Type 1 Topp-Leone $q-$Exponential Distribution and its Applications

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Abstract

The main purpose of this paper is to discuss a new lifetime distribution, called the type 1 Topp-Leone generated q-exponential distribution(Type 1 TLqE). Using the quantile approach various distributional properties, $L-$moments, order statistics, and reliability properties were established. We suggested a new reliability test plan, which is more advantageous and helps in making optimal decisions when the lifetimes follow this distribution. The new test plan is applied to illustrate its use in industrial contexts. Finally, we proved empirically the importance and the flexibility of the new model in model building by using a real data set.

Keywords: Type 1 Topp-Leone generated q-exponential distribution, Quantile density function, Quantile function, Hazard quantile function, $L-$moments, Reliability Test Plan.

1. Introduction

There are many statistical distributions which plays an important role in modeling survival and life time data such as exponential, weibull, logistic etc. Almost all these distributions with unbounded support. But there are situations in real life, in which observations can take values only in a limited range such as percentages, proportions or fractions. Papke and Wooldridge [12] claims that in many economic settings, such as fraction of total weekly hours spent working, pension plan participation rates, industry market shares, fraction of land area allocated to agriculture etc., the variable bounded between zero and one. Thus it is important to have models defined on the unit interval in order to have reasonable results.

A new distribution was introduced in 1955, called Topp Leone (TL) distribution, defined on finite support, proposed Topp and Leone [20] and used it as a model for failure data. A random variable X is distributed as the TL with parameter α denoted by $x \sim TL_{(\alpha)}$, with a cumulative distribution function

$$F_{TL}(x) = x^{\alpha}(2-x)^{\alpha}, 0 < x < 1, \alpha > 0. \tag{1}$$

The corresponding probability function is

$$f_{TL}(x) = 2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}. \tag{2}$$

Topp Leone distribution provides closed forms of cumulative density function (cdf) and the probability density function (pdf) and describes empirical data with J-shaped histogram such as powered tool band failures, automatic calculating machine failure. The Topp Leone distribution had been received little attention until Nadarajah and Kotz [10] discovered it. Further information and application of TL distribution can be obtained from Ghitany et al. [2], Kotz and Seier [7].

Lifetime data plays an important role in a wide range of applications such as medical, engineering an social sciences. When there is a need for more flexible distributions, almost all researchers are about to use the new one with more generalization. An excellent review of Lee et al. [9] has provided through knowledge of several methods for generating families of continuous univariate distributions. They discussed some noticeable developments after 1980, are method of generating skew distributions, beta generated method, method of adding parameters, transformed-transformer method, and composite method. The beta generated (BG) family of distributions belongs to a parameter adding method (Lee et al. [9], Kumaraswamy [8]). In a similar manner the relation of a random variable X having the TLG distribution and a random variable T having TL distribution is $X = G^{-1}(T)$, with $T \sim TL(\alpha)$. This relation demonstrates that the pdf of TL distribution, (2), is transformed into a new pdf through the function $G(\cdot)$ The cdf of TL generated random variable X is defined as

$$F_{TLG}(x) = \int_0^{G(x)} h(t)dt$$

where $h(t)$ is the pdf of TL variable and $G(x)$ is the cdf of any arbitrary random variable. Thus the cdf of TL generated random variable is

$$F_{TLG}(x) = 2\alpha \int_0^{G(x)} t^{\alpha-1}(1-t)(2-t)^{\alpha-1}dt = G(x)^{\alpha}(2-G(x))^{\alpha}. \tag{3}$$

By differentiating, we get the corresponding pdf,

$$f_{TLG}(x) = 2\alpha g(x)(1-G(x))G(x)^{\alpha-1}(2-G(x))^{\alpha-1}, \alpha > 0. \tag{4}$$

In reliability analysis, a frequently used distribution is exponential distribution having the characterizing property of constant hazard function. Due to this, exponential distribution is sometimes not suitable for analyzing data. This implies the need for more generalization. In such situations we use distribution called Topp-Leone Exponential distribution (TLE). TLE distribution comes as the combination of TL distribution and exponential distribution. Here TL distribution is the generator and exponential is the parent distribution. Sangsanit and Bodhisuwan [16] presented the Topp-Leone generated exponential (TLE) distribution as an example of the Topp-Leone generated distribution. A random variable X possessing TLE distribution having cdf and probability function defined respectively as

$$F_{TLE}(x) = (1-exp(-\lambda x))^{\alpha}(2-(1-exp(-\lambda x)))^{\alpha} = (1-exp(-2\lambda x))^{\alpha}$$

and

$$f_{TLE}(x) = 2\alpha\lambda exp(-2\lambda x)(1-exp(-2\lambda x))^{\alpha-1},$$

where α is the shape parameter and λ is the scale parameter .

Various entropy measures have been developed by mathematicians and physicists to describe several phenomena, depending on the field and the context in which it is being used. Tsallis [19], introduced a generalization of the Boltzmann-Gibbs entropy. Tsallis statistics have found applications in many areas such as physics, chemistry, biology, medicine, economics, geophysics, etc. By maximizing Tsallis entropy, subject to certain constraints, leads to the Tsallis distribution, also known as q-exponential distribution, which has the form $f(x) = c[1-(1-q)x]^{\frac{1}{(1-q)}}$ where c is the normalizing constant. Various applications and generalizations of the q-exponential distribution are given in Picoli et al. [13]. In the limit $q \to 1$, q-entropy converges to Boltzmann-Gibbs entropy.

