Single Server Retrial Queueing System with Catastrophe

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Abstract

The present paper analyses a retrial queueing system with Catastrophe. Primary and secondary customers follow Poisson processes. Inter arrival and service times are Exponentially distributed. Catastrophe occurs on a busy server and follows Poisson process. The server is sent for repair after its failure. The repair times are also Exponentially distributed. Steady state and time dependent solutions for number of customers in the system when the server is idle or busy are obtained. The probability of the server being under repair is obtained. Some performance measures are also evaluated. Numerical results are obtained and represented graphically.

Keywords: Queueing, Retrial, Catastrophe, Repair .

1. Introduction

We have seen in many real life situations that sometimes a customer on arrival does not get the service instantly . So he tries for the service after some random amount of time which is popularly known as retrial. Retrial queue is a model of this kind of system if the server is not free, the customer leaves the service area and joins the virtual queue known as orbit. Thereafter it retries from the orbit after a random amount of time to get service. The queueing systems with these repeated attempts have been used in many field such as telecommunication, computer networks, data transmission, etc. The analysis of such systems lead to the identificatio of a new class of queueing systems known as retrial queueing systems.
For example: In call centers where when customers call, if they are able to reach a live negotiator immediately , they are answered else they repeat the call after a couple of minutes.
The work on retrial queues in its early stages can be found in [1]. In [2] the author discussed some important single server retrial queueing models and represented analytic results. In [3] the single server retrial queue with finit number of sources is analyzed and customer 's arrival distribution, busy period and waiting time process is established. Time dependent probabilities for exact number of arrivals and departures from the system when the server is free or busy are obtained in [4]. An explanation of the retrial queueing system is shown in the following diagram.

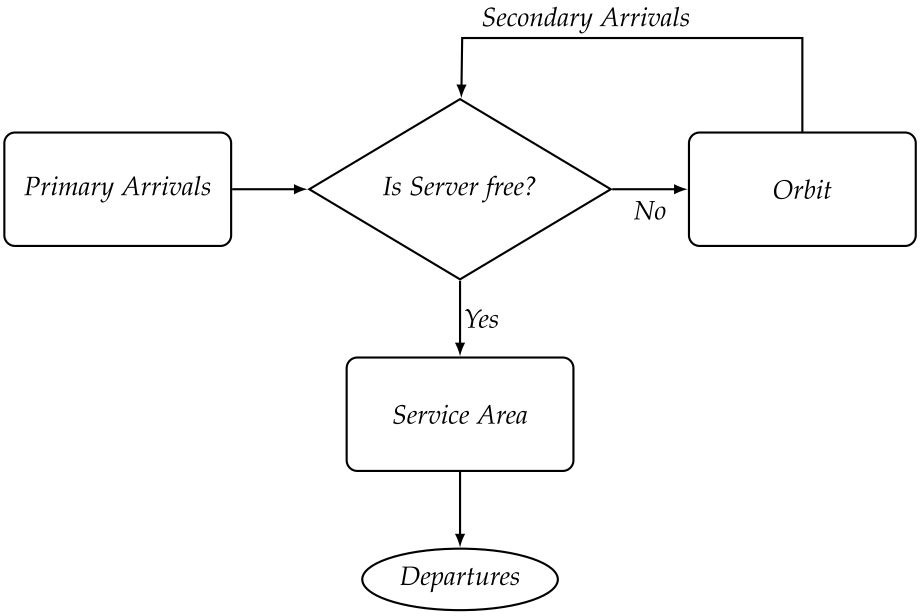

Figure 1: *Basic Structure of Retrial Queueing System*

Recently, a new concept of catastrophe has been introduced in queueing systems. The word catastrophe refers to a sudden, unexpected failure of a machine, computer network, electronic system, communication system, etc. Catastrophes occur randomly, eradicating all customers present in the system and temporarily inactivating the service facilities. Catastrophe resets the system from current state to zero state at random time intervals. Catastrophe may come from outside the system or from another service station. Retrial queueing models with catastrophe have applications in call centers, computer networks and in telecommunication systems that depend on satellites. In population dynamics, catastrophe can be considered as the natural disasters such as floods storms, etc. On the other hand when we talk of catastrophe in queueing systems, it deletes all the customers present there and causes breakdown of the server. A basic example of retrial queueing system with catastrophe is in call centers where if customers are able to reach a live negotiator immediately upon making a call, they are answered else they repeat the call after a couple of minutes. Furthermore, loss of all the customers and inactivation of the server will take place as a result of an incidental power failure or a virus attack. The diagram below shows the retrial queue with catastrophe.

