Stress-strength Reliability for Equi-correlated Multivariate Normal and its estimation

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Abstract

In this article it is mainly focused on discussion about estimation of stress-strength reliability under equi-correlated multivariate setup. It is seen in some situations that the components of a system are equi-correlated. Generally, the form of the equi-correlation structure within the components of a system is known for a given situation, however parameters that are involved in the equi-correlation structure always unknown. In this article, we propose a procedure to compute and estimate the stress-strength reliability R= Pr($\boldsymbol{a}'\boldsymbol{x} > \boldsymbol{b}'\boldsymbol{y}$) when $\boldsymbol{x}$ and $\boldsymbol{y}$ are distributed non-independently equi-correlated multivariate normal distribution, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are two known vectors. Here we have proposed the method of moments estimator to estimate these unknown parameters. Actually, we want to find out overall strength is larger than overall stress. In order to do that we take $\boldsymbol{a}'\boldsymbol{x}$ and $\boldsymbol{b}'\boldsymbol{y}$ as their representatives e.g. principal components of the respective vectors do the job approximately. An asymptotic distribution used to obtain confidence intervals for the stress-strength reliability. The performance of these intervals checked through the simulation study. Finally, we provide a real data analysis.

Keywords: Equi-correaled; Principal Component, Method of Moments Estimator (MOM); Asymptotic.

1. Introduction

The strength-stress model measured by R=Pr(X>Y), the lifetime of a component has a random strength X and it's subjected to random stress Y. In stress-strength model, at any time, the system fails if and only if, the stress is greater than its strength. First introduced to this model by Birnbaum [1] and then developed by Birnbaum and McCarty [2]. There has been a huge number of works as regards estimation of the reliability R= P(X>Y) in the field of stress-strength models. It has several applications particularly in engineering ideas, like structures, deterioration of rocket motors, static fatigue of ceramic parts, fatigue failure of craft structures, and also in mechanical, civil engineering. The R=Pr(X>Y) has been formulated for the huge majority of the popular statistical distributions when X and Y are independent random variables belonging to the same univariate family and also (X,Y) follows the bivariate distribution with dependence between X and Y. This form of R has been established for the bulk of popular statistical distributions, including Normal, uniform, exponential, gamma, beta, extreme value, Weibull, Laplace, logistic and the Pareto distributions…etc [3-7]. This model may be applied in clinical trial to comparing two treatment effects, it may be more useful to draw conclusions regarding the unit's free measure, rather than comparing the means [8]. Simonoff, Hochberg and Reiser [9] also used this model to find the effect of the treatment, if Y is the response for a control group, and X refers to a treatment group.

A numerical procedure obtained by Birnbaum and McCarty [2] based on the asymptotic distribution to find the sample size needed for setting up an upper confidence bound with the defined width and confidence coefficient. Sen [10] obtained the non-parametric confidence bounds for P(X<Y) based on independent samples. Govindarazulu [11] obtained two-sided confidence limits for R when X and Y are independent and also dependent normal variates. Church and Harris [12] obtained confidence intervals for R in case of independent normal varieties.

All these above existing works were done under the univariate or bivariate setup, Gupta and Gupta [13] first introduced the concept of estimating stress-strength reliability under multivariate normal setup. They considered the forms of R= $\Pr(\boldsymbol{a'x} > \boldsymbol{b'y})$, when $(\boldsymbol{x}_{p_1\text{x}1}, \boldsymbol{y}_{p_2\text{x}1})$ follows multivariate normal distribution with non-independent vector between $\boldsymbol{x}_{p_1\text{x}1}$ and $\boldsymbol{y}_{p_2\text{x}1}$, $\boldsymbol{a}'$ and $\boldsymbol{b}'$ are two known vectors. This problem arises when a system in the energy is supplied to the system by p_1 sources and is consumed through p_2 sources and the sources of energy supplied and consumed are linearly related with known vector $\boldsymbol{a}'$ and $\boldsymbol{b}'$. Under this set up Gupta and Gupta [13] considered only special cases of $\boldsymbol{a}'$ and $\boldsymbol{b}'$ and compared the MVUE and MLE estimates of R using given mean vector and dispersion matrix. Reiser and Farragi [14] derived the lower confidence bounds for R=P($\boldsymbol{a'x}^* > \boldsymbol{b'y}^*$) and solved it iteratively and also derived an approximate lower confidence bounds for R. Enis and Geisser [15] have demonstrated that, how to obtain the exact confidence bounds for R.

In many instances, it is seen that the components of a system are equi-correlated. Generally, the form of the equi-correlation structure is known for a given situation within the components of a system, however parameters that are involved in the equi-correlation structure are always unknown. Thus, we compute the stress strength reliability analytically for the special case of equi-correlated multivariate normal setup. We consider the principal component analysis to estimate the $\boldsymbol{a}'$ and $\boldsymbol{b}'$ where as Gupta and Gupta [13] considered only spatial cases of $\boldsymbol{a}'$ and $\boldsymbol{b}'$ and we present estimation of R using method of moment (MOM) estimates of the parameters for equi-correlated multivariate normal setup in Section 2.1. Determine the asymptotic distribution of $\hat{\delta}$ in Section 2.2. Finally, Simulation studies and data analysis are carried out in Section 3 for performance of MOM of R in teams of mean squared errors (MSE), relative bias (RB) and mean absolute error (MAE).

2. Estimation of stress-strength reliability (R)

Let, $\boldsymbol{x}_{p_1\text{x}1}$ and $\boldsymbol{y}_{p_2\text{x}1}$ be two random vector such that the distribution of $\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} \sim N_{p1+p2}(\mu, \Sigma)$

$$\text{where, } \mu = \begin{pmatrix}\mu_1 \mathbf{1}_{\mathbf{p_1}\text{x}1}\\ \mu_2 \mathbf{1}_{\mathbf{p_2}\text{x}1}\end{pmatrix}_{(p_1+p_2)\text{x}1} \text{ and } \Sigma = \sigma^2 \begin{pmatrix} \begin{pmatrix} 1\,\rho_1 \ldots.\rho_1 \\ \rho_1 1 \ldots.\rho_1 \\ \ldots\ldots\ldots\ldots \\ \rho_1\rho_1 \ldots.1 \end{pmatrix} & \begin{pmatrix} \rho_3\,\rho_3 \ldots..\rho_3 \\ \rho_3\,\rho_3 \ldots..\rho_3 \\ \ldots\ldots\ldots\ldots. \\ \rho_3\,\rho_3 \ldots..\rho_3 \end{pmatrix} \\ \begin{pmatrix} \rho_3\,\rho_3 \ldots..\rho_3 \\ \rho_3\,\rho_3 \ldots..\rho_3 \\ \ldots\ldots\ldots\ldots. \\ \rho_3\,\rho_3 \ldots..\rho_3 \end{pmatrix} & \begin{pmatrix} 1\,\rho_2 \ldots.\rho_2 \\ \rho_2 1 \ldots.\rho_1 \\ \ldots\ldots\ldots\ldots \\ \rho_2\rho_2 \ldots.1 \end{pmatrix} \end{pmatrix}_{(p_1+p_2)\text{x}(p_1+p_2)};$$

$$\Sigma_{11} = \sigma^2 \begin{pmatrix} 1\rho_1 \ldots\rho_1 \\ \rho_1 1 \ldots\rho_1 \\ \ldots\ldots\ldots\ldots \\ \rho_1\rho_1 \ldots 1 \end{pmatrix}_{(p_1 x p_1)}, \Sigma_{22} = \sigma^2 \begin{pmatrix} 1\rho_2 \ldots\rho_2 \\ \rho_2 1 \ldots\rho_2 \\ \ldots\ldots\ldots\ldots \\ \rho_2\rho_2 \ldots 1 \end{pmatrix}_{(p_2 x p_2)} \;\&\; \Sigma_{12} = \Sigma_{21}' = \sigma^2 \begin{pmatrix} \rho_3\rho_3 \ldots..\rho_3 \\ \rho_3\rho_3 \ldots..\rho_3 \\ \ldots\ldots\ldots\ldots \\ \rho_3\rho_3 \ldots..\rho_3 \end{pmatrix}_{(p_1 x p_2)};$$

$$-\frac{1}{p_1-1} < \rho_1 < 1, -\frac{1}{p_2-1} < \rho_2 < 1 \;\& -1 \le \rho_3 \le \frac{1}{\sqrt{p_1 p_2}}\left\{1 + \frac{(p_1-1)\rho_1 + (p_2-1)\rho_2}{2}\right\}$$

Now, we are interested to find out the overall strain vector is more than overall stress vector and a gross idea of doing this is to find that in terms of their principal components $\boldsymbol{a'x}$ and $\boldsymbol{b'y}$.

Then, we want to find the approximate reliability in terms of $\boldsymbol{a}'\boldsymbol{x}$ and $\boldsymbol{b}'\mathbf{y}$. Then, R= Pr($\boldsymbol{a}'\boldsymbol{x} > \boldsymbol{b}'\boldsymbol{y}$) = Pr($\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\mathbf{y} > 0$).