An important characteristic of q-exponential distribution is that it has two parameters q and λ providing more flexibility with regard to its decay, differently from exponential distribution. The q exponential distribution is defined by its cdf and pdf as,

$$F_{1qE}(x) = 1-[1-(1-q)\lambda x]^{\frac{(2-q)}{1-q}}. \tag{5}$$

$$f_{1qE}(x) = (2-q)\lambda[1-(1-q)\lambda x]^{\frac{1}{1-q}}, 1-\lambda(1-q)x > 0, \lambda > 0, q < 2, q \neq 0 \quad (6)$$

The parameter q is known as entropy index. As $q \to 1$, the q-exponential distribution becomes exponential distribution. In that sense q-exponential distribution is a generalization of exponential distribution. The parameters q and λ determine how quickly the pdf decays. In the reliability context, an important characteristic of the q-exponential distribution is its hazard rate, which is not necessarily constant as in exponential distribution.

The rest of the paper is organized as follows. In section 2 we will discuss the Type 1 Topp-Leone q-Exponential Distribution. In section 3 consists of the quantile properties of Type 1 TLqE distribution. Section 4, we described a new reliability test plan for type 1 TLqE distribution and its applications are also discussed. In Section 5, we apply the Type 1 TLqE distribution to a real data sets to show that it can be used quite effectively in analyzing lifetime data. Finally, concluding remarks and future work are addressed in Section 6.

2. Type 1 Topp-Leone q-Exponential Distribution

In this section we discuss the type 1 Topp-Leone generated $q-$ exponential (TLqE) distribution introduced by combining the TL distribution with $q-$ exponential distribution, for more details see Sebastian et al. [17]. Substituting (5) and (6) in (3) and (4) respectively we will get the distribution function and density function of TLqE distribution as follows:

$$F_{1TLqE}(x) = \{1-[1-(1-q)\lambda x]^{2(\frac{2-q}{1-q})}\}^{\alpha}, x > 0, \lambda, \alpha > 0, q < 2, q \neq 0.$$

and

$$f_{1TLqE}(x) = 2\alpha\lambda(2-q)[1-(1-q)\lambda x]^{\frac{3-q}{1-q}}\{1-[1-(1-q)\lambda x]^{2(\frac{2-q}{1-q})}\}^{\alpha-1},$$

where $1-[1-(1-q)\lambda x]^{2(\frac{2-q}{1-q})} > 0, \lambda, \alpha > 0, q < 2.$

Figure 1: *Plots of F(x) of TLqE distribution for* $\alpha = 1.1, \lambda = 0.1$ *(left) and for* $\lambda = 0.3, q = 1.1$ *(right).*

Figure 2: *Plots of f(x) of TLqE distribution for* $\alpha = 1.1, \lambda = 0.1$ *(left) and for* $\lambda = 0.3, q = 1.1$ *(right).*

In Figure 1 and Figure 2, we can see the plots of cdf and pdf of TLqE for different values of the shape parameters α and q. The survival function, the probability density function and the Hazard function are the three important functions that characterize the distribution of the survival times. Here

$$S(x) = 1 - \{1 - [1 - (1-q)\lambda x]^{2\left(\frac{2-q}{1-q}\right)}\}^{\alpha},$$

and

$$h(x) = \frac{2\alpha\lambda(2-q)[1-(1-q)\lambda x]^{\frac{3-q}{1-q}}\{1-[1-(1-q)\lambda x]^{2\left(\frac{2-q}{1-q}\right)}\}^{\alpha-1}}{1-\{1-[1-(1-q)\lambda x]^{2\left(\frac{2-q}{1-q}\right)}\}^{\alpha}}$$

respectively are the survival and the hazard function of TLqE distribution. Figure 3, gives the plots of $h(x)$ of TLqE distribution for different values of the shape parameters α and q.

Figure 3: *Plots of h(x) of TLqE distribution for* $\alpha = 1.1, \lambda = 0.1$ *(left) and for* $\lambda = 0.3, q = 1.1$ *(right).*

As $q \to 1$ then $f_{1TLqE}(x)$ goes to

$$f_{3TLqE}(x) \quad = \quad 2\alpha\lambda e^{-2\lambda x}\left(1 - e^{-2\lambda x}\right)^{\alpha-1}, \lambda, \alpha > 0, x > 0, \tag{7}$$

and the correspond cdf is

$$F_{3TLqE}(x) \quad = \quad \left(1 - e^{-2\lambda x}\right)^{\alpha}, x > 0, \lambda, \alpha > 0.$$

3. Quantile properties of type 1 Topp-Leone $q-$Exponential Distribution

3.1. Distributional characteristics

In modelling and analysis of statistical data, probability distribution can be specified either in terms of distribution function or by the quantile function. Quantile functions have several interesting properties that are not shared by distributions, which makes it more convenient for analysis. For example, the sum of two quantile functions is again a quantile function. For a nonnegative random variable X with distribution function F(x), the quantile function $Q(u)$ is defined by (see Nair et al. [11])

$$Q(u) = F^{-1}(u) = inf\{x : F(x) \geq u\}, \quad 0 \leq u \leq 1 \tag{8}$$

For every $-\infty < x < \infty$ and $0 < u < 1$, we have

$$F(x) \geq u \text{ if and only if } Q(u) \leq x.$$

Thus, if there exists an x such that $F(x) = u$, then $F(Q(u)) = u$ and $Q(u)$ is the smallest value of x satisfying $F(x) = u$. Further, if $F(x)$ is continuous and strictly increasing, $Q(u)$ is the unique value x such that $F(x) = u$, and so by solving the equation $F(x) = u$, we can find x in terms of u which is the quantile function of X.