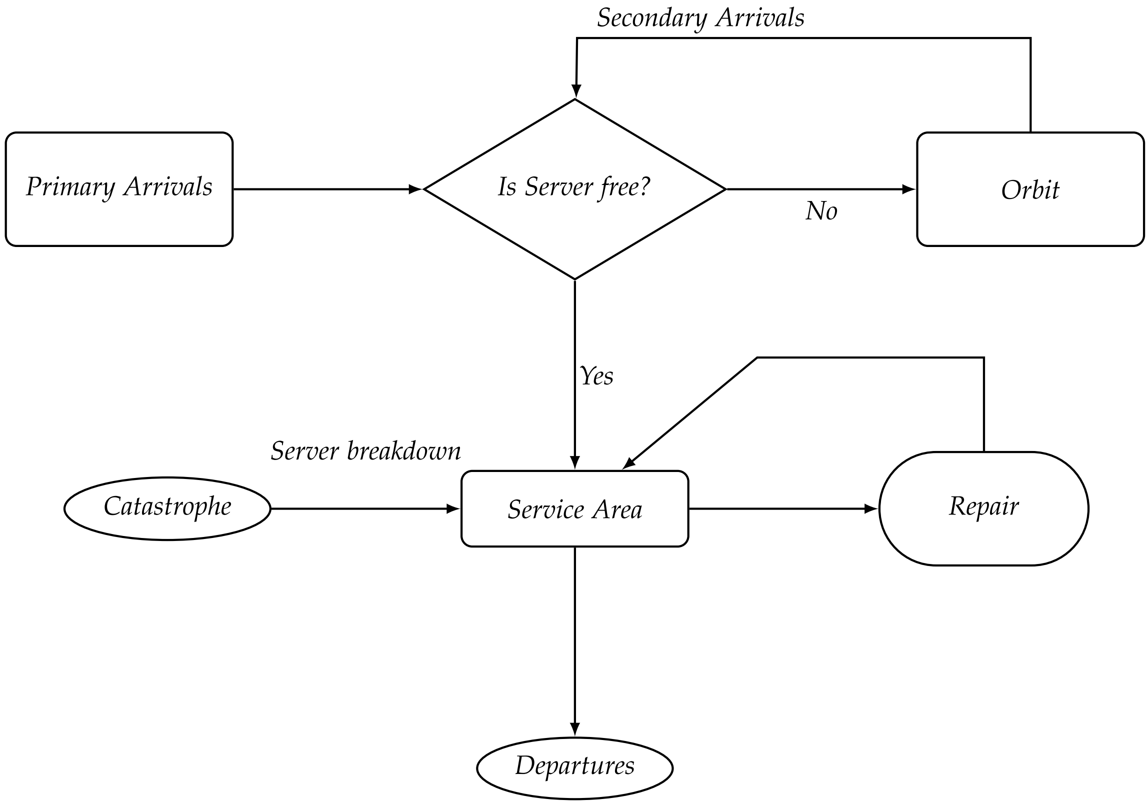

Figure 2: *Basic Structure of a Retrial Queueing System with Catastrophe*

The initial work on catastrophe occurring in a simple Markovian queue could be referred from [5] and [6]. In [7] the author discussed mean queue length and the asymptotic behavior of the probability of server being free. Also the steady state probabilities are obtained. The transient solution for the system with server failure and non-zero repair time on M/M/1 queueing system with catastrophe is obtained by [8] .
In this research paper when a server fails, it is sent for repair immediately and after getting repaired, the server comes back to its working position and the system becomes ready to accept new customers.
The paper has been organized in the following sections.
In section 2 the complete mathematical description of the model is defined Also, the difference-differential equations are derived in this section. Steady-state solution of the model along with the expected number of customers in the system is given in section 3. In section 4 the transient state probabilities and the probability of server being under repair are obtained. In section 5 verificatio of results is given. The numerical results are obtained and represented graphically in section 6. In section 7 the busy period probabilities of system and the server are obtained numerically and presented graphically . Section 8 discusses the conclusion and finall the references are listed.

2. Model Description

In this paper, a single server retrial queueing system with catastrophe is considered. In this system, the customers arrive according to Poisson process. On arrival if a customer find the server busy, he joins the orbit and retries from the orbit. These retrials are considered to be secondary arrivals. Catastrophe occurs on a busy server following Poisson process. It is assumed that the catastrophe occurs only when the system is non-empty and the server is busy. It has no effect on the system when the system is empty. Catastrophe makes system empty and also causes breakdown of the server. Once the system becomes empty and the server breaks down, it is sent for repair immediately . Further, it is assumed that during the repair time no arrival can take place.

Assumptions : The assumptions underlying the model are listed below.

- Arrival Process: The primary customers arrive at the system according to Poisson process with mean arrival rate λ.
- Retrial Process: The secondary customers arrive at the system according to Poisson process with mean arrival rate θ.
- Service Process: The service times are Exponentially distributed with parameter μ.
- Catastrophe: Catastrophe occur at the system according to Poisson process with rate ξ.
- Repair: The repair time is Exponentially distributed with parameter τ.

The input fl w of primary calls, intervals between repetitions, service times, catastrophes and repair times are statistically independent.

Shift operator E is used to increase the value of argument x by h so that $Ef(x) = f(x+h)$, $E^2 f(x) = E[Ef(x)] = E[f(x+h)] = f(x+2h)$ and so on. Here h is the equal interval of spacing. Laplace transformation $\bar{f}(s)$ of $f(t)$ is given by:

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt, \quad Re(s) > 0;$$

The Laplace inverse of

$$\frac{Q(p)}{P(p)} = \sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{t^{m_k - l} e^{a_k t}}{(m_k - l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}} \left(\frac{Q(p)}{P(p)} \right) (p - a_k)^{m_k} \quad \forall p = a_k, \quad a_i \neq a_k \; for \; i \neq k$$

where

$$P(p) = (p - a_1)^{m_1} (p - a_2)^{m_2} \ldots\ldots (p - a_n)^{m_n}$$

Q(p) is a polynomial of degree $< m_1 + m_2 + m_3 + \ldots\ldots\ldots + m_n - 1$

$$If \;\; L^{-1}\{f(s)\} = F(t) \;\; and \;\; L^{-1}\{g(s)\} = G(t) \;\; then$$
$$L^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)du = F * G$$

$F * G$ is called the convolution of F and G.

