 1 - \left( \frac{2}{\pi} \sin^{-1}(1-q) \right)^{\frac{1}{\alpha}} \right]$$

The median is obtained as

$$y_{0.5} = -\frac{1}{\theta} log \left[ 1 - \left( \frac{2}{\pi} \sin^{-1}(\frac{1}{2}) \right)^{\frac{1}{\alpha}} \right]$$

## 3.2. Moments

The $r^{th}$ moment of the SGEE distribution is given by

$$\begin{aligned} E(Y^r) &= \int_0^\infty y^r f(y) dy \\ &= \int_0^\infty y^r \alpha \frac{\pi}{2} \theta e^{-\theta y} (1-e^{-\theta y})^{\alpha-1} cos \left( \frac{\pi}{2} (1-e^{-\theta y})^\alpha \right) dy \end{aligned}$$

by putting $e^{-\theta y} = z$ and using the series expansions (7) and (5), we get

$$E(Y^r) = \frac{\alpha}{\theta^r} \sum_{j=0}^{\infty} \sum_{k=0}^{\alpha(2j+1)-1} \frac{(-1)^{j+k}}{2j!} (\frac{\pi}{2})^{2j+1} \binom{\alpha(2j+1)-1}{k} \int_0^1 (-logz)^r z^k dz \tag{8}$$

again putting $-logz = u$ in (8), we get $r^{th}$ moment as

$$E(Y^r) = \frac{\alpha}{\theta^r} \sum_{j=0}^{\infty} \sum_{k=0}^{\alpha(2j+1)-1} \frac{(-1)^{j+k}}{2j!} (\frac{\pi}{2})^{2j+1} \binom{\alpha(2j+1)-1}{k} \frac{\Gamma(r+1)}{(k+1)^{r+1}} \tag{9}$$

## 3.3. Moment Generating Function

The moment generating function of SGEE distribution is defined by

$$M_Y(t) = \int_0^\infty e^{ty} f(y) dy,$$

again using (4), (5) and (7) we have the final expression of MGF as

$$M_Y(t) = \alpha \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\alpha(2j+1)-1} \frac{(-1)^{j+k} (\frac{\pi}{2})^{2j+1}}{2j!(k+1)^{r+1}} \left( \frac{t}{\theta} \right)^r \binom{\alpha(j+1)-1}{k} \quad ; t < \theta$$

## 3.4. Mean Residual Life And Mean Waiting Time

The mean residual life function, say $\mu(t)$, is defined by

$$\mu(t) = \frac{1}{S(t)} \left( E(t) - \int_0^t y f(y) dy \right) - t \tag{10}$$