By using inversion method, we can generate a randon variate from TLqE distribution. We have already seen that the relationship between a random variable X, having TLqE distribution, and a random variable T, having the TL distribution, is

$$\begin{aligned} X &= G^{-1}(t) \\ &= \frac{1-(1-t)^{\frac{1-q}{2-q}}}{(1-q)\lambda} \end{aligned} \tag{9}$$

where $G^{-1}(\cdot)$ is related to inversion of the q-exponential cdf. The quantile function of the TL distribution is

$$t = 1 - \sqrt{1 - u^{\frac{1}{\alpha}}}, \tag{10}$$

where u is picked from the uniform distribution over $(0, 1)$. Then the quantile function of type 1 TLqE distribution is obtained by using equation (9) and (10),

$$Q(u) = \frac{1 - \left(\sqrt{1 - u^{\frac{1}{\alpha}}}\right)^{(\frac{1-q}{2-q})}}{(1-q)\lambda}, q < 2. \tag{11}$$

The quantile-based measures of the distributional characteristics like location, dispersion, skewness, and kurtosis are useful for estimating parameters of the model by matching population characteristics with corresponding sample characteristics. We can obtain the median as Median $= Q(\frac{1}{2})$. Dispersion is measured by the interquartile range, $IQR = Q(\frac{3}{4}) - Q(\frac{1}{4})$. Skewness is measured by Galton's coefficient, $S = \frac{Q(\frac{3}{4})+Q(\frac{1}{4})-2M}{IQR}$. Moors proposed a measure of kurtosis as, $T = \frac{Q(\frac{7}{8})-Q(\frac{5}{8})+Q(\frac{3}{8})-Q(\frac{1}{8})}{IQR}$.

If $f(x)$ is the probability function of X, then $f(Q(u))$ is called the density quantile function. The derivative of $Q(u)$,

$$q(u) = Q'(u),$$

is known as the quantile density function of X. If $F(x)$ is right continuous and strictly increasing, we have

$$F(Q(u)) = u \tag{12}$$

so that F(x) = u implies x = Q(u). When f(x) is the probability density function (PDF) of X; we have from (12)

$$q(u)f(Q(u)) = 1 \tag{13}$$

Quantile function has several properties that are not shared by distribution function. See Nair et al. [11] for details. Now the qunatile density of type 1 TLqE distribution is obtained as

$$q(u) = \frac{1}{2\alpha\lambda(2-q)} u^{\frac{1}{\alpha}-1}\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{q-3}{2(2-q)}}. \tag{14}$$

For the proposed family of distribution, the density function $f(x)$ can be written in terms of the distribution function as

$$f(x) = 2\alpha\lambda(2-q)\frac{F(x)^{1-\frac{1}{\alpha}}}{\left(1-(F(x))^{\frac{1}{\alpha}}\right)^{\frac{q-3}{2(2-q)}}}. \tag{15}$$

For all values of the parameters, the density is strictly decreasing in x and it tends to zero as $x \to \infty$.

3.2. $L-$moments

The L-moments are often found to be more desirable than the conventional moments in describing the characteristics of the distributions as well as for inference. A unified theory and a systematic study on $L-$ moments have been presented by Hosking [3].
The r^{th} $L-$ moment is given by

$$L_r = \int_0^1 \sum_{k=0}^{r-1}(-1)^{r-1-k}\binom{r-1}{k}\binom{r-1+k}{k}u^k Q(u)du. \tag{16}$$

Theorem 1. For the type 1 TLqE distribution, the r^{th} $L-$ moment can be obtained by using the following recurrence relation,

$$L_r = \sum_{k=0}^{r-1}(-1)^{r-1-k}\binom{r-1}{k}\binom{r-1+k}{k}\frac{\alpha}{\lambda(1-q)}\left[B(1,k\alpha) - B(\frac{(1-q)}{2(2-q)}+1,k\alpha)\right]. \tag{17}$$

So we can evaluate the $L-$coefficient of variation (τ_2), analogous to the coefficient of variation based on ordinary moments is given by, $\tau_2 = \frac{L_2}{L_1}$. Similarly the $L-$coefficient of skewness, (τ_3) and kurtosis, (τ_4) of type 1 Topp-Leone generated q-exponential quantile function respectively can be obtained as $\tau_3 = \frac{L_3}{L_2}$ and $\tau_4 = \frac{L_4}{L_3}$.

3.3. Order statistics of type 1 Topp-Leone q-Exponential Distribution

If $X_{r:n}$ is the rth order statistic in a random sample of size n, then the density function of $X_{r:n}$ can be written as

$$f_r(x) = \frac{1}{B(r,n-r+1)} f(x)F(x)^{r-1}(1-F(x))^{n-r} \tag{18}$$

From Eq.(15), we have

$$f_r(x) = \frac{2\alpha\lambda(2-q)}{B(r,n-r+1)}\frac{F(x)^{r-\frac{1}{\alpha}}(1-F(x))^{n-r}}{\left(1-(F(x))^{\frac{1}{\alpha}}\right)^{\frac{q-3}{2(2-q)}}}. \tag{19}$$

Hence,

$$\begin{aligned}\mu_{r:n} = E(X_{r:n}) &= \int x f_r(x)dx \\ &= \frac{2\alpha\lambda(2-q)}{B(r,n-r+1)}\int_0^\infty x\frac{F(x)^{r-\frac{1}{\alpha}}(1-F(x))^{n-r}}{\left(1-(F(x))^{\frac{1}{\alpha}}\right)^{\frac{q-3}{2(2-q)}}}dx. \end{aligned} \tag{20}$$