2.1. Notations

$P_{n,0}(t)$ = Probability that there are n customers in the system at time t and the server is free.
$P_{n,1}(t)$ = Probability that there are n customers in the system at time t and the server is busy.
$Q(t)$ = Probability that the server is under repair at time t.
$P_n(t)$ = Probability that there are n customers in the system at time t.

$$P_n(t) = P_{n,0}(t) + P_{n,1}(t) \;\; \forall \; n \geq 0;$$
$$and \qquad P_{n,1}(t) = 0 \;\; for \; n = 0;$$

Initially

$$P_{0,0}(0) = 1; \qquad P_{n,0}(0) = 0, \; n \neq 0; \qquad P_{n,1}(0) = 0, \;\; \forall n; \qquad Q(0) = 0 \,;$$

2.2. The Difference-Differential equations governing the system are:

$$\frac{d}{dt}P_{n,0}(t) = -(\lambda + n\theta)P_{n,0}(t) + \mu P_{n+1,1}(t) \qquad n \geq 1 \qquad (1)$$

$$\frac{d}{dt}P_{0,0}(t) = -\lambda P_{0,0}(t) + \tau Q(t) + \mu P_{1,1}(t) \qquad (2)$$

$$\frac{d}{dt}P_{n,1}(t) = -(\lambda + \mu + \xi)P_{n,1}(t) + \lambda P_{n-1,0}(t) + \lambda P_{n-1,1}(t)(1 - \delta_{n,1}) + n\theta P_{n,0}(t) \quad n \geq 1 \qquad (3)$$

$$\frac{d}{dt}Q(t) = -\tau Q(t) + \xi \sum_{n=1}^{\infty} P_{n,1}(t) \qquad (4)$$

wher e

$$\delta_{n,1} = \begin{cases} 1, \text{when } n = 1 \\ 0, \text{other wise} \end{cases}$$

3. The steady-state difference equations governing the system

Taking $P_n(t) \to P_n$ and $\frac{d}{dt}P_n(t) \to 0$ as $t \to \infty$

$$(\lambda + n\theta)P_{n,0} = \mu P_{n+1,1} \qquad n \geq 1 \qquad (5)$$

$$\lambda P_{0,0} = \mu P_{1,1} + \tau Q \qquad (6)$$

$$(\lambda + \mu + \xi)P_{n,1} = \lambda P_{n-1,0} + \lambda P_{n-1,1}(1 - \delta_{n,1}) + n\theta P_{n,0} \qquad n \geq 1 \qquad (7)$$

$$\tau Q = \xi \sum_{n=1}^{\infty} P_{n,1} \qquad (8)$$

wher e

$$\delta_{n,1} = \begin{cases} 1, \text{when } n = 1 \\ 0, \text{other wise} \end{cases}$$

3.1. Steady-state solution of the problem

Using $Ef(x) = f(x+1)$, equations (5) and (7) are repr esented as

$$[(\lambda + (n+1)\theta)E]P_{n,0} - \mu E^2 P_{n,1} = 0 \qquad n \geq 2 \qquad (9)$$

$$[\lambda + ((n+1)\theta)E]P_{n,0} + [\lambda - (\lambda + \mu + \xi)E]P_{n,1} = 0 \qquad n \geq 2 \qquad (10)$$

In order to fin the solution of the above system of equations, we need

$$E[\mu(n+1)\theta E^2 - (\lambda(\lambda + (n+1)\theta + \xi) + \theta((n+1)(\mu + \xi)))E + \lambda(\lambda + (n+1)\theta)] = 0 \quad n \geq 2 \qquad (11)$$

The values of $P_{n,0}$ and $P_{n,1}$ are giv en by

$$P_{n,0} = \sum_{i=0}^{2} a_i z_i^n \qquad n \geq 2$$

$$P_{n,1} = \sum_{i=0}^{2} b_i z_i^n \qquad n \geq 2$$

where z_0, z_1, z_2 are the roots of (11) with $z_0 = 0$ and a_i,b_i, $i = 0, 1, 2$ are the arbitrary constants to be evaluated. Other two roots of the equation (11) are given by

$$z_1, z_2 = \frac{1}{2\mu(\theta + n\theta)} \Bigg\{ \Big(\lambda(\lambda + (n+1)\theta + \xi) + \theta((n+1)(\mu + \xi)) \Big) \pm \Big[2\theta^2(\mu\xi - \mu\lambda + \xi\lambda) + 2n\theta^2(\mu^2 + \xi^2 + \lambda^2 + 2\mu\xi - 2\mu\lambda + 2\xi\lambda) + 2\lambda^2(2\theta\xi - \mu\theta) + 2n\lambda^2(2\theta\xi + \theta\lambda - \theta\mu) + 2\theta\mu\xi\lambda(n+1) + \lambda^2(\xi^2 + \theta^2 + \lambda^2) + 2\lambda^3(\theta + \xi) + 2\xi^2\theta\lambda(n+1) + \theta^2(\mu^2 + \xi^2) \Big]^{1/2} \Bigg\} \quad (12)$$

Clearly the root z_1 is always greater than 1 and the root z_2 is always less than 1. For the convergence of a solution, a root greater than or equal to 1 must be rejected. So when $z_i \geq 1$, a_i and b_i are taken as equal to zero.
Since here z_1 is greater than 1, so we take $a_1 = b_1 = 0$
As $z_0 = 0$ and $a_1 = b_1 = 0$, therefore the values of $P_{n,0}$ and $P_{n,1}$ are given by

$$P_{n,0} = a_2 z_2^n \qquad n \geq 2 \quad (13)$$
$$P_{n,1} = b_2 z_2^n \qquad n \geq 2 \quad (14)$$

From equations (5) and (7) for n=1 the probabilities $P_{1,0}$ and $P_{1,1}$ are given by

$$P_{1,0} = \frac{\mu}{(\lambda + \theta)} P_{2,1} = \frac{\mu}{(\lambda + \theta)} b_2 z_2^2 \quad (15)$$
$$P_{1,1} = \frac{\lambda}{\lambda + \mu + \xi} P_{0,0} + \frac{\mu\theta}{(\lambda + \mu + \xi)(\lambda + \theta)} b_2 z_2^2 \quad (16)$$