**Table 1:** *Average values of MLEs their corresponding MSEs and Bias.*

| Sample size | Parameter | | MLEs | | MSE | | Bias | |
|---|---|---|---|---|---|---|---|---|
| $n$ | $\alpha$ | $\theta$ | $\hat{\alpha}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\theta}$ |
| 30 | 0.8 | 0.5 | 0.86876 | 0.55268 | 0.04713 | 0.02634 | 0.06876 | 0.05268 |
| | | 1 | 0.88261 | 1.12988 | 0.04109 | 0.11373 | 0.08261 | 0.12988 |
| | | 1.5 | 0.88609 | 1.72108 | 0.04381 | 0.32292 | 0.08609 | 0.22108 |
| | | 2 | 0.88888 | 2.31489 | 0.05774 | 0.58598 | 0.08888 | 0.31489 |
| | 1 | 0.5 | 1.06555 | 0.53199 | 0.05840 | 0.03184 | 0.06555 | 0.03199 |
| | | 1 | 1.08661 | 1.13391 | 0.07117 | 0.08813 | 0.08661 | 0.13391 |
| | | 1.5 | 1.08243 | 1.68584 | 0.08902 | 0.28176 | 0.08243 | 0.18584 |
| | | 2 | 1.06703 | 2.15094 | 0.11414 | 0.50617 | 0.06702 | 0.15094 |
| | 1.5 | 0.5 | 1.65896 | 0.54400 | 0.23140 | 0.01787 | 0.15896 | 0.04400 |
| | | 1 | 1.56549 | 1.04447 | 0.14145 | 0.04484 | 0.06549 | 0.04447 |
| | | 1.5 | 1.65654 | 1.61426 | 0.30000 | 0.16226 | 0.15654 | 0.11426 |
| | | 2 | 1.72030 | 2.25512 | 0.27791 | 0.36792 | 0.22030 | 0.25512 |
| | 2 | 0.5 | 2.22021 | 0.53030 | 0.40033 | 0.01187 | 0.22021 | 0.03030 |
| | | 1 | 2.22111 | 1.06212 | 0.46756 | 0.05600 | 0.22111 | 0.06211 |
| | | 1.5 | 2.31075 | 1.69956 | 0.61034 | 0.18055 | 0.31075 | 0.19956 |
| | | 2 | 2.22600 | 2.17031 | 0.57051 | 0.26500 | 0.22600 | 0.17031 |
| 50 | 0.8 | 0.5 | 0.84206 | 0.54269 | 0.02796 | 0.02565 | 0.04206 | 0.04269 |
| | | 1 | 0.79948 | 1.00343 | 0.02079 | 0.06482 | -0.00051 | 0.00343 |
| | | 1.5 | 0.83571 | 1.61962 | 0.01981 | 0.13657 | 0.03571 | 0.11962 |
| | | 2 | 0.81309 | 2.05569 | 0.02020 | 0.24927 | 0.01309 | 0.05569 |
| | 1 | 0.5 | 1.03527 | 0.52180 | 0.03822 | 0.01277 | 0.03527 | 0.02180 |
| | | 1 | 1.08685 | 1.08659 | 0.04641 | 0.05628 | 0.08685 | 0.08659 |
| | | 1.5 | 1.06537 | 1.68051 | 0.05138 | 0.19718 | 0.06537 | 0.18051 |
| | | 2 | 1.01739 | 2.02631 | 0.02842 | 0.14193 | 0.01739 | 0.02630 |
| | 1.5 | 0.5 | 1.63833 | 0.53194 | 0.14694 | 0.00855 | 0.13833 | 0.03194 |
| | | 1 | 1.60152 | 1.07540 | 0.15908 | 0.04964 | 0.10152 | 0.07540 |
| | | 1.5 | 1.53020 | 1.53325 | 0.10391 | 0.06558 | 0.03020 | 0.03325 |
| | | 2 | 1.56678 | 2.08532 | 0.07309 | 0.14002 | 0.06678 | 0.08532 |
| | 2 | 0.5 | 2.15774 | 0.52840 | 0.24240 | 0.00829 | 0.15774 | 0.02840 |
| | | 1 | 2.09796 | 1.02776 | 0.16712 | 0.02854 | 0.09796 | 0.02776 |
| | | 1.5 | 2.11604 | 1.56729 | 0.21126 | 0.09217 | 0.11604 | 0.06729 |
| | | 2 | 2.19453 | 2.08430 | 0.35438 | 0.11187 | 0.19453 | 0.08430 |

where

$$E(t) = \frac{\alpha}{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{\alpha(2j+1)-1}\frac{(-1)^{j+k}}{(k+1)^2 2j!}(\frac{\pi}{2})^{2j+1}\binom{\alpha(2j+1)-1}{k} \tag{11}$$

and

$$\int_0^t xf(x)dx = \frac{\alpha}{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{\alpha(2j+1)-1}\frac{(-1)^{j+k}}{(k+1)^2 2j!}(\frac{\pi}{2})^{2j+1}\binom{\alpha(2j+1)-1}{k}\gamma\left(\theta t(k+1),2\right) \tag{12}$$

Substituting (3), (11) and (12) in (10), we get $\mu(t)$ as

$$\mu(t) = \frac{1}{1-\sin\left(\frac{\pi}{2}(1-e^{-\theta t})^{\alpha}\right)}\frac{\alpha}{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{\alpha(2j+1)-1}\frac{(-1)^{j+k}}{(k+1)^2 2j!}(\frac{\pi}{2})^{2j+1}\binom{\alpha(2j+1)-1}{k} \times\left[1-\gamma(\theta t(k+1))\right]-t$$

where $\gamma(a,b) = \int_0^a y^{b-1}e^{-y}dy$ is the lower incomplete gamma function.