In quantile terms, we have

$$E(X_{r:n}) = \frac{2\alpha\lambda(2-q)}{B(r,n-r+1)} \int_0^1 Q(u) \frac{u^{r-\frac{1}{\alpha}}(1-u)^{n-r}}{\left(1-(u)^{\frac{1}{\alpha}}\right)^{\frac{q-3}{2(2-q)}}} du. \tag{21}$$

For the type 1 TLqE distribution, the first-order statistic $X_{1:n}$ has the quantile function

$$Q_1(u) = Q(1-(1-u)^{\frac{1}{n}}) = \frac{1}{\lambda(1-q)} \left[1-\left(1-(1-u)^{\frac{1}{n}}\right)^{\frac{1}{\alpha}}\right]^{\frac{(1-q)}{2(2-q)}}, \tag{22}$$

and the nth order statistic $X_{n:n}$ has the quantile function

$$Q_n(u) = Q(u^{\frac{1}{n}}) = \frac{1}{\lambda(1-q)} \left[1-(1-u^{\frac{1}{n\alpha}})\right]^{\frac{(1-q)}{2(2-q)}}. \tag{23}$$

3.4. Hazard quantile function

One of the basic concepts employed for modeling and analysis of lifetime data is the hazard rate. In a quantile setup, Nair et al. [11] defined the hazard quantile function, which is equivalent to the hazard rate. The hazard quantile function $H(u)$ is defined as

$$H(u) = h(Q(u))) = (1-u)^{-1} f Q(u) = [(1-u)q(u)]^{-1}. \tag{24}$$

Thus $H(u)$ can be interpreted as the conditional probability of failure of a unit in the next small interval of time given the survival of the unit until $100(1-u)\%$ point of the distribution. Note that $H(u)$ uniquely determines the distribution using the identity,

$$Q(u) = \int_0^u \frac{dp}{(1-p)H(p)}. \tag{25}$$

The hazard quantile functions of type 1 TLqE distribution is

$$H(u) = \left((1-u)\frac{1}{2\alpha\lambda(2-q)} u^{\frac{1}{\alpha}-1} \left(1-u^{\frac{1}{\alpha}}\right)^{\frac{q-3}{2(2-q)}}\right)^{-1} \tag{26}$$

with $H(0) = \infty$ and $H(1) = 0$. Plots of hazard quantile function for different values of parameters are given in figure 4.

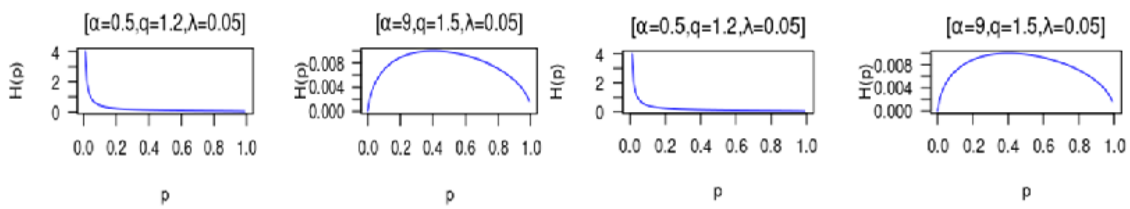

Figure 4: *Plots of hazard quantile function*

No.	Parameter region	Shape of hazard quantile function
1	$0 < \alpha < 1$ and $q < 2$	Decreasing hazard rate (DHR)
2	$\alpha > 0$ and $q < 2$	Upside-down Bathtub
3	$\alpha = 1$, $q < 2$ and $\lambda = 0$	Constant
4	$\alpha = 1$ and $q < 2$	DHR

Table 1: *Behavior of the hazard quantile function for different regions of parameter space.*

3.5. Mean residual quantile function

Another concept used in reliability is that of residual life $X_t = (X - t|X > t)$ with survival function

$$\bar{F}_t(x) = \bar{F}(t+x)/\bar{F}(t), \quad x \geq 0, 0 < t < T.$$

The mean residual life function is then

$$m(t) = E(X_t) = [\bar{F}(t)]^{-1} \int_t^{\infty} \bar{F}(x)dx.$$

Accordingly, the mean residual quantile function is defined by Nair et al. [11] as

$$M(u) = mQ(u) = (1-u)^{-1} \int_u^1 (Q(t) - Q(u))dt \tag{27}$$

which is the average remaining life beyond the $100(1-u)\%$ point of the distribution. For the type 1 TLqE distribution, $M(u)$ has the form

$$M(u) = \frac{1}{\lambda(1-q)} \left[\frac{\alpha}{1-u} B_{(1-u^{1/\alpha})} \left(\frac{(1-q)}{2(2-q)} + 1, \alpha \right) + (1 - u^{1/\alpha})^{\frac{(1-q)}{2(2-q)}} \right]. \tag{28}$$

where $B_u(a,b) = \int_0^u x^{a-1}(1-x)^{b-1}dx$ is the incomplete beta function.