Now, the distribution of $u = \boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\boldsymbol{y}$ follows $N(\mu_{u,}\sigma_u^2)$,

where, $\mu_u = E\,(\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\boldsymbol{y}) = \mu_1\boldsymbol{a}'\mathbf{1}_{\mathbf{p_1x1}} - \mu_2\boldsymbol{b}'\mathbf{1}_{\boldsymbol{p_2x1}}$

and $\sigma_u^2 = Var(\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\boldsymbol{y}) = \boldsymbol{a}'\sum_{\mathbf{11}}\boldsymbol{a} - \mathbf{2}\boldsymbol{a}'\sum_{\mathbf{12}}\boldsymbol{b} + \boldsymbol{b}'\sum_{\mathbf{22}}\boldsymbol{b}$

So, R= Pr($\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\boldsymbol{y} > 0$) = Pr($u > 0$)

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_u}\exp\left\{-\frac{1}{2}\left(\frac{u-\mu_u}{\sigma_u}\right)^2\right\}du = \int_{-\frac{\mu_u}{\sigma_u}}^\infty \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}z^2\right\}dz = \Phi\left(\frac{\mu_u}{\sigma_u}\right) \quad (1)$$

The overall representation of the two sets or vectors are related to vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, such that they are approximated by $\boldsymbol{a}'\boldsymbol{x}$ and $\boldsymbol{b}'\boldsymbol{y}$ as in principal component analysis. Principal component analysis explaining the variance-Covariance structure $\sum_{\mathbf{11}}$ & $\sum_{\mathbf{22}}$ of a set of variables $\boldsymbol{x}$ and $\boldsymbol{y}$ through a linear combination ($\boldsymbol{a}'$ & $\boldsymbol{b}'$) of these variables, i.e, explain maximum variability. It is noted that, the first principal component has the largest possible variance (that is, accounts for as much of the variability in the data as possible), and each succeeding component in turn has the highest variance possible under the constraint that it is orthogonal to the preceding components.

Let, the estimate of $\boldsymbol{a}'$ by $\boldsymbol{e}_1'$ normalized eigenvector of $\sum_{\mathbf{11}}$ corresponding to eigen value $\boldsymbol{\lambda_1}$ and $\boldsymbol{b}'$ by $\boldsymbol{l}_1'$ normalized eigenvector of $\sum_{\mathbf{22}}$ corresponding to eigen value $\boldsymbol{\lambda_1}$.

Thus, we have $\boldsymbol{e}_1' = \frac{1}{\sqrt{p_1}}\mathbf{1}'_{\mathbf{p_1x1}}$ and $\boldsymbol{l}_1' = \frac{1}{\sqrt{p_2}}\mathbf{1}'_{\mathbf{p_2x1}}$ [16]

Then, $\mu_u = \frac{1}{\sqrt{p_1}}\mu_1\,\mathbf{1}'_{\mathbf{p_1x1}}\mathbf{1}_{\mathbf{p_1x1}} - \frac{1}{\sqrt{p_2}}\mu_2\,\mathbf{1}'_{\mathbf{p_2x1}}\mathbf{1}_{\mathbf{p_2x1}} = \sqrt{p_1}\mu_1 - \sqrt{p_2}\mu_2$

and,

$$\sigma_u^2 = \frac{1}{p_1}\mathbf{1}'_{\mathbf{p_1x1}}\sum\nolimits_{\mathbf{11}}\mathbf{1}_{\mathbf{p_1x1}} - 2\frac{1}{\sqrt{p_1p_2}}\mathbf{1}'_{\mathbf{p_1x1}}\sum\nolimits_{\mathbf{12}}\mathbf{1}_{\mathbf{p_2x1}} + \frac{1}{p_2}\mathbf{1}'_{\mathbf{p_2x1}}\sum\nolimits_{\mathbf{22}}\mathbf{1}_{\mathbf{p_2x1}}$$

$$= \sigma^2\left[\frac{p_1(1+(p_1-1)\rho_1)}{p_1} - 2\frac{p_1p_2}{\sqrt{p_1p_2}}\rho_3 + \frac{p_2(1+(p_2-1)\rho_2)}{p_2}\right]$$

$$= \sigma^2\left(2 + (p_1-1)\rho_1 + (p_2-1)\rho_2 - 2\sqrt{p_1p_2}\rho_3\right)$$

Then, from equation (1)

$$\text{R= Pr}(\boldsymbol{a}'\boldsymbol{x} > \boldsymbol{b}'\boldsymbol{y}) = \Phi\left[\frac{\sqrt{p_1}\mu_1 - \sqrt{p_2}\mu_2}{\sigma\sqrt{(2+(p_1-1)\rho_1+(p_2-1)\rho_2-2\sqrt{p_1p_2}\rho_3)}}\right] = \Phi(\delta) \quad (2)$$

where Φ = Distribution function of univariate standard normal distribution.

Now to estimate the R, we need to estimate the parameters of $\mu_1, \mu_2, \rho_1, \rho_2, \rho_3$ and σ^2 in equation (2). Thus, we obtain the method of moments estimator (MOM) of these unknow parameters to estimate δ, denoted by $\hat{\delta}$ and obtain its asymptotic distribution.

2.1.Method of Moments Estimation

Suppose, $m_1, m_2, \ldots, m_6$ are the sample moments of the random sample of $\binom{\boldsymbol{x_\alpha}}{\boldsymbol{y_\alpha}}$, $\alpha = 1,2,\ldots,n$, where $\binom{\boldsymbol{x_\alpha}}{\boldsymbol{y_\alpha}} \sim N_{p1+p2}(\mu,\Sigma)$. The sample moments are defined as

$$m_1 = \frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n}x_{\text{ir}}\right)$$
$$m_2 = \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n}y_{\text{jr}}\right)$$
$$m_3 = \frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n}x_{\text{ir}}^2\right)$$
$$m_4 = \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n}y_{\text{jr}}^2\right)$$

$$m_5 = \frac{2}{p_1(p_1-1)}\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)$$
$$m_6 = \frac{2}{p_2(p_2-1)}\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$
$$m_7 = \frac{1}{p_1p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$

Then the moment estimators (MOM) of $\mu_1, \mu_2, \rho_1, \rho_2, \rho_3$ and σ^2 are define as

$$\hat{\mu}_1 = m_1 = \frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}\right)$$
$$\hat{\mu}_2 = m_2 = \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}\right)$$
$$\begin{aligned}\hat{\sigma}^2 &= \frac{1}{2}\left[\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right) + \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right) - \hat{\mu}_1{}^2 - \hat{\mu}_2{}^2\right]\\
&= \frac{1}{2}\left[\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right) + \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right) -\right.\\
&\qquad \left.\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}\right)\right)^2 - \left(\frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}\right)\right)^2\right]\\
&= \frac{1}{2}(m_3 + m_4 - m_1^2 - m_2^2)\end{aligned}$$
$$\begin{aligned}\hat{\rho}_1 &= \frac{1}{\hat{\sigma}^2}\left(\frac{2}{p_1(p_1-1)}\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{\mathrm{n}}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right) - \hat{\mu}_1{}^2\right)\\
&= \frac{2}{(m_3 + m_4 - m_1^2 - m_2^2)}(m_5 - m_1^2)\end{aligned}$$
$$\begin{aligned}\hat{\rho}_2 &= \frac{1}{\hat{\sigma}^2}\left(\frac{2}{p_2(p_2-1)}\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right) - \hat{\mu}_2{}^2\right)\\
&= \frac{2}{(m_3 + m_4 - m_1^2 - m_2^2)}(m_6 - m_2^2)\end{aligned}$$
$$\begin{aligned}\hat{\rho}_3 &= \frac{1}{\hat{\sigma}^2}\left(\frac{1}{p_1p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right) - \hat{\mu}_1\hat{\mu}_2\right)\\
&= \frac{2}{(m_3 + m_4 - m_1^2 - m_2^2)}(m_7 - m_1m_2)\end{aligned}$$

Thus, the method of moment estimator of R by $\hat{R} = \Phi(\hat{\delta})$

where, $\hat{\delta} = \frac{\sqrt{p_1}\hat{\mu}_1 - \sqrt{p_2}\hat{\mu}_2}{\hat{\sigma}\sqrt{\{2+(p_1-1)\hat{\rho}_1+(p_2-1)\hat{\rho}_2-2\sqrt{p_1p_2}\hat{\rho}_3\}}}$

$$= \frac{m_1\sqrt{p_1} - m_2\sqrt{p_2}}{\sqrt{-m_1^2 - m_2^2 + m_3 + m_4}\sqrt{\frac{-m_3 - m_4 + m_5 + m_6 + m_1^2p_1 - m_5p_1 + m_2^2p_2 - m_6p_2 - 2m_1m_2\sqrt{p_1p_2} + 2m_7\sqrt{p_1p_2}}{m_1^2 + m_2^2 - m_3 - m_4}}}$$

2.2. Asymptotic distribution of $\hat{\delta}$

In this Section, we obtain the asymptotic distribution of $\hat{\delta}$ using delta method [17]. Using this we may determine the confidence intervals.