**Table 2:** *Average values of MLEs their corresponding MSEs and Bias.*

| Sample size | Parameter | | MLEs | | MSE | | Bias | |
|---|---|---|---|---|---|---|---|---|
| $n$ | $\alpha$ | $\theta$ | $\hat{\alpha}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\theta}$ | $\hat{\alpha}$ | $\hat{\theta}$ |
| 100 | 0.8 | 0.5 | 0.82267 | 0.52370 | 0.01321 | 0.01083 | 0.02267 | 0.02370 |
| | | 1 | 0.79728 | 0.99739 | 0.00716 | 0.02731 | -0.00271 | -0.00260 |
| | | 1.5 | 0.82011 | 1.54754 | 0.01276 | 0.06654 | 0.02011 | 0.04754 |
| | | 2 | 0.79681 | 1.99112 | 0.01290 | 0.12323 | -0.00318 | -0.00887 |
| | 1 | 0.5 | 1.03990 | 0.51737 | 0.01768 | 0.00528 | 0.03990 | 0.01737 |
| | | 1 | 1.01765 | 1.03115 | 0.01432 | 0.02452 | 0.01765 | 0.03115 |
| | | 1.5 | 1.02663 | 1.55358 | 0.01549 | 0.04853 | 0.02663 | 0.05358 |
| | | 2 | 1.03361 | 2.07698 | 0.02073 | 0.10262 | 0.03360 | 0.07698 |
| | 1.5 | 0.5 | 1.54919 | 0.51586 | 0.04621 | 0.00518 | 0.04919 | 0.01586 |
| | | 1 | 1.55740 | 1.02647 | 0.04768 | 0.01628 | 0.05740 | 0.02646 |
| | | 1.5 | 1.56732 | 1.56766 | 0.03581 | 0.03419 | 0.06731 | 0.06766 |
| | | 2 | 1.57208 | 2.10929 | 0.04142 | 0.08105 | 0.07208 | 0.10929 |
| | 2 | 0.5 | 2.04186 | 0.50132 | 0.07637 | 0.00288 | 0.04186 | 0.00132 |
| | | 1 | 2.04210 | 1.02675 | 0.08509 | 0.01862 | 0.04210 | 0.02675 |
| | | 1.5 | 2.10625 | 1.54551 | 0.13007 | 0.03257 | 0.10625 | 0.04551 |
| | | 2 | 2.06746 | 2.04980 | 0.08307 | 0.06683 | 0.06746 | 0.04980 |
| 200 | 0.8 | 0.5 | 0.79883 | 0.50439 | 0.00455 | 0.00428 | -0.00116 | 0.00439 |
| | | 1 | 0.80354 | 1.01873 | 0.00364 | 0.01240 | 0.00354 | 0.01873 |
| | | 1.5 | 0.79510 | 1.48724 | 0.00360 | 0.02438 | -0.00489 | -0.01275 |
| | | 2 | 0.79160 | 2.01861 | 0.00621 | 0.06748 | -0.00839 | 0.01861 |
| | 1 | 0.5 | 0.99809 | 1.50664 | 0.00558 | 0.01866 | -0.00190 | 0.00664 |
| | | 1 | 1.00884 | 2.03002 | 0.00997 | 0.05265 | 0.00884 | 0.03002 |
| | | 1.5 | 1.00041 | 1.51007 | 0.00670 | 0.02119 | 0.00041 | 0.01007 |
| | | 2 | 1.00884 | 2.03002 | 0.00997 | 0.05265 | 0.00884 | 0.03002 |
| | 1.5 | 0.5 | 1.56024 | 0.51511 | 0.02153 | 0.00224 | 0.06024 | 0.01511 |
| | | 1 | 1.51925 | 1.01414 | 0.01833 | 0.00859 | 0.01925 | 0.01414 |
| | | 1.5 | 1.53288 | 1.52218 | 0.02876 | 0.02280 | 0.03288 | 0.02218 |
| | | 2 | 1.52145 | 2.01055 | 0.02441 | 0.03646 | 0.02145 | 0.01055 |
| | 2 | 0.5 | 2.03758 | 0.50909 | 0.03859 | 0.00208 | 0.03758 | 0.00909 |
| | | 1 | 2.07367 | 1.03099 | 0.05998 | 0.00920 | 0.07367 | 0.03099 |
| | | 1.5 | 2.01058 | 1.50662 | 0.04448 | 0.01675 | 0.01054 | 0.00662 |
| | | 2 | 2.02562 | 2.01343 | 0.03551 | 0.02001 | 0.02562 | 0.01343 |