3.6. Reversed hazard quantile function

The reversed hazard quantile function is (Nair et al. [11]) defined by

$$A(u) = \frac{1}{uq(u)} \tag{29}$$

and it determines the distribution through the formula

$$Q(u) = \int_0^u \frac{1}{pA(p)} dp. \tag{30}$$

For type 1 TLqE distribution,

$$A(u) = q(u) = \frac{1}{2\alpha\lambda(2-q)} u^{\frac{1}{\alpha}-2} \left(1 - u^{\frac{1}{\alpha}}\right)^{\frac{q-3}{2(2-q)}}. \tag{31}$$

4. Reliability Test Plan

Acceptance sampling plan is an inspection procedure used to determine whether to accept or reject a specific quantity of material. (See Kantam et al. [6], Rao et al. [15], Jose and Joseph [4], Joseph and Jose [5] etc.) If it is applied to a series of lots, it prescribes a procedure that will give a specified probability of accepting lots of given quality.

In statistical quality control, acceptance sampling plan is concerned with the inspection of a sample of products taken from a lot and the decision whether to accept or reject the lot based on the quality of the product. Here we discuss the reliability test, with its operating characteristic function plan for accepting or rejecting a lot where the lifetime of the product follows type 1 Topp - Leone $q-$ exponential distribution. In a life testing experiment, the procedure is to terminate the test by a predetermined time 't' and note the number of failures. If the number of failures at the end of time 't' does not exceed a given number 'c', called acceptance number then we accept the lot with a given probability of at least 'p'. But if the number of failures exceeds 'c' before

time 't', we reject the lot. For such truncated life test and the associated decision rule, we are interested to obtain the smallest sample size to make at a decision. Even though a large number of distributions belonging to Topp-Leone generated family have been developed with wide range of applications, none of these have been applied in acceptance sampling to develop reliability test plans.This motivated the present study.
Assume that the lifetime of a product T follows the type 1 Topp-Leone $q-$ exponential distribution with cumulative distribution function (cdf)

$$F(t) = \{1 - [1 - (1-q)\frac{t}{\lambda}]^{2(\frac{2-q}{1-q})}\}^{\alpha}, t > 0, \lambda, \alpha > 0, q < 2. \tag{32}$$

Let λ_0 be the required minimum average life time and the shape parameters α and q are known. Then

$$F_{TLqE}(t;\alpha,q,\lambda) \leq G_{TLqE}(t;\alpha,q,\lambda_0) \Leftrightarrow \lambda \geq \lambda_0. \tag{33}$$

A sampling plan is specified by the number of units n on test, the acceptance number c, the maximum test duration t and the minimum average lifetime represented by λ_0.

The probability of accepting a bad lot (consumer 's risk) should not exceed the value $1-p^*$, where p^* is a lower bound for the probability that a lot of true value λ below λ_0 is rejected by the sampling plan. For fixed p^* the sampling plan is characterized by $(n, c, t/\lambda_0)$. Binomial distribution can be used to find the acceptance probability for sufficiently large lots. The aim is to determine the smallest positive integer n for given values of c and t/ λ_0 such that

$$L(p_0) = \sum_{i=0}^{c} \binom{n}{i} {p_0}^i (1-p_0)^{n-i} \leq 1 - p^* \tag{34}$$

where $p_0 = F_{TLqE}(t;\alpha,q,\lambda_0)$ given by (32) which indicates failure probability before time 't' which depends only on the ratio t/ λ_0. The operating characteristic function $L(p)$ is the acceptance probability of the lot as a function of the failure probability $p(\lambda) = F_{TLqE}(t;\alpha,q,\lambda)$

The average life time of the product is increasing with λ and the failure probability $p(\lambda)$ decreases implying that the operating characteristic function is increasing in λ. The minimum values of n satisfying (34) are obtained for $\alpha = 2, q = 1.1$ and p^*=0.75, 0.95, 0.99 and t/λ_0 = 0.248, 0.361, 0.482, 0.602, 0.903, 1.204, 1.505 and 1.806. The results are displayed in Table 2.

If $p_0 = F_{TLqE}(t;\alpha,q,\lambda_0)$ is small and n is very large, the binomial probability may be approximated by Poisson probability with parameter $\theta = np_0$ so that (34) becomes

$$L_1(p_0) = \sum_{i=0}^{c} \frac{\theta^i}{i!} e^{-\theta} \leq 1 - p^* \tag{35}$$

The minimum values of n satisfying (35) are obtained for the same combination of values of α, q, p^* and t/λ_0 and are displayed in Table 3.

The operating characteristic function of the sampling plan (n,c,t/ λ_0) gives the probability L(p) of accepting the lot with

$$L(p) = \sum_{i=0}^{c} \binom{n}{i} {p_0}^i (1-p_0)^{n-i} \tag{36}$$

where $p = F(t, \lambda)$ is considered as a function of λ.

Table 2: *Minimum sample size using binomial approximation*

p^*	c	t/λ_0							
		0.248	0.361	0.482	0.602	0.903	1.204	1.505	1.806
	0	11	6	4	3	2	1	1	1
	1	22	12	8	6	4	3	3	2
	2	32	17	11	9	6	4	4	4
	3	41	22	15	11	7	6	5	5
	4	51	27	18	14	9	7	6	6
0.75	5	60	33	22	17	11	9	8	7
	6	70	38	25	19	13	10	9	8
	7	79	43	29	22	14	12	10	9
	8	88	48	32	24	16	13	11	11
	9	97	52	35	27	18	14	13	12
	10	106	57	39	30	20	16	14	13
	0	24	12	8	6	4	3	2	2
	1	38	20	13	10	6	4	4	3
	2	50	27	17	13	8	6	5	5
	3	62	33	22	16	10	8	7	6
	4	73	39	26	19	12	9	8	7
0.95	5	84	45	30	22	14	11	9	8
	6	95	51	34	25	16	12	11	10
	7	106	56	37	28	18	14	12	11
	8	116	62	41	31	20	15	13	12
	9	126	68	45	34	22	17	15	13
	10	136	73	49	37	24	18	16	14
	0	36	19	12	9	5	4	3	2
	1	52	27	18	13	13	6	5	4
	2	66	35	23	17	17	8	6	5
	3	80	42	27	20	20	9	8	7
	4	92	49	32	24	24	11	9	8
0.99	5	104	55	36	27	27	13	11	9
	6	116	61	41	30	30	14	12	11
	7	128	68	45	33	33	16	13	12
	8	139	74	49	36	36	17	15	13
	9	150	80	53	39	39	19	16	14
	10	162	86	57	42	42	21	17	16