Let us define as

$$\hat{\delta} = g\left(\underset{\sim}{m}\right), \underset{\sim}{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \end{pmatrix} \text{ and } \delta = h\left(\underset{\sim}{\mu}\right), \underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ {\mu_1}^2 + \sigma^2 \\ {\mu_2}^2 + \sigma^2 \\ {\mu_1}^2 + \sigma^2\rho_1 \\ {\mu_2}^2 + \sigma^2\rho_2 \\ \mu_1\mu_2 + \sigma^2\rho_3 \end{pmatrix}$$

By using the central limit theorem,

$$\left(\sqrt{n}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \\ {\mu_1}^2 + \sigma^2 \\ {\mu_2}^2 + \sigma^2 \\ {\mu_1}^2 + \sigma^2\rho_1 \\ {\mu_2}^2 + \sigma^2\rho_2 \\ \mu_1\mu_2 + \sigma^2\rho_3 \end{pmatrix}\right) \xrightarrow{d} N_7\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma^*\right)$$

where, $$\Sigma^* = D\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \end{pmatrix} = \begin{pmatrix} Var(m_1) & Cov(m_1, m_2) & Cov(m_1, m_3) & \ldots\ldots & Cov(m_1, m_7) \\ Cov(m_1, m_2) & Var(m_2) & Cov(m_2, m_3) & \ldots\ldots & Cov(m_2, m_7) \\ Cov(m_1, m_3) & Cov(m_2, m_3) & Var(m_3) & \ldots\ldots & Cov(m_3, m_7) \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots \\ Cov(m_1, m_7) & Cov(m_2, m_7) & Cov(m_3, m_7) & \ldots\ldots & Var(m_7) \end{pmatrix}$$

$$Var(m_1) = Var\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}\right)\right)$$

$$= \frac{1}{n^2{p_1}^2}\left[\sum_{i=1}^{p_1} Var\left(\sum_{r=1}^{n} x_{\text{ir}}\right) + 2\sum_{\substack{i=1 \\ i<j}}^{p_1}\sum_{j=1}^{p_1} Cov\left(\sum_{r=1}^{n} x_{\text{ir}}, \sum_{r=1}^{n} x_{\text{jr}}\right)\right]$$

$$= \frac{\sigma^2}{p_1}[1 + (p_1 - 1)\rho_1]$$

$$Var(m_2) = Var\left(\frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{jr}}\right)\right)$$

$$= \frac{\sigma^2}{p_2}[1 + (p_2 - 1)\rho_2]$$

$$Var(m_3) = Var\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}^2\right)\right)$$

$$= \frac{1}{n^2{p_1}^2}\left[\sum_{i=1}^{p_1} \text{Var}\left(\sum_{r=1}^{n} x_{\text{ir}}^2\right) + 2\sum_{\substack{i=1 \\ i<j}}^{p_1}\sum_{j=1}^{p_1} \text{Cov}\left(\sum_{r=1}^{n} x_{\text{ir}}^2, \sum_{r=1}^{n} x_{\text{jr}}^2\right)\right]$$

$$= \frac{1}{p_1}[4\sigma^2{\mu_1}^2 + 2\sigma^4 + (p_1 - 1)(4{\mu_1}^2\rho_1\sigma^2 + 2{\rho_1}^2\sigma^4)]$$

$$Var(m_4) = Var\left(\frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{jr}}^2\right)\right)$$

$$= \frac{1}{p_2}[4\sigma^2{\mu_2}^2 + 2\sigma^4 + (p_2 - 1)(4{\mu_2}^2\rho_2\sigma^2 + 2{\rho_2}^2\sigma^4)]$$

$$Var(m_5) = Var\left(\frac{2}{p_1(p_1-1)}\sum_{i=1}^{p_1}\sum_{j=1 i<j}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)\right)$$

$$= \frac{4}{n^2{p_1}^2(p_1-1)^2}\left[\sum_{i=1}^{p_1}\sum_{j=1 i<j}^{p_1}\left(Var\left(\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)\right) + \right.$$

$$\left.\sum_{i=1}^{p_1}\sum_{\substack{j=1 \\ i<j,k<l(i,j)\neq(k,l)}}^{p_1}\sum_{k=1}^{p_1}\sum_{l=1}^{p_1}\left(Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}, \sum_{r=1}^{n} x_{\mathrm{kr}}x_{\mathrm{lr}}\right)\right)\right]$$

$$= \frac{4}{({p_1}^2(p_1-1)^2)}\left[\frac{p_1(p_1-1)}{2}\{2{\mu_1}^2\sigma^2 + 2{\mu_1}^2\rho_1\sigma^2 + \sigma^4 + {\rho_1}^2\sigma^4\}\right.$$
$$+\frac{1}{4}\{((p_1-1)(p_1-2))((p_1-1)(p_1-2)-2)(4{\mu_1}^2\rho_1\sigma^2 + 2{\rho_1}^2\sigma^4)\}$$
$$+\frac{1}{4}\Big\{\{(p_1(p_1-1))(p_1(p_1-1)-2) - ((p_1-1)(p_1-2))((p_1-1)(p_1-2)-2)\}$$
$$\left.(3{\mu_1}^2\rho_1\sigma^2 + {\mu_1}^2\sigma^2 + {\rho_1}^2\sigma^4 + \rho_1\sigma^4)\}\right]$$

$$Var(m_6) = Var\left(\frac{2}{p_2(p_2-1)}\sum_{i=1}^{p_2}\sum_{j=1 i<j}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$

$$= \frac{4}{n^2{p_2}^2(p_2-1)^2}\left[\sum_{i=1}^{p_2}\sum_{j=1 i<j}^{p_2}\left(\mathrm{Var}\left(\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right) + \right.$$

$$\left.\sum_{i=1}^{p_2}\sum_{\substack{j=1 \\ i<j,k<l(i,j)\neq(k,l)}}^{p_2}\sum_{k=1}^{p_2}\sum_{l=1}^{p_2}\left(Cov\left(\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}, \sum_{r=1}^{n} y_{\mathrm{kr}}y_{\mathrm{lr}}\right)\right)\right]$$

$$= \frac{4}{({p_2}^2(p_2-1)^2)}\left[\frac{p_2(p_2-1)}{2}\{2{\mu_2}^2\sigma^2 + 2{\mu_2}^2\rho_2\sigma^2 + \sigma^4 + {\rho_2}^2\sigma^4\} + \right.$$
$$\frac{1}{4}\{((p_2-1)(p_2-2))((p_2-1)(p_2-2)-2)(4{\mu_2}^2\rho_2\sigma^2 + 2{\rho_2}^2\sigma^4)\} +$$
$$\frac{1}{4}\Big\{\{(p_2(p_2-1))(p_2(p_2-1)-2) - ((p_2-1)(p_2-2))((p_2-1)(p_2-2)-2)\}$$
$$\left.(3{\mu_2}^2\rho_2\sigma^2 + {\mu_2}^2\sigma^2 + {\rho_2}^2\sigma^4 + \rho_2\sigma^4)\}\right]$$

$$Var(m_7) = Var\left(\frac{1}{p_1p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$

$$= \frac{1}{n^2{p_1}^2{p_2}^2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\mathrm{Var}\left(\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right) + \right.$$

$$2\sum_{i}\sum_{\substack{j \\ i\neq j\neq k}}\sum_{k} Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}, \sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{kr}}\right) +$$

$$2\sum_{\substack{i \\ i\neq j\neq k}}\sum_{j}\sum_{k} Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}, \sum_{r=1}^{n} x_{\mathrm{kr}}y_{\mathrm{jr}}\right) +$$

$$\left.2\sum_{\substack{i \\ i\neq j\neq k\neq l}}\sum_{j}\sum_{k}\sum_{l} Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}, \sum_{r=1}^{n} x_{\mathrm{kr}}y_{\mathrm{lr}}\right)\right]$$

$$= \frac{1}{p_1p_2}[({\mu_1}^2\sigma^2 + {\mu_2}^2\sigma^2 + 2\mu_1\mu_2\sigma^2\rho_3 + \sigma^4 + {\rho_3}^2\sigma^4) +$$
$$(p_1-1)({\mu_1}^2\sigma^2 + 2\mu_1\mu_2\sigma^2\rho_3 + {\mu_2}^2\rho_1\sigma^2 + \sigma^4\rho_1 + \sigma^4{\rho_3}^2) +$$
$$(p_2-1)({\mu_2}^2\sigma^2 + 2\mu_1\mu_2\sigma^2\rho_3 + {\mu_1}^2\rho_2\sigma^2 + \sigma^4\rho_2 + \sigma^4{\rho_3}^2) +$$
$$(p_1-1)(p_2-1)({\mu_1}^2\sigma^2\rho_2 + 2\mu_1\mu_2\sigma^2\rho_3 + {\mu_2}^2\sigma^2\rho_1 + \sigma^4\rho_1\rho_2 + \sigma^4{\rho_3}^2)]$$

$$Cov(m_1, m_2) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}\right), \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{jr}}\right)\right)$$
$$= \frac{1}{n^2 p_1 p_2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} \text{Cov}\left(\sum_{r=1}^{n} x_{\text{ir}}, \sum_{r=1}^{n} y_{\text{jr}}\right)\right]$$
$$= \sigma^2 \rho_3$$

$$Cov(m_1, m_3) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}\right), \frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}^2\right)\right)$$
$$= \frac{1}{n^2 {p_1}^2}\left[\sum_{i=1}^{p_1} Cov\left(\sum_{r=1}^{n} x_{\text{ir}}, \sum_{r=1}^{n} x_{\text{ir}}^2\right) + \ 2\sum_{\substack{i=1 \\ i<j}}^{p_1}\sum_{j=1}^{p_1} Cov\left(\sum_{r=1}^{n} x_{\text{ir}}, \sum_{r=1}^{n} x_{\text{jr}}^2\right)\right]$$
$$= \frac{2}{p_1}[\mu_1\sigma^2 + (p_1 - 1)\sigma^2\mu_1\rho_1]$$