The mean waiting time of $Y$, say $\bar{\mu}(t)$, is given by

$$\bar{\mu}(t) = t - \frac{1}{F(t)}\int_0^t yf(y)dy$$

$$\bar{\mu}(t) = t - \frac{1}{\sin\left(\frac{\pi}{2}(1-e^{-\theta t})^{\alpha}\right)}\frac{\alpha}{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{\alpha(2j+1)-1}\frac{(-1)^{j+k}}{(k+1)^2 2j!}\left(\frac{\pi}{2}\right)^{2j+1}\binom{\alpha(2j+1)-1}{k} \times \gamma\left(\theta t(k+1), 2\right)$$

## 3.5. Order Statistics

Let $Y_1, Y_2, ..., Y_n$ be a random sample of size $n$, and if $Y_{i:n}$ denote the $i^{th}$ order statistic, then the PDF of $Y_{i:n}$, say $f_{i:n}(y)$ is given by

$$f_{i:n}(y) = \frac{n!}{(i-1)!(n-i)!}F(y)^{i-1}f(y)(1-G(y))^{n-i}.$$

We can write the PDF $f_{i:n}(y)$ of $i^{th}$ order statistic of SGEE distribution as

$$f_{i:n}(y) = \frac{\alpha\pi\theta e^{-\theta y}(1-e^{-\theta y})^{\alpha-1}}{2B(i,n-i+1)}\frac{sin^n\left(\frac{\pi}{2}(1-e^{-\theta y})^{\alpha}\right)}{tan\left(\frac{\pi}{2}(1-e^{-\theta y})^{\alpha}\right)}cosec^{n-i}\left(\frac{\pi}{2}(1-e^{-\theta y})^{\alpha}-1\right)$$

Where $B(a,b)$ is the beta function.