Table 3: *Minimum sample size using poisson approximation*

p^*	c	t/λ_0							
		0.248	0.361	0.482	0.602	0.903	1.204	1.505	1.806
	0	12	7	5	4	3	2	2	2
	1	23	13	9	7	5	4	4	3
	2	33	18	13	10	7	6	5	5
	3	43	23	16	13	9	7	7	6
	4	52	29	20	15	11	9	8	7
0.75	5	62	34	23	18	12	10	9	9
	6	71	39	27	21	14	12	11	10
	7	80	44	30	23	16	13	12	11
	8	89	49	34	26	18	15	13	12
	9	99	54	37	29	20	16	15	14
	10	108	59	40	31	21	18	16	15
	0	25	14	10	8	5	4	4	4
	1	40	22	15	12	8	7	6	6
	2	52	29	20	15	11	9	8	7
	3	64	36	24	19	13	11	10	9
	4	76	42	29	22	15	13	11	11
0.95	5	87	48	33	25	17	14	13	12
	6	98	54	37	28	20	16	14	14
	7	109	60	41	32	22	18	16	15
	8	119	65	45	35	24	20	18	17
	9	130	71	49	38	26	21	19	18
	10	140	77	52	41	28	23	21	19
	0	38	21	15	11	8	7	9	6
	1	55	30	21	16	11	9	12	8
	2	70	38	26	20	14	12	15	10
	3	83	46	31	24	17	14	18	12
	4	96	53	36	28	19	16	21	13
0.99	5	109	59	41	31	22	18	24	15
	6	120	66	45	35	24	20	26	17
	7	132	72	49	38	26	22	29	18
	8	144	79	54	42	28	23	31	20
	9	156	85	58	45	31	25	34	21
	10	167	91	62	48	33	27	36	23

Table 4: *Values of the Operating Characteristic function for the sampling plan (n,c,t/λ_0)*

				λ/λ_0						
p^*	n	c	t/ λ_0	2	2.5	3	3.5	4	4.5	5
0.75	32	2	0.241	0.8821	0.9541	0.9804	0.9909	0.9954	0.9975	0.9986
	17	2	0.361	0.8665	0.9453	0.9758	0.9884	0.9940	0.9967	0.9981
	11	2	0.482	0.8594	0.9403	0.9728	0.9866	0.9930	0.9961	0.9977
	9	2	0.602	0.8113	0.9144	0.9591	0.9792	0.9888	0.9937	0.9962
	6	2	0.903	0.7429	0.8711	0.9334	0.9640	0.9797	0.9880	0.9927
	4	2	1.204	0.7926	0.8952	0.9449	0.9696	0.9825	0.9895	0.9935
	4	2	1.505	0.6387	0.7926	0.8801	0.9291	0.9568	0.9729	0.9825
	4	2	1.806	0.4854	0.6703	0.7926	0.8688	0.9158	0.9449	0.9632
0.95	50	2	0.241	0.7096	0.8689	0.9389	0.9699	0.9843	0.9913	0.9949
	27	2	0.361	0.6638	0.8383	0.9208	0.9595	0.9782	0.9877	0.9928
	17	2	0.482	0.6579	0.8306	0.9149	0.9555	0.9756	0.9861	0.9917
	13	2	0.602	0.6111	0.7968	0.8936	0.9425	0.9677	0.9812	0.9886
	8	2	0.903	0.5499	0.7448	0.8567	0.9181	0.9518	0.9708	0.9817
	6	2	1.204	0.4955	0.6944	0.8184	0.8911	0.9334	0.9582	0.9731
	5	2	1.505	0.4409	0.6410	0.7756	0.8595	0.9109	0.9423	0.9619
	5	2	1.806	0.2792	0.4788	0.6410	0.7572	0.8359	0.8883	0.9231
0.99	66	2	0.241	0.5459	0.7689	0.8838	0.9398	0.9675	0.9817	0.9892
	35	2	0.361	0.4989	0.7292	0.8572	0.9232	0.9573	0.9753	0.9852
	23	2	0.482	0.4582	0.6921	0.8307	0.9059	0.9462	0.9683	0.9806
	17	2	0.602	0.4242	0.6588	0.8055	0.8887	0.9349	0.9608	0.9757
	17	2	0.903	0.3109	0.5421	0.7126	0.8221	0.8892	0.9298	0.9546
	8	2	1.204	0.2681	0.4858	0.6596	0.7792	0.8567	0.9061	0.9374
	6	2	1.505	0.2866	0.4955	0.6612	0.7764	0.8523	0.9015	0.9334
	5	2	1.806	0.2792	0.4788	0.6410	0.7572	0.8359	0.8883	0.9231

The values of n and c are determined by means of operating characteristics (OC) function for given value of p^* and t/ λ_0 are displayed in Table 4 by considering the fact that $p = F(\frac{t}{\lambda_0}/\frac{\lambda}{\lambda_0})$.