By substituting the above values in equation (7) for $n = 2$ and for $n = 3$, we have

$$b_2 z_2^2 = \frac{\mu\lambda}{(\lambda + \theta)(\lambda + \mu + \xi)} \left(1 + \frac{\theta}{\lambda + \mu + \xi}\right) b_2 z_2^2 + \frac{\lambda^2}{(\lambda + \mu + \xi)^2} P_{0,0} + \frac{2\theta}{\lambda + \mu + \xi} a_2 z_2^2 \quad (17)$$

$$b_2 z_2^3 = \frac{\lambda}{\lambda + \mu + \xi} \left(1 + \frac{2\theta}{\lambda + \mu + \xi}\right) a_2 z_2^2 + \frac{\mu\lambda^2}{(\lambda + \theta)(\lambda + \mu + \xi)^2} \left(1 + \frac{\theta}{\lambda + \mu + \xi}\right) b_2 z_2^2 + \frac{3\theta}{\lambda + \mu + \xi} a_2 z_2^3 + \left(\frac{\lambda}{\lambda + \mu + \xi}\right)^3 P_{0,0} \quad (18)$$

On solving equations (17) and (18) we get

$$a_2 = \frac{\left(\frac{\lambda}{\lambda+\mu+\xi}\right)^2 \left[\left(\frac{\lambda}{\lambda+\mu+\xi}\right) B - A\right]}{z_2^2 \left[2AC - B\left(\frac{\lambda}{\lambda+\mu+\xi}(1 + 2C) + 3z_2 C\right)\right]} P_{0,0} \quad (19)$$

$$b_2 = \frac{1}{B z_2^2} \left\{ \frac{2C\left(\frac{\lambda}{\lambda+\mu+\xi}\right)^2 \left[B\left(\frac{\lambda}{\lambda+\mu+\xi}\right) - A\right]}{2AC - B\left(\frac{\lambda}{\lambda+\mu+\xi}(1 + 2C) + 3z_2 C\right)} + \left(\frac{\lambda}{\lambda + \mu + \xi}\right)^2 \right\} P_{0,0} \quad (20)$$

and the value of $P_{0,0}$ can be found by using the relation

$$P_{0,0} + \sum_{n=1}^{\infty} (P_{n,0} + P_{n,1}) + Q = 1$$

After simplificatio

$$
\begin{aligned}
P_{0,0} = & \Bigg\{ 1 + \left(1+\frac{\xi}{\tau}\right)\left(\frac{\lambda}{\lambda+\mu+\xi}\right) + \left(\frac{\mu}{\lambda+\theta} + \left(1+\frac{\xi}{\tau}\right)\left(\frac{\mu\theta}{(\lambda+\theta)(\lambda+\mu+\xi)}\right)\right) \\
& \left[\frac{2C\left(\frac{\lambda}{\lambda+\mu+\xi}\right)^2\left[B\left(\frac{\lambda}{\lambda+\mu+\xi}\right)-A\right]}{2ABC - B^2\left(\frac{\lambda}{\lambda+\mu+\xi}(1+2C)+3z_2C\right)} + \frac{1}{B}\left(\frac{\lambda}{\lambda+\mu+\xi}\right)^2\right] \\
& + \frac{1}{1-z_2}\left[\frac{\left(\frac{\lambda}{\lambda+\mu+\xi}\right)^2\left[\left(\frac{\lambda}{\lambda+\mu+\xi}\right)B - A\right]}{\left(2AC - B\left(\frac{\lambda}{\lambda+\mu+\xi}(1+2C)+3z_2C\right)\right)}\right] \\
& + \frac{1+\frac{\xi}{\tau}}{1-z_2}\left[\frac{2C\left(\frac{\lambda}{\lambda+\mu+\xi}\right)^2\left[B\left(\frac{\lambda}{\lambda+\mu+\xi}\right)-A\right]}{2ABC - B^2\left(\frac{\lambda}{\lambda+\mu+\xi}(1+2C)+3z_2C\right)} + \frac{1}{B}\left(\frac{\lambda}{\lambda+\mu+\xi}\right)^2\right]\Bigg\}^{-1} \qquad (21)
\end{aligned}
$$

wher e

$$
\begin{aligned}
A &= z_2 - \frac{\mu\lambda^2}{(\lambda+\theta)(\lambda+\mu+\xi)^2}\left(1+\frac{\theta}{\lambda+\mu+\xi}\right) \\
B &= 1 - \frac{\mu\lambda}{(\lambda+\theta)(\lambda+\mu+\xi)}\left(1+\frac{\theta}{\lambda+\mu+\xi}\right) \\
C &= \frac{\theta}{\lambda+\mu+\xi}
\end{aligned}
$$

Hence by using the values of a_2, b_2 and $P_{0,0}$, the probabilities $P_{n,0}$, $P_{n,1}$ and Q are completely known for various values of n.

3.2. Expected number of customers in the system

Expected number of customers in the system is giv en by

$$L_s = L_{s,0} + L_{s,1}$$

wher e
$L_{s,0}$ denotes the expected number of customers in the system when the ser ver is free.
Ther efor e, by definitio of expectation

$$
\begin{aligned}
L_{s,0} &= \sum_{n=1}^{\infty} nP_{n,0} \\
&= P_{1,0} + \sum_{n=2}^{\infty} nP_{n,0} \\
&= P_{1,0} + a_2 \sum_{n=2}^{\infty} nz_2^n \\
&= P_{1,0} + a_2 z_2\left[\frac{1}{(1-z_2)^2} - 1\right] \qquad (22)
\end{aligned}
$$

Similarly
$L_{s,1}$ denotes the expected number of customers in the system when the ser ver is busy.