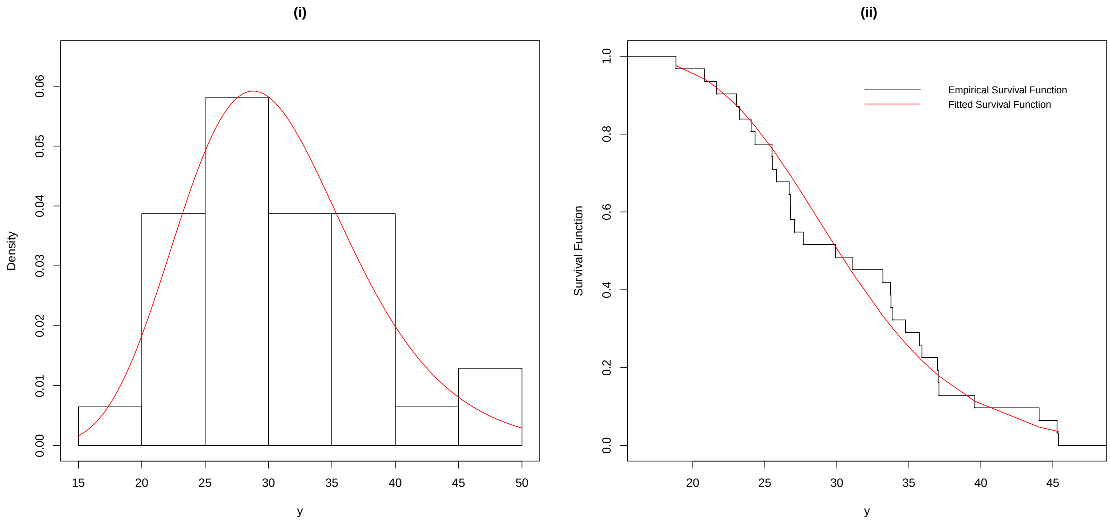


**Figure 2:** *(i) The relative histogram and the fitted SGEE distribution. (ii) The empirical survival function and fitted SGEE survival function for data set I.*

## 4. Statistical Inference

### 4.1. Maximum Likelihood Estimation

Let $y_1, y_2, ..., y_n$ be a random sample from SGEE distribution, then the logarithm of the likelihood function is given by

$$l = n log(\frac{\alpha\pi}{2}) + n log\theta - \theta \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} log(1 - e^{-\theta y_i})^{\alpha-1} + \sum_{i=1}^{n} log\left[cos\left(\frac{\pi}{2}(1 - e^{-\theta y_i})^{\alpha}\right)\right], \quad (13)$$

by differentiating partially (13) with respect to the parameters $\alpha$ and $\theta$ and equating the derivatives to zero, we get

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log(1 - e^{-\theta y_i})\left[1 - \frac{\pi}{2}(1 - e^{-\theta y_i})^{\alpha} tan\left(\frac{\pi}{2}(1 - e^{-\theta y_i})^{\alpha}\right)\right] = 0$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\left(\frac{y_i e^{-\theta y_i}}{1 - e^{-\theta y_i}}\right)\left[(\alpha - 1) - \frac{\alpha\pi}{2}(1 - e^{-\theta y_i})^{\alpha} tan\left(\frac{\pi}{2}(1 - e^{-\theta y_i})^{\alpha}\right)\right] = 0$$

It is clear that these equations cannot be solved analytically, so the MLEs of parameters are obtained through R software.

**Theorem 1:** *If the parameter $\theta$ is known, then the MLE of $\alpha$ exists and is unique.*

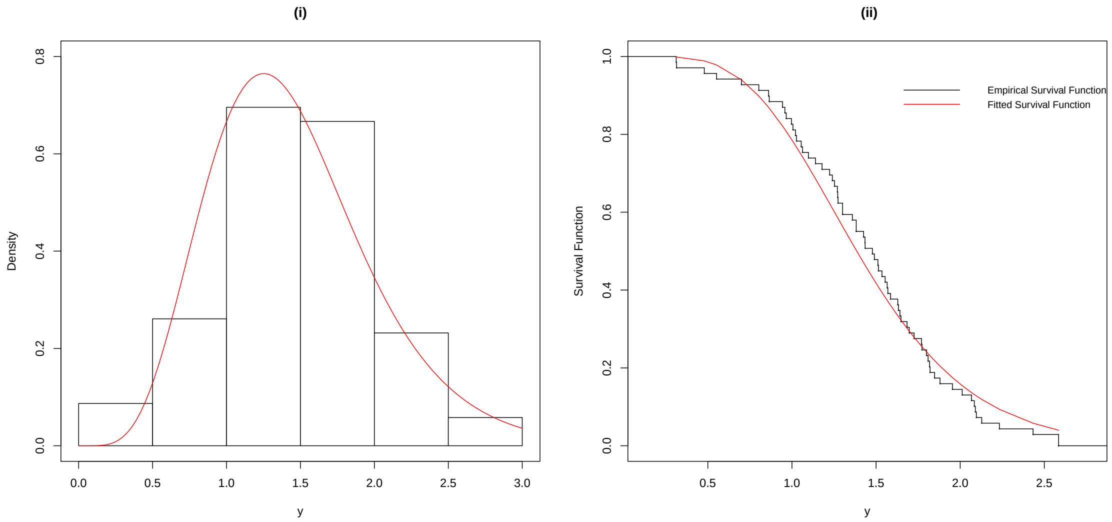


**Figure 3:** *(i) The relative histogram and the fitted SGEE distribution. (ii) The empirical survival function and fitted SGEE survival function for data set II.*