The producer 's risk is the probability of rejecting a lot when $\lambda > \lambda_0$.We can compute the producer 's risk by first finding $p = F(t;\lambda)$ and then using the binomial distribution function. For the given value of producer 's risk say 0.05 we obtain p from the sampling plan given in Table 1 subject to the condition that

$$\sum_{i=0}^{c}\binom{n}{i}{p_0}^i(1-p_0)^{n-i} \geq 0.95 \tag{37}$$

The minimum value of λ/λ_0 satisfying (37) for the sampling plan (n,c,t/ λ_0) and for the given p^* are listed in Table 5.

4.1. Explanation of the tables

Assume that the lifetime follows type 1 TLqE distribution with α=2 and q=1.1. Suppose that the experimenter is interested in establishing that the true unknown average life is at least 1000 hours with confidence p^* = 0.75. It is desired to stop the experiment at t = 602 hours. Then, for an acceptance number c = 2, the required n is 9 (Table 2) . If during 602 hours, no more than 2 failures out of 9 are observed, then the experimenter can assert that the average life is at least 1000 hours with a confidence level of 0.75. If the Poisson approximation to binomial probability is used, the value of n is 10 (Table 3) . For this sampling plan (n = 9, c = 2, t/ λ_0=0.602) under the type 1 TLqE distribution, the operating characteristic values from Table 3 are given below.
Comparing with Reliability Test Plans for Marshall- Olkin Extended Exponential distribution (see Rao et al. [15]), for α=2, acceptance number c=9, for the specified ratio t/ λ_0=0.482 and confidence level p^*=0.75, the minimum sample size is 49 using binomial approximation, whereas for type 1 TLqE distribution it is 35. Similarly , if we are considering each value of c and each vale of t/ λ_0, the scaled termination time is uniformly smaller than those for the present reliability test plans.This improvement makes the new test plan more advantageous and helps in making optimal decisions.

Table 5: *Minimum ratio of true λ and required λ_0 for the acceptability of a lot with producer's risk of 0.05 for $\alpha = 2, q = 1.1$*

p^*	c	t/ λ_0							
		0.241	0.361	0.482	0.602	0.903	1.204	1.505	1.806
0.75	0	6.6277	6.9167	7.4538	8.5749	10.0789	9.02959	10.8794	12.9175
	1	3.3083	3.3984	3.6148	3.8266	4.5121	4.9581	6.1036	5.0452
	2	2.5084	2.6096	2.6871	2.9107	3.3462	3.1597	3.8907	4.6549
	3	2.1518	2.7074	2.3305	2.3933	2.5223	2.9791	3.0960	3.6261
	4	1.9886	2.0125	2.0707	2.2308	2.3344	2.4468	2.5469	3.0898
	5	1.8547	1.9365	1.9647	2.0913	2.2343	2.4174	2.6911	2.7487
	6	1.7632	1.8542	1.8795	1.9327	2.0847	2.1529	2.3735	2.4319
	7	1.7019	1.7454	1.8105	1.8628	1.9042	2.1529	2.1756	2.2671
	8	1.6283	1.6914	1.7471	1.7379	1.8576	2.0156	2.0471	2.4124
	9	1.5945	1.6413	1.6744	1.7379	1.8576	1.8908	2.1317	2.3129
	10	1.5624	1.5946	1.6406	1.7093	1.8130	1.8908	2.0063	2.1783
0.95	0	9.2357	10.6374	11.5682	12.5609	13.9643	16.5278	16.0189	19.2227
	1	4.3401	4.7867	4.9101	5.1202	5.8541	6.0161	7.2767	7.3615
	2	3.1957	3.3984	3.5699	3.7542	4.1055	4.3640	4.8318	5.6912
	3	2.6899	2.8940	3.0059	3.1139	3.2874	3.8654	4.2043	4.4687
	4	2.4550	2.5494	2.6871	2.7369	2.9548	3.0668	3.5219	3.7152
	5	2.2687	2.3403	2.4084	2.5180	2.6971	2.8961	3.0217	3.2294
	6	2.1163	2.1899	2.2584	2.3362	2.4825	2.5905	3.0217	3.2294
	7	2.0186	2.0965	2.1290	2.1820	2.3698	2.5389	2.7524	2.9591
	8	1.9320	2.0125	2.0707	2.0913	2.3001	2.3603	2.5195	2.7204
	9	1.8547	1.9365	1.9647	2.0490	2.1721	2.3055	2.5195	2.5581
	10	1.8074	1.8806	1.9163	2.0086	2.1721	2.2019	2.3635	2.4565
0.99	0	12.2925	13.5082	13.2554	14.7459	17.3020	20.5092	20.6597	21.2298
	1	5.3685	5.5826	5.9984	6.0065	9.0097	7.6532	8.7403	9.0242
	2	3.7269	3.9328	4.1647	4.3518	6.5277	5.3182	5.4550	5.7982
	3	3.1957	3.2789	3.4039	3.4961	5.2442	4.1826	4.8318	5.0452
	4	2.7589	2.8940	3.0059	3.1819	4.6708	3.7256	3.8335	4.2262
	5	2.5084	2.6096	2.7086	2.8206	4.2051	3.3635	3.7239	3.6261
	6	2.3569	2.4134	2.5856	2.6591	3.9886	2.9791	3.2546	3.6261
	7	2.2279	2.3172	2.4084	2.4538	3.6622	2.8961	2.9504	3.3029
	8	2.1163	2.2099	2.2865	2.3251	3.4877	2.6037	2.9504	3.0234
	9	2.0498	2.1326	2.1914	2.2308	3.3462	2.5389	2.7524	2.8362
	10	1.9886	2.0619	2.1290	2.1820	3.2308	2.5389	2.5748	2.8362