$$Cov(m_1, m_4) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}\right), \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{jr}}^2\right)\right)$$
$$= \frac{1}{n^2 p_1 p_2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} Cov\left(\sum_{r=1}^{n} x_{\text{ir}}, \sum_{r=1}^{n} y_{\text{jr}}^2\right)\right]$$
$$= 2\sigma^2 \rho_3 \mu_2$$

$$Cov(m_1, m_5) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}\right), \frac{2}{p_1(p_1 - 1)}\sum_{\substack{i=1 \\ i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}x_{\text{jr}}\right)\right)$$
$$= \frac{2}{n^2 p_1^2 (p_1 - 1)} Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\text{ir}}, \sum_{\substack{i=1 \\ i<j}}^{p_1}\sum_{j=1}^{p_1}\sum_{r=1}^{n} x_{\text{ir}}x_{\text{jr}}\right)$$
$$= \frac{2}{p_1}\mu_1\sigma^2[\rho_1(p_1 - 1) + 1]$$
$$Cov(m_1, m_6) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}\right), \frac{2}{p_2(p_2 - 1)}\sum_{\substack{i=1 \\ i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{ir}}y_{\text{jr}}\right)\right)$$
$$= \frac{2}{n^2 p_1 p_2 (p_2 - 1)}\left(\sum_{i}\sum_{\substack{j \\ j<k}}\sum_{k} Cov\left(\sum_{r=1}^{n} x_{\text{ir}}, \sum_{r=1}^{n} y_{\text{jr}}y_{\text{kr}}\right)\right)$$
$$= 2\mu_2\sigma^2\rho_3$$
$$Cov(m_1, m_7) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}\right), \frac{1}{p_1 p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}y_{\text{jr}}\right)\right)$$
$$= \frac{1}{n^2 p_1^2 p_2} Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\text{ir}}, \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{r=1}^{n} x_{\text{ir}}y_{\text{jr}}\right)$$
$$= \frac{1}{p_1}[\mu_1\sigma^2\rho_3 + \mu_2\sigma^2 + (p_1 - 1)(\mu_1\sigma^2\rho_3 + \mu_2\sigma^2\rho_1)]$$

$$Cov(m_2, m_3) = Cov\left(\frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{jr}}\right), \frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}^2\right)\right)$$
$$= \frac{1}{n^2 p_1 p_2} Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\text{ir}}^2, \sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\text{jr}}\right)$$
$$= 2\sigma^2\rho_3\mu_1$$

$$Cov(m_2, m_4) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{ir}}\right), \frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{ir}}^2\right)\right)$$

$$= \frac{1}{n^2 p_2^2}\left[\sum_{i=1}^{p_2} \text{Cov}\left(\sum_{r=1}^{n} y_{\text{ir}}, \sum_{r=1}^{n} y_{\text{ir}}^2\right) + 2\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2} \text{Cov}\left(\sum_{r=1}^{n} y_{\text{ir}}, \sum_{r=1}^{n} y_{\text{jr}}^2\right)\right]$$

$$= \frac{2}{p_2}[\mu_2\sigma^2 + (p_2 - 1)\sigma^2\mu_2\rho_2]$$

$$Cov(m_2, m_5) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{ir}}\right), \frac{2}{p_1(p_1 - 1)}\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}x_{\text{jr}}\right)\right)$$

$$= \frac{2}{n^2 p_1 p_2 (p_1 - 1)} Cov\left(\sum_{i=1}^{p_2}\sum_{r=1}^{n} y_{\text{ir}}, \sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\sum_{r=1}^{n} x_{\text{ir}}x_{\text{jr}}\right)$$

$$= 2\mu_1\sigma^2\rho_3$$

$$Cov(m_2, m_6) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{ir}}\right), \frac{2}{p_2(p_2 - 1)}\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{ir}}y_{\text{jr}}\right)\right)$$

$$= \frac{2}{n^2 p_2^2 (p_2 - 1)} Cov\left(\sum_{i=1}^{p_2}\sum_{r=1}^{n} y_{\text{ir}}, \sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\text{ir}}y_{\text{jr}}\right)$$

$$= \frac{2\mu_2\sigma^2}{p_2}[\rho_2(p_2 - 1) + 1]$$

$$Cov(m_2, m_7) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{ir}}\right), \frac{1}{p_1 p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}y_{\text{jr}}\right)\right)$$

$$= \frac{1}{n^2 p_1 p_2^2} Cov\left(\sum_{i=1}^{p_2}\sum_{r=1}^{n} y_{\text{ir}}, \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{r=1}^{n} x_{\text{ir}}y_{\text{jr}}\right)$$

$$= \frac{1}{p_2}[\mu_2\sigma^2\rho_3 + \mu_2\sigma^2 + (p_2 - 1)(\mu_2\sigma^2\rho_3 + \mu_2\sigma^2\rho_2)]$$

$$Cov(m_3, m_4) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}^2\right), \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{jr}}^2\right)\right)$$

$$= \frac{1}{n^2 p_1 p_2} Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\text{ir}}^2, \sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\text{jr}}^2\right)$$

$$= 4\mu_1\mu_2\sigma^2\rho_3 + 2\sigma^4{\rho_3}^2$$

$$Cov(m_3, m_5) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}^2\right), \frac{2}{p_1(p_1 - 1)}\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}x_{\text{jr}}\right)\right)$$

$$= \frac{2}{n^2 p_1^2 (p_1 - 1)} Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\text{ir}}^2, \sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\sum_{r=1}^{n} x_{\text{ir}}x_{\text{jr}}\right)$$

$$= \frac{1}{p_1}[4({\mu_1}^2\sigma\rho_1 + {\mu_1}^2\sigma^2 + \sigma^4\rho_1) + 4{\mu_1}^2\sigma^2\rho_1 + 2\sigma^4{\rho_1}^2]$$

$$Cov(m_3, m_6) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right), \frac{2}{p_2(p_2-1)}\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{2}{n^2p_1p_2(p_2-1)}Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}^2, \sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$
$$= 4\mu_1\mu_2\sigma^2\rho_3 + 2\sigma^4{\rho_3}^2$$

$$Cov(m_3, m_7) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right), \frac{1}{p_1p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{1}{n^2p_1^2p_2}Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}^2, \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$
$$= \frac{2}{p_1}[({\mu_1}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2 + \sigma^4\rho_3) + (p_1-1)({\mu_1}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2\rho_1 + \sigma^4\rho_1\rho_3)]$$

$$Cov(m_4, m_5) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right), \frac{2}{p_1(p_1-1)}\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)\right)$$
$$= \frac{2}{n^2p_1p_2(p_1-1)}\mathrm{Cov}\left(\sum_{i=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{jr}}^2, \sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)$$
$$= 4\mu_1\mu_2\sigma^2\rho_3 + 2\sigma^4{\rho_3}^2$$

$$Cov(m_4, m_6) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right), \frac{2}{p_2(p_2-1)}\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{2}{n^2p_2^2(p_2-1)}\mathrm{Cov}\left(\sum_{i=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{jr}}^2, \sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$
$$= \frac{1}{p_2}[4({\mu_2}^2\sigma\rho_2 + {\mu_2}^2\sigma^2 + \sigma^4\rho_2) + 4{\mu_2}^2\sigma^2\rho_2 + 2\sigma^4{\rho_2}^2]$$

$$Cov(m_4, m_7) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right), \frac{1}{p_1p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{1}{n^2p_1p_2^2}Cov\left(\sum_{i=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{jr}}^2, \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$
$$= \frac{2}{p_2}[({\mu_2}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2 + \sigma^4\rho_3) + (p_2-1)({\mu_2}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2\rho_2 + \sigma^4\rho_2\rho_3)]$$
$$Cov(m_5, m_6) =$$
$$Cov\left(\frac{2}{p_1(p_1-1)}\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right), \frac{2}{p_2(p_2-1)}\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{4}{n^2p_1p_2(p_1-1)(p_2-1)}Cov\left(\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}, \sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$
$$= 4\mu_1\mu_2\sigma^2\rho_3 + 2\sigma^4{\rho_3}^2$$

$$Cov(m_5, m_7) = Cov\left(\frac{2}{p_1(p_1-1)}\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}x_{\text{jr}}\right), \frac{1}{p_1p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}y_{\text{jr}}\right)\right)$$

$$= \frac{2}{n^2p_1^2(p_1-1)p_2} Cov\left(\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\sum_{r=1}^{n} x_{\text{ir}}x_{\text{jr}}, \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{r=1}^{n} x_{\text{ir}}y_{\text{jr}}\right)$$

$$= \frac{2}{p_1}[(2{\mu_1}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2\rho_1 + \mu_1\mu_2\sigma^2 + \sigma^4\rho_3 + \sigma^4\rho_1\rho_3) +$$
$$(p_1-2)({\mu_1}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2\rho_1 + \sigma^4\rho_1\rho_3)]$$

$$Cov(m_6, m_7) = Cov\left(\frac{2}{p_2(p_2-1)}\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\text{ir}}y_{\text{jr}}\right), \frac{1}{p_1p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\text{ir}}y_{\text{jr}}\right)\right)$$

$$= \frac{2}{n^2p_1p_2^2(p_2-1)} Cov\left(\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\text{ir}}y_{\text{jr}}, \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{r=1}^{n} x_{\text{ir}}y_{\text{jr}}\right)$$

$$= \frac{2}{p_2}[(2{\mu_2}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2\rho_2 + \mu_1\mu_2\sigma^2 + \sigma^4\rho_3 + \sigma^4\rho_2\rho_3) +$$
$$(p_2-2)({\mu_2}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2\rho_2 + \sigma^4\rho_2\rho_3)]$$