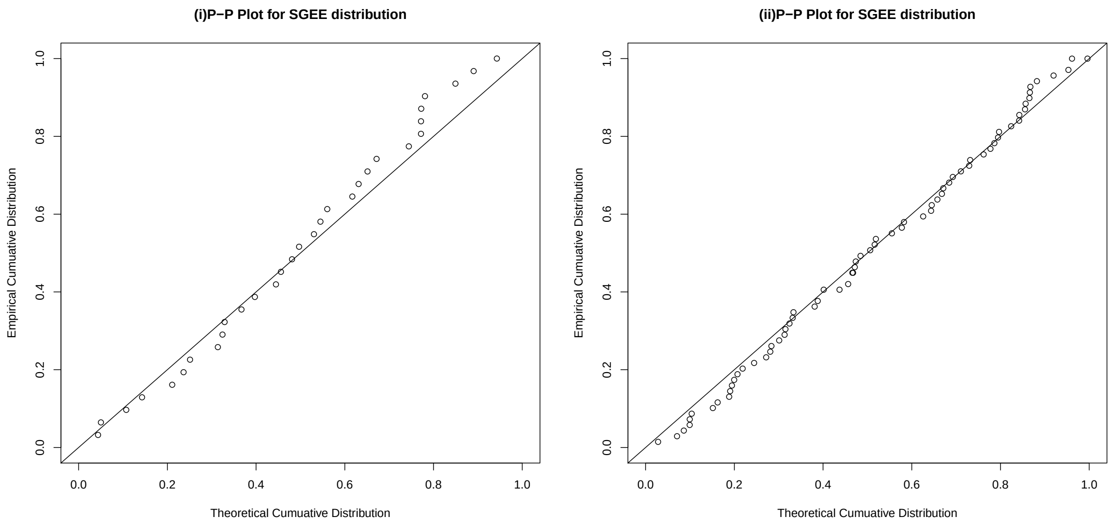


**Figure 4:** *P-P plot for the SGEE distribution for data set I and data set II*

**Proof:** Since,

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log(1-e^{-\theta y_i}) \left[1 - \frac{\pi}{2}(1-e^{-\theta y_i})^{\alpha} tan\left(\frac{\pi}{2}(1-e^{-\theta y_i})^{\alpha}\right)\right]$$

$$\lim_{\alpha \to 0} \frac{\partial l}{\partial \alpha} = \infty + \sum_{i=1}^{n} log(1-e^{-\theta y_i}) \left[1 - \frac{\pi}{2} tan\frac{\pi}{2}\right] = \infty$$