$$L_{s,1} = P_{1,1} + b_2 z_2\left[\frac{1}{(1-z_2)^2} - 1\right] \qquad (23)$$

By using (24) and (25)

$$L_s = P_{1,0} + P_{1,1} + (a_2 + b_2)\left(\frac{1}{(1-z_2)^2} - 1\right)z_2$$
$$= P_1 + (a_2 + b_2)\left(\frac{1}{(1-z_2)^2} - 1\right)z_2 \tag{24}$$

4. Laplace transform of difference-differential equations

Using the Laplace transfor mation $\bar{f}(s)$ of $f(t)$ giv en by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt, \quad Re(s) > 0;$$

in the equations (1)-(4) along with the initial conditions, we have

$$(s + \lambda + n\theta)\bar{P}_{n,0}(s) = \mu \bar{P}_{n+1,1}(s) \qquad n \geq 1 \tag{25}$$
$$(s + \lambda)\bar{P}_{0,0}(s) - 1 = \tau \bar{Q}(s) + \mu \bar{P}_{1,1}(s) \tag{26}$$
$$(s + \lambda + \mu + \xi)\bar{P}_{n,1}(s) = \lambda \bar{P}_{n-1,0}(s) + \lambda \bar{P}_{n-1,1}(s)(1 - \delta_{n,1}) + n\theta \bar{P}_{n,0}(s) \quad n \geq 1 \tag{27}$$
$$(s + \tau)\bar{Q}(s) = \xi \sum_{n=1}^{\infty} \bar{P}_{n,1}(s) \tag{28}$$

wher e

$$\delta_{n,1} = \begin{cases} 1, \text{when } n = 1 \\ 0, \text{other wise} \end{cases}$$

4.1. Transient solution of the Problem

Solving equations (25)-(28) recursiv ely, we have

$$\bar{P}_{0,0}(s) = \frac{1}{(s+\lambda)} + \frac{\tau}{(s+\lambda)}\bar{Q}(s) + \frac{\mu}{(s+\lambda)}\bar{P}_{1,1}(s) \tag{29}$$
$$\bar{P}_{n,0}(s) = \frac{\mu}{s+\lambda+n\theta}\left[\sum_{k=1}^{n+1}\left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^{n-k+1}\eta_k'(s)\bar{P}_{k,0}(s) + \left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^{n}\bar{P}_{1,1}(s)\right] \qquad n \geq 1 \tag{30}$$

wher e

$$\eta_k'(s) = \begin{cases} 1 & \text{if } k = 1 \\ 1 + \frac{k\theta}{s+\lambda+\mu+\xi} & \text{if } k = 2 \text{ } to \text{ } n \\ \frac{k\theta}{s+\lambda+\mu+\xi} & \text{if } k = n+1 \end{cases}$$

$$\bar{P}_{1,1}(s) = \frac{\lambda}{s+\lambda+\mu+\xi}\left[\frac{1}{(s+\lambda)} + \frac{\tau}{(s+\lambda)}\bar{Q}(s) + \frac{\mu}{(s+\lambda)}\bar{P}_{1,1}(s)\right] + \frac{\theta}{s+\lambda+\mu+\xi}\bar{P}_{1,0}(s) \tag{31}$$
$$\bar{P}_{n,1}(s) = \sum_{k=1}^{n}\left[\left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^{n-k}\eta_k'(s)\bar{P}_{k,0}(s)\right] + \left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^{n-1}\bar{P}_{1,1}(s) \qquad n \geq 2 \tag{32}$$

wher e

$$\eta_k'(s) = \begin{cases} 1 & \text{if } k = 1 \\ 1 + \frac{k\theta}{s+\lambda+\mu+\xi} & \text{if } k = 2 \text{ } to \text{ } n-1 \\ \frac{k\theta}{s+\lambda+\mu+\xi} & \text{if } k = n \end{cases}$$

$$\bar{Q}(s) = \frac{\xi}{(s+\tau)} \sum_{n=1}^{\infty} \bar{P}_{n,1}(s) \tag{33}$$

Taking the Inverse Laplace transfor m of equations of (29)-(33), we have

$$P_{0,0}(t) = e^{-\lambda t} + \tau e^{-\lambda t} * Q(t) + \mu e^{-\lambda t} * P_{1,1}(t) \tag{34}$$

$$\begin{aligned} P_{n,0}(t) = {} & \mu\lambda^n e^{-(\lambda+n\theta)t}\left[\frac{1}{(\mu+\xi)^n} - e^{-(\mu+\xi)t}\sum_{r=0}^{n-1}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{n-r}}\right] * P_{1,0}(t) + e^{-(\lambda+n\theta)t}\sum_{k=2}^{n}\mu\lambda^{n-k+1} \\ & \left[\frac{1}{(\mu+\xi)^{n-k+1}} - e^{-(\mu+\xi)t}\sum_{r=0}^{n-k}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{n-k-r+1}}\right] * P_{k,0}(t) + e^{-(\lambda+n\theta)t}\sum_{k=2}^{n}(\mu k\theta)\lambda^{n-k+1} \\ & \left[\frac{1}{(\mu+\xi)^{n-k+2}} - e^{-(\mu+\xi)t}\sum_{r=0}^{n-k+1}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{n-k-r+2}}\right] * P_{k,0}(t) + e^{-(\lambda+n\theta)t}(n+1)\mu\theta \\ & \left[\frac{1}{(\mu+\xi)} - \frac{e^{-(\mu+\xi)t}}{(\mu+\xi)}\right] * P_{n+1,0}(t) + \mu\lambda^n e^{-(\lambda+n\theta)t}\left[\frac{1}{(\mu+\xi)^n} - e^{-(\mu+\xi)t}\sum_{r=0}^{n-1}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{n-r}}\right] \\ & * P_{1,1}(t) \qquad n \geq 1 \end{aligned} \tag{35}$$