Thus we find,

$$\underset{\sim}{g}^{/}(\hat{\delta}) = \left(\frac{\partial g}{\partial m_1}, \frac{\partial g}{\partial m_2}, \frac{\partial g}{\partial m_3}, \frac{\partial g}{\partial m_4}, \frac{\partial g}{\partial m_5}, \frac{\partial g}{\partial m_6}, \frac{\partial g}{\partial m_7}\right)_{\underset{\sim}{m}=\underset{\sim}{\mu}}$$

where,

$$\left(\frac{\partial g}{\partial m_1}\right)_{m_1=\mu_1} = \frac{p_1\mu_1(\sqrt{p_1}\mu_1 - \sqrt{p_2}\mu_2)}{\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{3/2}}$$

$$\left(\frac{\partial g}{\partial m_2}\right)_{m_2=\mu_2} = \frac{p_2\mu_2(\sqrt{p_1}\mu_1 - \sqrt{p_2}\mu_2)}{\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{3/2}}$$

$$\left(\frac{\partial g}{\partial m_3}\right)_{m_3=({\mu_1}^2+\sigma^2)} = \frac{-\sqrt{p_1}\mu_1 + \sqrt{p_2}\mu_2}{2\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{3/2}}$$

$$\left(\frac{\partial g}{\partial m_4}\right)_{m_4=({\mu_2}^2+\sigma^2)} = \frac{-\sqrt{p_1}\mu_1 + \sqrt{p_2}\mu_2}{2\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{3/2}}$$

$$\left(\frac{\partial g}{\partial m_5}\right)_{m_5=({\mu_1}^2+\sigma^2\rho_1)} = \frac{-(-1+p_1)(\sqrt{p_1}\mu_1 - \sqrt{p_2}\mu_2)}{2\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{3/2}}$$

$$\left(\frac{\partial g}{\partial m_6}\right)_{m_6=({\mu_2}^2+\sigma^2\rho_2)} = \frac{(-1+p_2)(-\sqrt{p_1}\mu_1 + \sqrt{p_2}\mu_2)}{2\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{\frac{3}{2}}}$$

$$\left(\frac{\partial g}{\partial m_7}\right)_{m_7=(\mu_1\mu_2+\sigma^2\rho_3)} = \frac{\sqrt{p_1p_2}\left(\sqrt{p_1}\mu_1-\sqrt{p_2}\mu_2\right)}{\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{3/2}}$$

Using the delta method, we have

$$\sqrt{n}\left(\hat{\delta}-\delta\right)\xrightarrow{d} N(0,\sigma_\delta^2)$$

or,

$$\hat{\delta}\xrightarrow{d} N\left(\delta,\frac{\sigma_\delta^2}{n}\right)$$

where,

$$\sigma_\delta^2 = \underset{\sim}{g}'\left(\hat{\delta}\right)\Sigma^*\underset{\sim}{g}\left(\hat{\delta}\right)$$

$$\begin{gathered}
= ((\sigma^3p_1^4p_2\rho_1^2(\mu_1^2+2\sigma^2\rho_1)-2p_2^2\mu_2^2\rho_1((2+4\sigma)\mu_1^2+3\sigma^3\rho_1)+ \\
\sigma^3\sqrt{p_1p_2}\mu_2(-1+\rho_1)(-2+\rho_1-(-1+p_2)\rho_2))- \\
2\sigma^3p_1^{7/2}p_2^{3/2}\rho_1^2(\mu_1\mu_2+2\sigma^2\rho_3)+p_1p_2(-4(1+2\sigma)\mu_1^4\rho_1+ \\
2\mu_1^2\rho_1(2(1+3\sigma)p_2\mu_2^2-3\sigma^3\rho_1)+\sigma^3p_2^3\rho_2^2(\mu_2^2+2\sigma^2\rho_2)- \\
4\sigma\sqrt{p_1p_2}\mu_1\mu_2(p_2^2\mu_2^2\rho_2-\sigma^2(-1+\rho_1)(-2+\rho_1+\rho_2)+ \\
p_2(-\mu_2^2(-1+\rho_2)+\sigma^2(-1+\rho_1)\rho_2))- \\
2(2(1+2\sigma)\mu_2^4\rho_2+\sigma^3\mu_2^2(-4\sqrt{p_1p_2}-2\sqrt{p_1p_2}\rho_1 \\
(-1+\rho_2)+6\sqrt{p_1p_2}\rho_2+(3-2\sqrt{p_1p_2})\rho_2^2)+ \\
\sigma^5(-2+\rho_1+\rho_2)^2(-2+\rho_1+\rho_2+4\sqrt{p_1p_2}\rho_3))- \\
2\sigma p_2^2(\sigma^2\mu_2^2\rho_2(-2+\rho_1-2(-1+\sqrt{p_1p_2})\rho_2)+\mu_2^4(-1+\rho_1-2 \\
(-1+\sqrt{p_1p_2})\rho_2)+\sigma^4\rho_2^2(-6+3\rho_1+3\rho_2+4\sqrt{p_1p_2}\rho_3))+ \\
p_2(4\mu_2^4(\sigma\sqrt{p_1p_2}+(1-\sigma(-3+\sqrt{p_1p_2}))\rho_2)+ \\
\sigma^3\mu_2^2(4+9\rho_1^2+4(-1+3\sqrt{p_1p_2})\rho_2+(9-8\sqrt{p_1p_2}) \\
\rho_2^2+\rho_1(-4+(2-4\sqrt{p_1p_2})\rho_2))+ \\
2\sigma^5\rho_2(-2+\rho_1+\rho_2)(-6+3\rho_1+3\rho_2+8\sqrt{p_1p_2}\rho_3)))- \\
2p_1^{3/2}\sqrt{p_2}(-2\mu_1\mu_2(-2\sigma^3\sqrt{p_1p_2}-\sigma^3\sqrt{p_1p_2}\rho_1(-1+\rho_2)+ \\
(3\sigma^3\sqrt{p_1p_2}+(2+4\sigma)\mu_2^2)\rho_2-\sigma^3(-3+\sqrt{p_1p_2})\rho_2^2)+ \\
\sigma^3p_2^3\rho_2^2(\mu_1\mu_2+2\sigma^2\rho_3)+p_2(4(1+3\sigma)\mu_1^3\mu_2\rho_1+ \\
4\sigma\sqrt{p_1p_2}\mu_1^2\mu_2^2(-1+\rho_2)-2\sigma^3\sqrt{p_1p_2}\mu_2^2\rho_1(-3+2\rho_1+\rho_2)+ \\
\mu_1\mu_2(4\mu_2^2(\sigma\sqrt{p_1p_2}+(1-\sigma(-3+\sqrt{p_1p_2}))\rho_2)+\sigma^3 \\
(4+9\rho_1^2+(-4+6\sqrt{p_1p_2})\rho_2+(9-4\sqrt{p_1p_2})\rho_2^2- \\
2\rho_1(2+(-1+\sqrt{p_1p_2})\rho_2)))+ \\
2\sigma^5(-2+\rho_1+\rho_2)\rho_3(-2+\rho_1+\rho_2+4\sqrt{p_1p_2}\rho_3))- \\
2\sigma p_2^2(2\sqrt{p_1p_2}\mu_1^2\mu_2^2\rho_2+\mu_1\mu_2(\sigma^2\rho_2(-2+\rho_1- \\
(-2+\sqrt{p_1p_2})\rho_2)+\mu_2^2(-1+\rho_1-2(-1+\sqrt{p_1p_2})\rho_2))+ \\
2\sigma^4\rho_2\rho_3\left(-2+\rho_1+\rho_2+2\sqrt{p_1p_2}\rho_3\right)-\sigma^2\mu_2^2 \\
(-2\rho_3+\rho_1(\sqrt{p_1p_2}\rho_2+2\rho_3))))+ \\
\sigma p_1^3p_2(-2\mu_1^4(-1+2\rho_1+\rho_2-p_2\rho_2)+4p_2\mu_1^3\mu_2\rho_3+ \\
4\sigma^2\mu_1\mu_2\rho_1\left(\sqrt{p_1p_2}\rho_1-2p_2\rho_3\right)+ \\
\mu_1^2(-4\sigma^2\rho_1^2+2\rho_1(-\sigma^2(-2+\rho_2)+p_2(\mu_2^2+\sigma^2\rho_2))- \\
4\sigma^2p_2\rho_3^2)+\sigma^2\rho_1(-6\sigma^2\rho_1^2+8\sigma^2p_2\rho_3^2+ \\
\rho_1(p_2(\mu_2^2+6\sigma^2\rho_2)-2\sigma^2(-6+3\rho_2+4\sqrt{p_1p_2}\rho_3))))- \\
4\sigma p_1^{5/2}\sqrt{p_2}(-\sigma^2\sqrt{p_1p_2}\mu_1\mu_2\rho_1(-1+\rho_2)+ \\
p_2(-\mu_1^3\mu_2(-1+2\rho_1+\rho_2)+ \\
\sigma^2\mu_1\mu_2\left(-2\rho_1^2+\rho_1\left(2+\left(-1+\sqrt{p_1p_2}\right)\rho_2\right)+2(-1+\rho_2)\rho_3\right)+ \\
\sigma^2\rho_1(\sqrt{p_1p_2}\mu_2^2\rho_1-2\sigma^2\rho_3(-2+\rho_1+\rho_2+2\sqrt{p_1p_2}\rho_3)))+ \\
p_2^2(\mu_1^3\mu_2\rho_2+2\mu_1^2\mu_2^2\rho_3+2\sigma^2\rho_3(-\mu_2^2\rho_1+\sigma^2(\rho_1\rho_2+2\rho_3^2))+ \\
\mu_1\mu_2(\mu_2^2\rho_1+\sigma^2(\rho_1\rho_2-2\rho_3(\rho_2+\rho_3))))))+ \\
p_1^2(-2\mu_1^2\rho_2((2+4\sigma)\mu_2^2+3\sigma^3\rho_2)+p_2(4(1+3\sigma)\mu_1^4\rho_1+ \\
4\sigma\sqrt{p_1p_2}\mu_1^3\mu_2(-1+\rho_2)-4\sigma^3\sqrt{p_1p_2}\mu_1\mu_2\rho_1
\end{gathered}$$