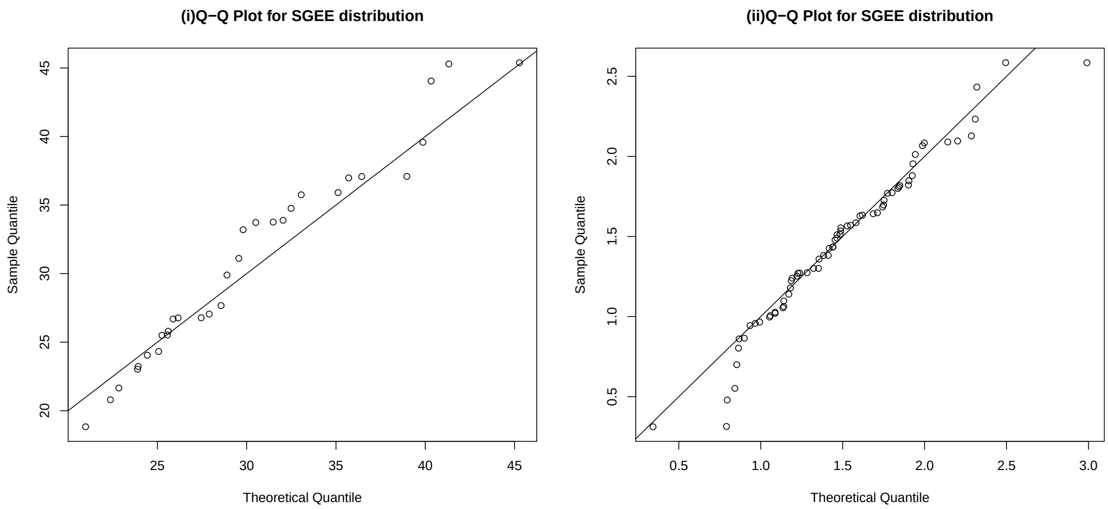


**Figure 5:** *Q-Q plot for the SGEE distribution for data set I and data set II*

Also

$$\lim_{\alpha\to\infty}\frac{\partial l}{\partial \alpha} = 0+\sum_{i=1}^{n} log(1-e^{-\theta y_i}) < 0,$$

therefore, there exists atleast one root say $\hat{\alpha}(0,\infty)$, such that $\frac{\partial l}{\partial \alpha}=0$
For uniqueness of root, we have

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \frac{\pi}{2}(1-e^{-\theta y_i})^{\alpha}\sum_{i=1}^{n}\left(log(1-e^{-\theta y_i})\right)^2\left[\frac{\pi}{2}sec^2\left(\frac{\pi}{2}(1-e^{-\theta y_i})^{\alpha}\right)+tan\left(\frac{\pi}{2}(1-e^{-\theta y_i})^{\alpha}\right)\right] < 0$$

Hence the proof.

■

**Theorem 2:** *If the parameter $\alpha$ is known, then the MLE of $\theta$ exists and is unique.*
**Proof:** Since,

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\left(\frac{y_i e^{-\theta y_i}}{1-e^{-\theta y_i}}\right)\left[(\alpha-1)-\alpha\frac{\pi}{2}(1-e^{-\theta y})^{\alpha} tan\left(\frac{\pi}{2}(1-e^{-\theta y})^{\alpha}\right)\right]$$

$$\lim_{\theta\to 0}\frac{\partial l}{\partial \theta} = \infty$$

Also

$$\lim_{\theta\to\infty}\frac{\partial l}{\partial \theta} = 0-\sum_{i=1}^{n} y_i < 0$$

Therefore, there exists atleast one root say $\hat{\theta}(0,\infty)$, such that $\frac{\partial l}{\partial \theta}=0$
For uniqueness of root, we have

$$\frac{\partial^2 l}{\partial \theta^2} = -\frac{n}{\theta^2} - \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} \left( \frac{y_i e^{-\theta y_i}}{1 - e^{-\theta y_i}} \right)^2 \left[ \alpha\ x \left\{ \alpha\ x \sec^2(x) + (\alpha - 1)\tan(x) \right\} + (\alpha - 1) e^{-\theta y_i} \right] < 0$$

where,

$$x = \frac{\pi}{2}(1 - e^{-\theta y})^{\alpha}$$

Hence proved.

■

**Table 3:** *MLEs and -2l, AIC, AICC, BIC , K-S statistic and P-value for data set I.*

| Model | $\hat{\alpha}$ | $\hat{\theta}$ | -2l | AIC | AICC | BIC | K-S statistic | p-value |
|---|---|---|---|---|---|---|---|---|
| SGEE | 0.07832 (1.29403) | 0.11465 (0.01650) | 208.3183 | 212.3183 | 212.7468 | 215.1862 | 0.13013 | 0.6235 |
| G | 0.93208 (1.76667) | 0.61445 (0.15676) | 208.3212 | 212.3212 | 212.7691 | 215.2292 | 0.13189 | 0.6232 |
| APE | 1.26948 (1.18703) | 9.55912 (0.89405) | 222.5222 | 226.5222 | 226.9508 | 229.3902 | 0.1757 | 0.2619 |
| NAPTE | 1.91332 (1.67776) | 9.99821 (0.70944) | 218.8185 | 222.8185 | 223.2470 | 225.6864 | 0.15716 | 0.3877 |
| EE | 1.47335 (0.95528) | 0.14956 (0.01917) | 208.8087 | 212.8087 | 213.2372 | 215.6766 | 0.13588 | 0.5697 |
| R | 2.36356 (0.00830) | - | 236.4447 | 238.4447 | 238.5826 | 239.8787 | 0.31888 | 0.0026 |
| E | 0.032455 (0.00582) | - | 250.5289 | 252.5289 | 252.6668 | 253.9629 | 0.39128 | 0.0012 |