$$\begin{aligned} P_{n,1}(t) = {} & \lambda^{n-1}e^{-(\lambda+\mu+\xi)t}\frac{t^{n-2}}{(n-2)!} * P_{1,0}(t) + e^{-(\lambda+\mu+\xi)t}\sum_{k=2}^{n-1}\lambda^{n-k}\frac{t^{n-k-1}}{(n-k-1)!} * P_{k,0}(t) + e^{-(\lambda+\mu+\xi)t} \\ & \sum_{k=2}^{n-1}(k\theta)\lambda^{n-k}\frac{t^{n-k}}{(n-k)!} * P_{k,0}(t) + n\theta e^{-(\lambda+\mu+\xi)t} * P_{n,0}(t) + \lambda^{n-1}e^{-(\lambda+\mu+\xi)t}\frac{t^{n-2}}{(n-2)!} * P_{1,1}(t) \\ & n \geq 2 \end{aligned} \tag{36}$$

$$P_{1,1}(t) = \lambda e^{-(\lambda+\mu+\xi)t} * P_{0,0}(t) + \theta e^{-(\lambda+\mu+\xi)t} * P_{1,0}(t) \tag{37}$$

$$Q(t) = \xi e^{-\tau t}\sum_{n=1}^{\infty} P_{n,1}(t) \tag{38}$$

5. Verification of Results

- Summing equations (29)-(33) over n we get,

$$\sum_{n=0}^{\infty}[\bar{P}_{n,0}(s) + \bar{P}_{n,1}(s)] + \bar{Q}(s) = \frac{1}{s}$$

and hence

$$\sum_{n=0}^{\infty}[P_{n,0}(t) + P_{n,1}(t)] + Q(t) = 1$$

which is a verificatio for our results.

6. Numerical Solution and Graphical Representation

The Numerical results are generated using MATLAB programming for the case $\rho = (\frac{\lambda}{\mu}) = 0.8$, $\eta = (\frac{\theta}{\mu}) = 0.9$, $\tau' = (\frac{\tau}{\mu}) = 0.6$, $\xi' = (\frac{\xi}{\mu}) = 0.4$. In following tables, we list some significan probabilities at various time instants.

Table 1: *At time t = 1*

t	$P_{0,0}$	$P_{1,0}$	$P_{2,0}$	$P_{3,0}$	$P_{4,0}$	$P_{5,0}$	$P_{1,1}$	$P_{2,1}$	$P_{3,1}$
1	0.5904	0.0216	0.0033	0.0004	0.0001	0	0.226	0.0701	0.0161

$P_{4,1}$	$P_{5,1}$	Sum
0.0029	0.0005	0.9314

Table 2: *At time t = 5*

t	$P_{0,0}$	$P_{1,0}$	$P_{2,0}$	$P_{3,0}$	$P_{4,0}$	$P_{5,0}$	$P_{1,1}$	$P_{3,1}$	$P_{4,1}$
5	0.3644	0.0556	0.0187	0.0069	0.0032	0	0.1572	0.0489	0.0247

$P_{5,1}$	$Q(t)$	Sum
0.0145	0.2126	0.9067

Table 3: *At time t = 15*

t	$P_{0,0}$	$P_{1,0}$	$P_{3,0}$	$P_{4,0}$	$P_{5,0}$	$P_{1,1}$	$P_{2,1}$	$P_{3,1}$	$P_{4,1}$
15	0.3571	0.0528	0.0078	0.0041	0	0.1514	0.0897	0.0491	0.0274

$P_{5,1}$	$Q(t)$	Sum
0.018	0.2237	0.9811

Table 4: *At time t = 25*

t	$P_{0,0}$	$P_{1,0}$	$P_{2,0}$	$P_{3,0}$	$P_{4,0}$	$P_{5,0}$	$P_{1,1}$	$P_{2,1}$	$P_{3,1}$
25	0.3571	0.0528	0.0189	0.0078	0.0041	0	0.1514	0.0897	0.0491

$P_{4,1}$	$Q(t)$	Sum
0.0274	0.2237	0.982

Table 5: *At time t = 40*

t	$P_{0,0}$	$P_{1,0}$	$P_{2,0}$	$P_{3,0}$	$P_{5,0}$	$P_{1,1}$	$P_{2,1}$	$P_{3,1}$	$P_{4,1}$
40	0.3571	0.0528	0.0189	0.0078	0	0.1514	0.0897	0.0491	0.0274

$P_{5,1}$	$Q(t)$	Sum
0.018	0.2237	0.9959

In the follo wing figu es, the pr obabilities ar e graphed against time. Figur e 3 to figu e 5 are plotted for the case $\rho = 0.8$, $\eta = 0.9$, $\tau' = 0.6$, $\xi' = 0.4$.

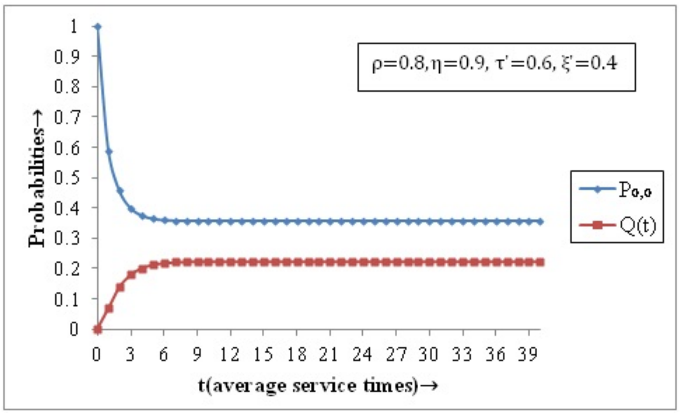

Figure 3: *Probabilities $P_{0,0}$ and $Q(t)$ against average service times t*

In figu e 3, the probabilities $P_{0,0}$ and $Q(t)$ (probability of the server being under repair) are plotted against time t. From the graph, we observe that the probability $P_{0,0}$ decreases rapidly from its initial value 1 at $t = 0$ and thereafter becomes steady. On the other hand, the probability $Q(t)$ increases in the beginning and then becomes stable.