$$(-3 + 2\rho_1 + \rho_2) + \mu_1^2(4\mu_2^2(\sigma\sqrt{p_1p_2} + (1 + 3\sigma - \sigma\sqrt{p_1p_2})$$
$$\rho_2) + \sigma^3(4 + 9\rho_1^2 + 2\rho_1(-2 + \rho_2) - 4\rho_2 + 9\rho_2^2)) +$$
$$2\sigma^3\rho_1(-2\sqrt{p_1p_2}\mu_2^2(-1 + \rho_2) + \sigma^2(-2 + \rho_1 + \rho_2)$$
$$(-6 + 3\rho_1 + 3\rho_2 + 8\sqrt{p_1p_2}\rho_3))) + \sigma p_2^3(2\mu_2^4\rho_1 +$$
$$4\mu_1\mu_2^3\rho_3 + \sigma^2\rho_2(\mu_1^2\rho_2 + 2\sigma^2(3\rho_1\rho_2 + 4\rho_3^2)) +$$
$$2\mu_2^2(\mu_1^2\rho_2 + \sigma^2(\rho_1\rho_2 - 2\rho_3(2\rho_2 + \rho_3)))) -$$
$$2\sigma p_2^2(2\sqrt{p_1p_2}\mu_1^3\mu_2\rho_2 + \mu_1^2(\sigma^2\rho_2(-2 + \rho_1 + 2\rho_2) +$$
$$\mu_2^2(-2 + 3\rho_1 + (3 - 2\sqrt{p_1p_2})\rho_2)) +$$
$$\sigma^2\mu_2^2\left(2\rho_1^2 + \rho_1\left(-2 + \left(1 - 2\sqrt{p_1p_2}\right)\rho_2\right) - 4(-1 + \rho_2)\rho_3\right) -$$
$$2\sigma^2\mu_1\mu_2\left(-2\rho_3 + \rho_1\left(\sqrt{p_1p_2}\rho_2 + 2\rho_3\right)\right) +$$
$$2\sigma^4(3\rho_1^2\rho_2 + 2(-2 + \rho_2)\rho_3^2 + \rho_1$$
$$(3\rho_2^2 + 2\rho_3^2 + \rho_2(-6 + 4\sqrt{p_1p_2}\rho_3))))))/$$
$$(2n\sigma^5p_1p_2\left(2 + (-1 + p_1)\rho_1 + (-1 + p_2)\rho_2 - 2\sqrt{p_1p_2}\rho_3\right)^3))$$

2.3. Asymptotic Confidence Intervals for R

Based on the asymptotic distribution of $\hat{\delta}$, we obtain the asymptotic confidence interval of R. Here, the estimate of R by $\hat{R} = \Phi(\hat{\delta})$, i.e. $\hat{\delta} = \Phi^{-1}(R)$ and we have $\hat{\delta} \xrightarrow{d} N\left(\delta, \frac{\sigma_\delta^2}{n}\right)$ as $n \to \infty$. In order to determine the two sided confidence Intervals, we find out the two numbers $L_1$and L_2 $(L_1 < L_2)$, such that, for a given α, we have

$$P(L_1 \leqslant \Phi(\delta) \leqslant L_2) = 1 - \alpha$$

or,
$$P(\Phi^{-1}(L_1) \leqslant \delta \leqslant \Phi^{-1}(L_2)) = 1 - \alpha \quad (3)$$

Then, an asymptotic (1-α) level confidence Intervals for δ is given by

$$P\left(-z_{\alpha/2} \leqslant \frac{\sqrt{n}(\hat{\delta} - \delta)}{\sigma_\delta} \leqslant z_{\alpha/2}\right) = 1 - \alpha$$

or,
$$P\left(-\frac{z_{\alpha/2}\sigma_\delta}{\sqrt{n}} \leqslant (\hat{\delta} - \delta) \leqslant \frac{z_{\alpha/2}\sigma_\delta}{\sqrt{n}}\right) = 1 - \alpha$$

or,
$$P\left(\hat{\delta} - \frac{z_{\alpha/2}\sigma_\delta}{\sqrt{n}} \leqslant \delta \leqslant \hat{\delta} + \frac{z_{\alpha/2}\sigma_\delta}{\sqrt{n}}\right) = 1 - \alpha$$

We can replace $\sigma_\delta$by $\hat{\sigma}_\delta$ to obtain asymptotic confidence Intervals for δ. Thus, we can write

$$P\left(\hat{\delta} - \frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}} \leqslant \delta \leqslant \hat{\delta} + \frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}\right) = 1 - \alpha \quad (4)$$

Comparing (3) and (4), we have $L_1$and L_2respectively as

$$\Phi^{-1}(L_1) = \hat{\delta} - \frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}$$

or,
$$L_1 = \Phi\left(\hat{\delta} - \frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}\right)$$

and,
$$L_2 = \Phi\left(\hat{\delta} + \frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}\right)$$

Then, an asymptotic (1-α) level confidence Intervals for R is represented by

$$(L_1, L_2) = \left\{\Phi\left(\hat{\delta} - \frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}\right), \Phi\left(\hat{\delta} + \frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}\right)\right\}$$

where, $z_{\alpha/2}$ upper critical value for the standard normal distribution.

Thus, an asymptotic (1-α) confidence lower bound for R as

$$L_B = \Phi\left(\hat{\delta} - \frac{z_{\alpha}\hat{\sigma}_\delta}{\sqrt{n}}\right)$$

3. Simulation study and Data analysis

3.1. Simulation study

Now, we compute convergence and performance of MOM estimator, we considered different scenarios, each corresponding to a different combination of distributional parameters with different reliabilities for p_1=10 and p_2=10, reported in Table 1. We set the six parameters in order to get a high value (>0.5) for the reliability, since one typically looks for high reliability for the study component or system in real practice. Through these scenarios we cover the large range of reliability, since the range of R from 0.5825 to 0.9736.

For this above purpose we compute the following measures:

(i) Sample mean of $\hat{R}$ using MOM

(ii) Mean square error (MSE) of $\hat{R}$: $E(\hat{R} - R)^2$

(iii) Mean Relative Bias (RB) of $\hat{R}$: $\frac{E(\hat{R})-R}{R}$

(iv) Mean absolute error (MAE) of $\hat{R}$: $E(|\hat{R} - R|)$

It is difficult to obtain the analytical form of the equation (1) for various 'R'. So, we figure out these by using simulation study. Hence, we generate the random samples of size n from $\binom{\boldsymbol{x}}{\boldsymbol{y}} \sim N_{p1+p2}(\mu, \Sigma)$ for different scenarios. For each of sample drown of size n, we compute the R using MOM by taking 500 replications each time and also compute the above measures. Here we consider the different sample sizes (n) as 10, 30, 50 and 100. For this purpose, here, R programming language is used. The simulation results are reported in Table 2. It is noted that, the MSE, RB and MAE of $\hat{R}$ are reduces as the sample size increases decrease as expected and when n=100, $\hat{R}$ achieved the true value of R under each scenario. Thus, the result seems to be supportive for R in larger sample. Also, the performance of the R using MOM is quite satisfactory in terms of MSE, RB and MAE for small sample sizes. Hence, it is satisfying the consistency property of the MOM of R.

We take the components as p_1=14, p_2=12 and set the parameters are μ_1=4, μ_2=3.5, ρ_1=0.8, ρ_2=0.6, ρ_3=0.7, σ^2=4 in order to verify the asymptotic distribution of $\hat{\delta}$ as follows normal distribution, described in section 2.2. Then, we have $\hat{\delta} \sim N(1.538, 0.0196)$ and generate n=500 samples using this as theoretical quantiles. For each of sample drown of size n=500, we compute the $\hat{\delta}$ using MOM by taking 500 replications each time is treated as sample quantiles. Figure 1, Q-Q plot [18] and Shapiro-Wilk normality test [19] (result as W = 0.99686, p-value = 0.4469) provided satisfactory result that the $\hat{\delta}$ follows asymptotic normal distribution.