## 4.2. Simulation Study

To acertain the consistency and stability of the estimates, the simulation study was performed by taking samples of size (n=30, 50, 100 and 200) each replicated 100 times for different parameter vectors $\alpha = (0.8, 1, 1.5, 2)$, $\theta = (0.5, 1, 1.5, 2)$, were obtained from SGEE distribution. In each case, the average values of MLEs (estimates) and the corresponding empirical mean squared errors (MSEs) and bias were considered. The simulation results are displayed in table 1 and table 2. From tables 1 and 2, it is obvious that as the sample size increases the MSE and bais decreases in all the cases.

# 5. Applications

To justify the validity and applicability of the SGEE distribution two real data sets have been used. The data set I is the strength of glass of the aircraft window taken from [4] and was also recently

**Table 4:** *MLEs and -2l, AIC, AICC, BIC , K-S statistic and P-value for data set II.*

| Model | $\hat{\alpha}$ | $\hat{\theta}$ | -2l | AIC | AICC | BIC | K-S statistic | p-value |
|---|---|---|---|---|---|---|---|---|
| SGEE | 1.75535 (1.39591) | 1.36922 (0.14288) | 106.0445 | 110.0445 | 110.2263 | 114.5127 | 0.08801 | 0.6590 |
| G | 0.99694 (1.68238) | 4.82104 (1.02421) | 106.1653 | 110.1653 | 110.2250 | 114.6335 | 0.08977 | 0.6275 |
| APE | 1.29960 (1.68238) | 2.04747 (1.02421) | 108.8658 | 112.8658 | 113.0476 | 117.3340 | 0.09277 | 0.5925 |
| NAPTE | 0.82534 (1.15467) | 1.89659 (1.66053) | 113.3371 | 117.3371 | 117.5190 | 121.8054 | 0.11192 | 0.3531 |
| EE | 0.82839 (2.03418) | 1.89659 (0.18852) | 112.5386 | 116.5386 | 116.5983 | 120.7727 | 0.10192 | 0.4031 |
| R | 1.08350 (0.06521) | - | 118.8298 | 120.8298 | 120.8895 | 123.0639 | 0.31888 | 0.0026 |
| E | 0.68902 (0.08294) | - | 189.4026 | 191.4026 | 191.4623 | 193.6367 | 0.36224 | 0.0001 |

reported by [2] The data set II finds its source in [14] and was also reported by [7] It is about the tensile strength (with unit in GPa) for single carbon fibers.

For comparision purpose, we compared the proposed SGEE distribution with several other models namely, gamma (G), alpha power exponential (APE) [9], noval alpha power transformed exponential (NAPTE) [6], exponentiated exponential (EE) [5], Rayleigh (R) and Exponential (E) distributions. From table (3) and (4) it is ostensive that SGEE distribution has the smallest values of the criteria -2l, AIC, AICC, BIC, K-S statistic and maximum p-value among all other distributions. Hence, we can say that the proposed model fits better for these data sets. Figure 2(i) and 3(i) display relative histograms for data set I and II respectively. Also, the Figure 2(ii) and 3(ii) shows the plots of the fitted SGEE survival function and empirical survival function of the data set I and II, respectively.

## 6. Concluding Remarks

A new continuous probability model based on a trigonometric function was introduced having symmetric, decreasing and positively skewed density function. Some of the well-known mathematical properties of the introduced model were also discussed. The authenticity and applicability of the introduced model was examined by considering two real data sets, it was perceived that the SGEE distribution is more appropriate for the given data sets than all other competitive models.

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