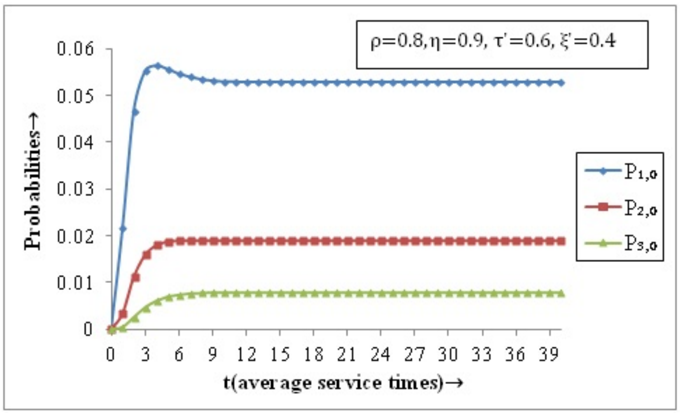

Figure 4: *Probabilities $P_{1,0}$, $P_{2,0}$ and $P_{3,0}$ against average service times t*

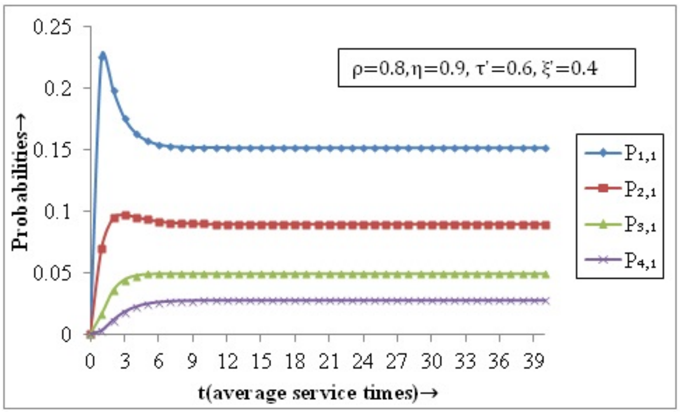

Figure 5: *Probabilities $P_{1,1}$, $P_{2,1}$, $P_{3,1}$ and $P_{4,1}$ against average service times t*

The probabilities $P_{1,0}$, $P_{2,0}$, $P_{3,0}$ are plotted against time t in figu e 4 and the probabilities $P_{1,1}$, $P_{2,1}$, $P_{3,1}$, $P_{4,1}$ are plotted against time t in figu e 5. The graphs clearly indicate that all the probabilities increase rapidly in the beginning, gradually decline to a certain extent and finall become stable. Also, it can be seen from figu e 4 that the probability of more customers in the system achieves a lower highest value. In addition, figu e 5 shows that the probability of server being busy attains a lower steady value when there are more customers in the system.

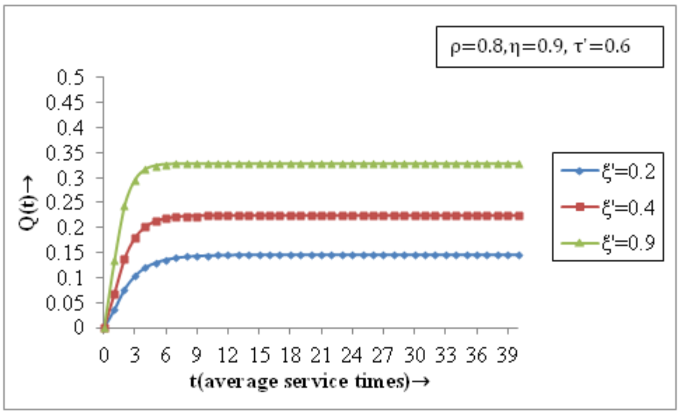

Figure 6: *Effect of change in* ξ' *on probability* $Q(t)$

In figu e 6, we study the effect of change in ξ'(catastrophe rate per unit service time) on the probability $Q(t)$(probability of server being under repair). From the graph it can be seen that whenever the catastrophe rate per unit service time increases, the probability $Q(t)$ also increases which is as desired.

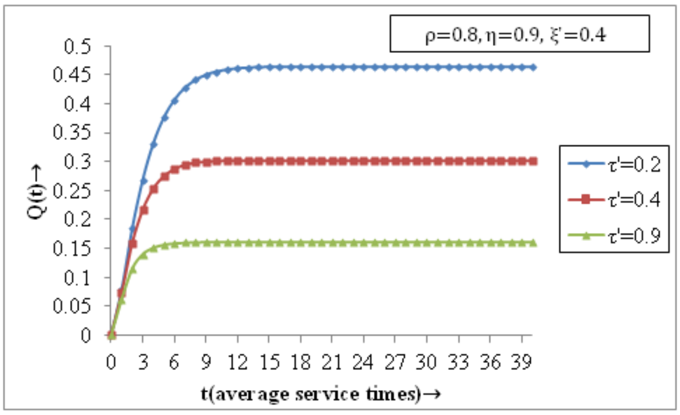

Figure 7: *Effect of change in* τ' *on probability* $Q(t)$

In figu e 7, the effect of change in τ'(repair rate per unit service time) on the probability $Q(t)$ is studied. From the graph it is clearly visible that whenever the repair rate per unit service time increases, the probability $Q(t)$ decreases.