The results of the simulation study for the confidence intervals as lower (L_1) limit, upper (L_2) limit and lower bound (L_B) are recorded in Table 3. Table 1 and 2, represent the asymptotic and bootstrap confidence belt at 90%, 95% and 99% levels. It has been observed that for a small sample size, the estimate of R is getting high and also confidence intervals in case of asymptotic. The results get better as the sample sizes increase and the reliability R gets closer to true value. The overall band of asymptotic confidence is going to sink as the sample sizes increase and it has consistent variation.

Table 1: *Parameters for the equi-correlated set-up*

Scenario	Parameters and values						
	μ_1	μ_2	ρ_1	ρ_2	ρ_3	σ^2	R
1	1.5	1	0.2	0.2	0.1	16	0.5825
2	2.5	1.5	0.3	0.2	0.1	16	0.6453
3	3	2	0.3	0.2	0.2	16	0.6915
4	4	2.5	0.5	0.4	0.3	16	0.7209
5	4	2.5	0.5	0.4	0.3	8	0.7962
6	4	2.5	0.6	0.4	0.4	8	0.8335
7	4	2.5	0.6	0.5	0.5	8	0.8881
8	4	2.5	0.7	0.6	0.5	4	0.8912
9	4	2.5	0.8	0.6	0.6	4	0.9293
10	4	2.5	0.8	0.7	0.7	4	0.9736

Table 2: *Simulation results: Coverage Probability, MSE, RB and MAE*

Scenario	Sample Size	Sample Mean ($\hat{R}$)	MSE	RB	MAE
1	10	0.5840	0.0177	0.0025	0.1064
	30	0.5833	0.0054	0.0013	0.0576
	50	0.5831	0.0034	0.0010	0.0467
	100	0.5828	0.0017	0.0006	0.0328
2	10	0.6491	0.0154	0.0060	0.1002
	30	0.6456	0.0048	0.0005	0.0537
	50	0.6455	0.0029	0.0004	0.0431
	100	0.6454	0.0015	0.0001	0.0315
3	10	0.6918	0.0171	0.0006	0.1034
	30	0.6918	0.0052	0.0005	0.0582
	50	0.6917	0.0027	0.0004	0.0423
	100	0.6917	0.0015	0.0003	0.0309
4	10	0.7238	0.0161	0.0040	0.1024
	30	0.7222	0.0046	0.0017	0.0538
	50	0.7209	0.0026	-0.0001	0.0409
	100	0.7206	0.0013	-0.0004	0.0295
5	10	0.7977	0.0110	0.0018	0.0856
	30	0.7973	0.0033	0.0014	0.0457
	50	0.7969	0.0022	0.0008	0.0385
	100	0.7965	0.0012	0.0004	0.0271
6	10	0.8376	0.0091	0.0048	0.0776
	30	0.8347	0.0033	0.0014	0.0457
	50	0.8341	0.0020	0.0007	0.0356
	100	0.8336	0.0009	0.0001	0.0245
7	10	0.8903	0.0057	0.0024	0.0603
	30	0.8895	0.0019	0.0016	0.0354
	50	0.8888	0.0014	0.0008	0.0298
	100	0.8884	0.0006	0.0003	0.0192
8	10	0.8934	0.0057	0.0025	0.0622
	30	0.8926	0.0020	0.0016	0.0355
	50	0.8918	0.0013	0.0007	0.0284
	100	0.8917	0.0006	0.0005	0.0205

Scenario	Sample Size	Sample Mean ($\hat{R}$)	MSE	RB	MAE
9	10	0.9310	0.0031	0.0018	0.0434
	30	0.9304	0.0013	0.0012	0.0278
	50	0.9298	0.0008	0.0005	0.0224
	100	0.9296	0.0004	0.0003	0.0159
10	10	0.9740	0.0008	0.0004	0.0213
	30	0.9737	0.0003	0.0001	0.0139
	50	0.9736	0.0002	0.0000	0.0119
	100	0.9735	0.0001	-0.0001	0.0085

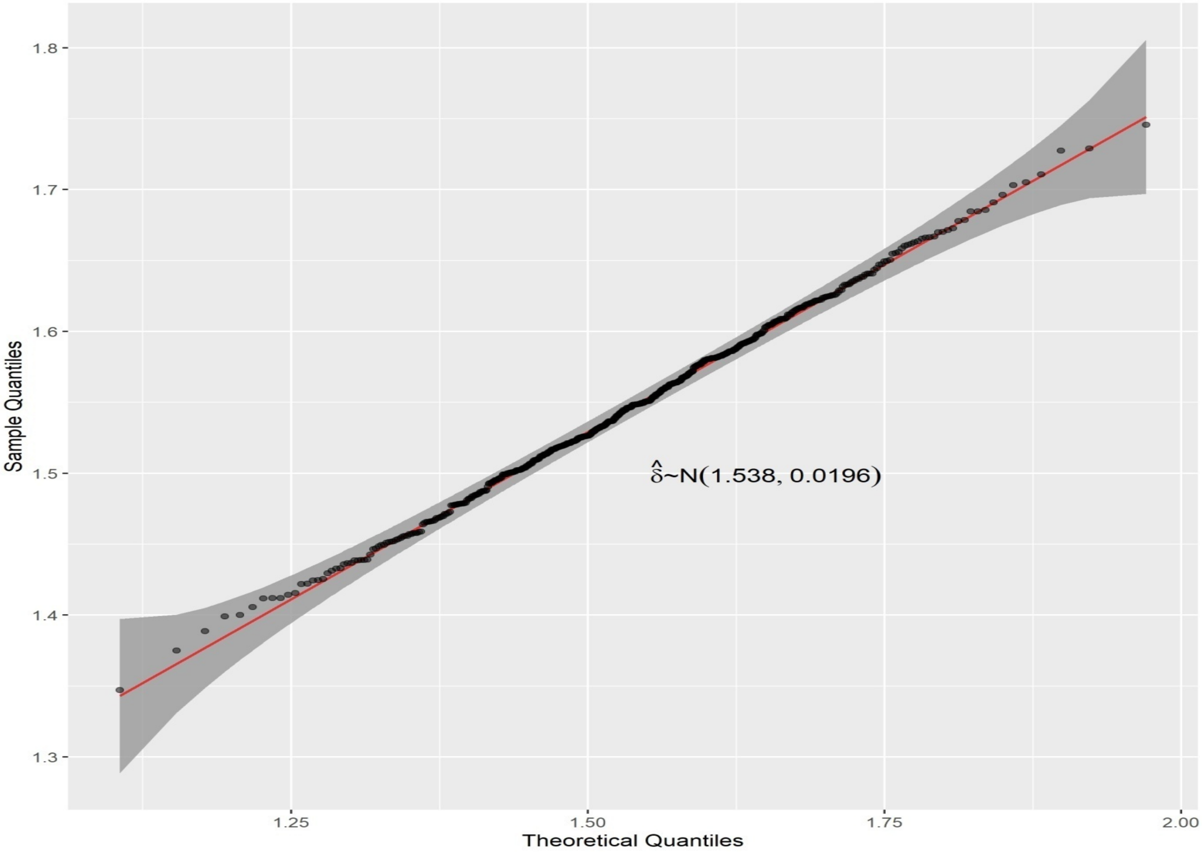

Figure 1: *Normal Q-Q Plot*

Table 3: *Asymptotic Confidence Intervals*

Scen-ario	Sample Size	$\hat{R}$	90% L	90% U	90% LB	95% L	95% U	95% LB	99% L	99% U	99% LB
1	10	0.5884	0.5167	0.6572	0.5326	0.5028	0.6699	0.5167	0.4757	0.6942	0.4867
	30	0.5842	0.5624	0.6058	0.5672	0.5582	0.6099	0.5624	0.5499	0.6179	0.5533
	50	0.5838	0.5706	0.5969	0.5735	0.5681	0.5994	0.5706	0.5631	0.6042	0.5651
	100	0.5830	0.5762	0.5897	0.5777	0.5749	0.5910	0.5762	0.5724	0.5935	0.5734
2	10	0.7109	0.6465	0.7692	0.6612	0.6336	0.7796	0.6465	0.6079	0.7991	0.6184
	30	0.6978	0.6712	0.7235	0.6771	0.6659	0.7283	0.6712	0.6557	0.7375	0.6598
	50	0.6876	0.6744	0.7005	0.6774	0.6719	0.7029	0.6744	0.6669	0.7077	0.6689
	100	0.6444	0.6371	0.6516	0.6387	0.6357	0.6530	0.6371	0.6330	0.6557	0.6341
3	10	0.7583	0.6551	0.8420	0.6793	0.6335	0.8555	0.6551	0.5902	0.8797	0.6079
	30	0.7235	0.6925	0.7530	0.6995	0.6863	0.7584	0.6925	0.6742	0.7688	0.6792
	50	0.7135	0.6859	0.7400	0.6921	0.6805	0.7449	0.6859	0.6698	0.7543	0.6741