Table 6: *Values of the operating characteristic function L(p) for values of $\lambda/\ \lambda_0$.*

$\lambda/\ \lambda_0$	2	2.5	3	3.5	4	4.5	5
L(p)	0.8113	0.9144	0.9591	0.9792	0.9888	0.9937	0.9962

4.2. Application

Consider the following ordered failure times of the release of a software given in terms of hours from starting of the execution of the software up to the time at which a failure of the software is occurred (see Wood [21]). This data can be regarded as an ordered sample of size n = 9 consisting of the observations {254, 788, 1054, 1393, 2216, 2880, 3593, 4281, 5180}. Let the required average lifetime be 1000 hours and the testing time be t = 602 hours, which leads to ratio of t/ λ_0 = 0.602 with a corresponding sample size n = 9 and an acceptance number c = 2, which are obtained from Table 1 for p^* = 0.75. Therefore, the sampling plan for the above sample data is (n =9, c = 2, t/ λ_0 = 0.602). Based on the observations, we have to decide whether to accept the product or reject it. We accept the product only if the number of failures before 602 hours is less than or equal to 2. However, the confidence level is assured by the sampling plan only if the given life times follow type 1 TLqE distribution. In order to confirm that the given sample is generated by lifetimes following the type 1 TLqE distribution, we have compared the sample quantiles and the corresponding population quantiles and found a satisfactory agreement. Thus, the adoption of the decision rule of the sampling plan seems to be justified. In the sample of 9 units, there is only one failure at 254 hours before t = 602 hours. Therefore we accept the product.

5. Numerical Illustration

The data consists of the number of successive failure for the air conditioning system reported of each member in a fleet of thirteen Boeing 720 jet air planes. The pooled data with 214 observations was considered by Proschan [14]. 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71.

Table 7: *The values of estimated parameters of Dataset 1*

The model	Estimate and Standard Error (in paranthesis) of Dataset 1
Type 1 TLqE	α=1.2150 (0.2422),λ=0.0149 (0.0079), q=1.3883(0.1279)
TLE	α=0.9036 (0.0884),λ=0.0052 (0.0005)
E	λ =0.0112 (0.0008)

Table 8: *Goodness of fit of collection of different distributions for the data set.*

	AIC	CAIC	BIC	HQIC	A^*	w^*	K-S	p value	-log L
Type 1 TLqE	1964.01	1964.18	1973.60	1967.92	0.46	0.06	0.04	0.78	979.02
TLE	1968.37	1968.44	1974.74	1970.95	1.37	0.25	0.07	0.28	982.18
E	1967.47	1967.49	1970.65	1968.76	2.08	0.40	0.08	0.13	982.73

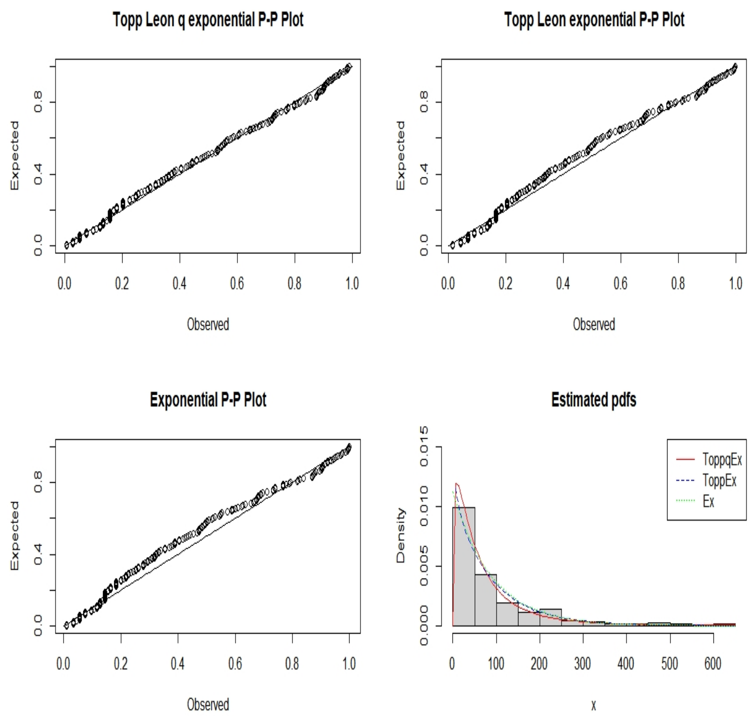

Figure 5: *The P - P plot and estimated pdf of data set.*

According to the Table 8 the type 1 TLqE model is more appropriate as compared to the TLE and exponential distribution.

6. Conclusion and future work

Introduced a quantile function associated with the type 1 Topp-Leone q-exponential distribution. The estimation of parameters of the model using $L-$moments is studied. Also, a reliability test plan was derived on the basis that the lifetime distribution of the test item follows the type 1 TLqE distribution. Besides, we find the minimum sample size needed for the acceptance or rejection of a lot based on percentiles. Some useful tables were provided and applied to establish the test plan. The new test plan is applied to illustrate its use in industrial contexts. We proved empirically the importance and flexibility of the new model in the model building by using a real data set. One can develop a parallel theory for type 2 TLqE distribution using the type 2 beta generated form given in Sebastian et al. [18].

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