7. Busy Period Probabilities

This section discusses the busy period probabilities of server and system.
In terms of probability, busy servers are determined as follows:

$$P(Server\ is\ busy) = \sum_{n\geq 1} P_{n,1}(t) \tag{39}$$

And busy systems are determined as follows:

$$P(System\ is\ busy\ ) = \sum_{n>0} (P_{n,0}(t) + P_{n,1}(t)) + Q(t) \tag{40}$$

7.1. Numerical and Graphical Representation of Busy Period Probabilities

The numerical results are obtained using MATLAB programming and following [9]. The Probabilities of system busy and server busy are obtained for different values of ρ keeping the other parameters constant and are presented in the table below:

Table 6: *Probabilities of system busy and server busy for different values of* ρ

	Probability(System Busy)			Probability(Server Busy)		
t	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$
0	0	0	0	0	0	0
1	0.1916	0.2711	0.341	0.1833	0.2548	0.3156
2	0.2361	0.3283	0.4042	0.212	0.2853	0.3434
3	0.2514	0.3463	0.4203	0.2169	0.2884	0.3432
4	0.2577	0.3525	0.4235	0.2178	0.288	0.3407
5	0.2604	0.3544	0.4229	0.2179	0.2872	0.3387
6	0.2615	0.3548	0.4217	0.2177	0.2865	0.3373
7	0.262	0.3547	0.4207	0.2176	0.2861	0.3366
8	0.2621	0.3546	0.4201	0.2175	0.2859	0.3361
9	0.2622	0.3545	0.4197	0.2175	0.2857	0.3359
10	0.2622	0.3544	0.4194	0.2174	0.2857	0.3357

A graph depicting the probabilities of server and system busy is also included.

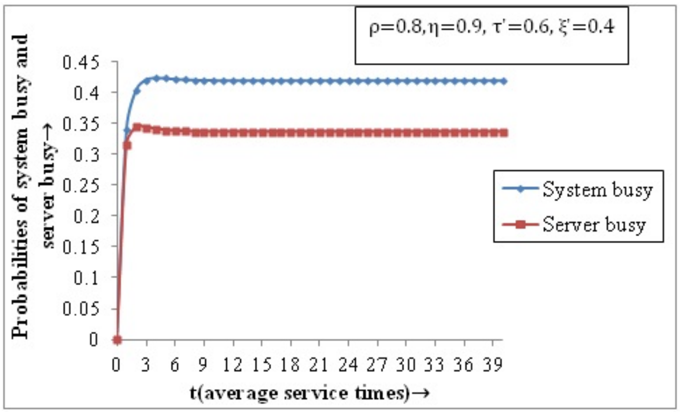

Figure 8: *Probabilities of system busy and server busy against average service times t*

In figu e 8, the probabilities of system busy and server busy are plotted against time t for the case $\rho = 0.8$, $\eta = 0.9$, $\tau' = 0.6$, $\xi' = 0.4$. It is clearly visible from the graph that probability of system busy is higher than probability of server busy. Both probabilities increase rapidly in the beginning and then become stable with time.

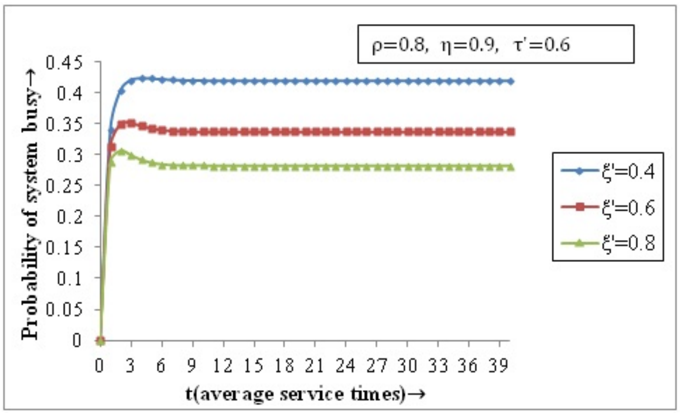

Figure 9: *Probability of system busy for different values of* ξ'

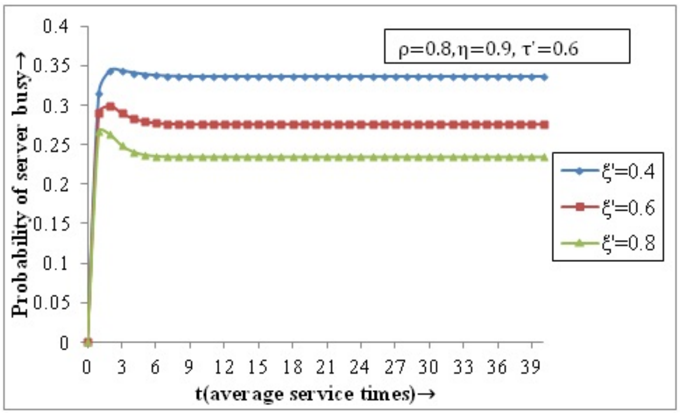

Figure 10: *Probability of server busy for different values of* ξ'

In figu es 9 and 10, the probability of system busy and the probability of server busy are plotted respectively against time t for different values of ξ' keeping other parameters constant. As we know when catastrophe occurs, the server breaks down and system becomes empty. So the probability that both system and server remain busy attains lower value for greater values of catastrophe rate per unit service time.

8. Conclusion

We have modeled a single server retrial queueing system with catastrophe to quantify various performance measures and understand characteristics of related systems. The catastrophe has significan impact on businesses, computer networks, etc. It is very important to manage the risk of catastrophe for the smooth functioning of the system. In this paper, the steady-state and transient state probabilities for the number of customers in the system when the server is busy or idle are obtained by solving difference-differential equations. In addition, the probability that server is under repair is also obtained. Some performance measures are given. Numerical solutions and busy period probabilities are obtained by using MATLAB programming and presented graphically . This model is applicable in call centers, computer networks, etc.

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