Scen-ario	Sample Size	$\hat{R}$	90%			95%			99%		
			L	U	LB	L	U	LB	L	U	LB
	100	0.6954	0.6853	0.7054	0.6875	0.6833	0.7073	0.6853	0.6795	0.7110	0.6810
4	10	0.8041	0.6534	0.9063	0.6903	0.6202	0.9202	0.6534	0.5530	0.9429	0.5805
	30	0.7571	0.7004	0.8074	0.7134	0.6889	0.8162	0.7004	0.6659	0.8328	0.6753
	50	0.7420	0.7226	0.7606	0.7269	0.7188	0.7641	0.7226	0.7113	0.7709	0.7143
	100	0.7243	0.7140	0.7344	0.7163	0.7121	0.7363	0.7140	0.7081	0.7400	0.7097
5	10	0.8613	0.7002	0.9502	0.7418	0.6619	0.9603	0.7002	0.5823	0.9753	0.6151
	30	0.8570	0.7952	0.9048	0.8101	0.7818	0.9124	0.7952	0.7542	0.9260	0.7656
	50	0.8352	0.8057	0.8616	0.8125	0.7997	0.8663	0.8057	0.7877	0.8751	0.7926
	100	0.7996	0.7854	0.8133	0.7886	0.7826	0.8158	0.7854	0.7771	0.8208	0.7793
6	10	0.8800	0.7306	0.9586	0.7699	0.6940	0.9673	0.7306	0.6170	0.9799	0.6490
	30	0.8613	0.7941	0.9117	0.8104	0.7793	0.9195	0.7941	0.7488	0.9334	0.7614
	50	0.8506	0.8098	0.8852	0.8193	0.8012	0.8911	0.8098	0.7839	0.9020	0.7910
	100	0.8340	0.8111	0.8551	0.8163	0.8064	0.8589	0.8111	0.7972	0.8662	0.8010
7	10	0.9310	0.5747	0.9973	0.6824	0.4762	0.9988	0.5747	0.2930	0.9998	0.3639
	30	0.9253	0.7169	0.9896	0.7780	0.6581	0.9934	0.7169	0.5327	0.9975	0.5846
	50	0.9079	0.8218	0.9586	0.8442	0.8008	0.9650	0.8218	0.7557	0.9752	0.7746
	100	0.8839	0.8527	0.9101	0.8601	0.8462	0.9145	0.8527	0.8328	0.9228	0.8383
8	10	0.9146	0.6030	0.9934	0.6935	0.5194	0.9964	0.6030	0.3570	0.9991	0.4214
	30	0.8957	0.7818	0.9587	0.8117	0.7538	0.9662	0.7818	0.6940	0.9776	0.7190
	50	0.8939	0.8218	0.9421	0.8399	0.8051	0.9490	0.8218	0.7699	0.9605	0.7846
	100	0.8937	0.8723	0.9124	0.8772	0.8678	0.9157	0.8723	0.8589	0.9218	0.8626
9	10	0.9602	0.7672	0.9973	0.8304	0.7032	0.9985	0.7672	0.5597	0.9996	0.6200
	30	0.9486	0.7554	0.9949	0.8157	0.6954	0.9970	0.7554	0.5633	0.9990	0.6186
	50	0.9451	0.8894	0.9759	0.9043	0.8751	0.9797	0.8894	0.8438	0.9857	0.8571
	100	0.9306	0.9004	0.9532	0.9078	0.8936	0.9568	0.9004	0.8795	0.9631	0.8853
10	10	0.9893	0.0092	1.0000	0.0922	0.0006	1.0000	0.0092	0.0000	1.0000	0.0000
	30	0.9821	0.6637	0.9999	0.7862	0.5403	1.0000	0.6637	0.2991	1.0000	0.3926
	50	0.9770	0.8131	0.9990	0.8715	0.7509	0.9995	0.8131	0.6038	0.9999	0.6668
	100	0.9768	0.9252	0.9945	0.9409	0.9091	0.9960	0.9252	0.8705	0.9979	0.8873

3.2. Data Analysis

In this section, we apply the above methods to find out values of the estimators as $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\sigma}^2$and $\hat{R}$ from a given data set. The secondary data set of "Wave Energy Converters Data Set" is taken from the UCI Machine Learning site. The data set can be downloaded at https://archive.ics.uci.edu/ml/datasets/Wave+Energy+Converters. This data set consists of positions and absorbed power outputs of wave energy converters (WECs) in four real wave scenarios from the southern coast of Australia (Sydney, Adelaide, Perth and Tasmania). From this date set we take only two place of data set as Adelaide and Perth. We consider the eleven variables names as WECs absorbed power from each data set, which are consistent with the positive correlation among the variables. Then, we find out the stress strength reliability of absorbed power between the Adelaide and Perth respectively. Here we select the number of variables as p_1=11, p_2=11 and the MOM estimates as $\hat{\mu}_1$ =88175.2, $\hat{\mu}_2$=87244.27, $\hat{\rho}_1$=0.06567, $\hat{\rho}_2$=0.05251, $\hat{\rho}_3$=-0.04049, $\hat{\sigma}^2$ =107224128 and $\hat{R}$=0.55872. Jennrich test [20] used to examine the differences between correlation matrices of elevens variables of Adelaide and Perth data sets. The result shows that, the sample and estimated correlations by MOM are equal, reported in Table 4 and 5. This means that there is an equi-correlation between variables of the above data sets. The mean vectors of each data set reported in Table 6 and all are mostly equal. The performance of MOM quite good for sample size. The confidence intervals result on "Wave Energy Converters Data Set" shows in Table 7. The asymptotic confidence intervals in terms of lower limit, upper limits and lower bound are almost same and also band at different levels, but confidence interval and band of bootstrap is lesser then asymptotic confidence intervals values.

Table 4. *Correlation Matrix and Estimated Correlation Matrix of Adelaide data set*

Correlation Matrix											
Variable	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11
V1	1	0.022	0.009	0.044	0.051	0.022	0.018	0.12	0.07	0.115	0.07
V2	0.022	1	0.017	0.032	0.019	0.069	0.013	0.049	0.0004	0.07	0.039
V3	0.009	0.017	1	0.048	0.047	0.047	0.082	0.07	0.062	0.091	0.056
V4	0.044	0.032	0.048	1	0.052	0.067	0.06	0.041	0.059	0.085	0.098
V5	0.051	0.019	0.047	0.052	1	0.05	0.101	0.101	0.118	0.092	0.045
V6	0.022	0.069	0.047	0.067	0.05	1	0.063	0.068	0.078	0.098	0.026
V7	0.018	0.013	0.082	0.06	0.101	0.063	1	0.093	0.14	0.125	0.065
V8	0.12	0.049	0.07	0.041	0.101	0.068	0.093	1	0.071	0.145	0.052
V9	0.07	0.0004	0.062	0.059	0.118	0.078	0.14	0.071	1	0.071	0.092
V10	0.115	0.07	0.091	0.085	0.092	0.098	0.125	0.145	0.071	1	0.12
V11	0.07	0.039	0.056	0.098	0.045	0.026	0.065	0.052	0.092	0.12	1
Estimated Correlation Matrix using MOM											
V1	1	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066
V2	0.066	1	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066
V3	0.066	0.066	1	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066
V4	0.066	0.066	0.066	1	0.066	0.066	0.066	0.066	0.066	0.066	0.066
V5	0.066	0.066	0.066	0.066	1	0.066	0.066	0.066	0.066	0.066	0.066
V6	0.066	0.066	0.066	0.066	0.066	1	0.066	0.066	0.066	0.066	0.066
V7	0.066	0.066	0.066	0.066	0.066	0.066	1	0.066	0.066	0.066	0.066
V8	0.066	0.066	0.066	0.066	0.066	0.066	0.066	1	0.066	0.066	0.066
V9	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	1	0.066	0.066
V10	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	1	0.066
V11	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	1
Jennrich test: χ^2= 30.1314, p-value= 0.9974615 (H_0: all the correlations are equal)											

Table 5. *Correlation Matrix and Estimated Correlation Matrix of Perth data set*

Correlation Matrix											
Variable	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11
V1	1	0.052	0.063	0.121	0.113	0.03	0.023	0.117	0.003	0.075	0.109
V2	0.052	1	0.048	0.037	0.031	0.05	0.018	0.065	0	0.018	0.052
V3	0.063	0.048	1	0.11	0.05	0.032	0.04	0.034	0.048	0.029	0.044
V4	0.121	0.037	0.11	1	0.086	0.048	0.029	0.096	0.019	0.068	0.038
V5	0.113	0.031	0.05	0.086	1	0.022	0.002	0.118	0.003	0.128	0.049
V6	0.03	0.05	0.032	0.048	0.022	1	0.024	0.045	0.029	0.041	0.061
V7	0.023	0.018	0.04	0.029	0.002	0.024	1	0.026	0.088	0.068	0.09
V8	0.117	0.065	0.034	0.096	0.118	0.045	0.026	1	0.008	0.069	0.114
V9	0.003	0	0.048	0.019	0.003	0.029	0.088	0.008	1	0.011	0.029
V10	0.075	0.018	0.029	0.068	0.128	0.041	0.068	0.069	0.011	1	0.077
V11	0.109	0.052	0.044	0.038	0.049	0.061	0.09	0.114	0.029	0.077	1
Estimated Correlation Matrix using MOM											

V1	1	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053
V2	0.053	1	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053
V3	0.053	0.053	1	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053
V4	0.053	0.053	0.053	1	0.053	0.053	0.053	0.053	0.053	0.053	0.053
V5	0.053	0.053	0.053	0.053	1	0.053	0.053	0.053	0.053	0.053	0.053
V6	0.053	0.053	0.053	0.053	0.053	1	0.053	0.053	0.053	0.053	0.053
V7	0.053	0.053	0.053	0.053	0.053	0.053	1	0.053	0.053	0.053	0.053
V8	0.053	0.053	0.053	0.053	0.053	0.053	0.053	1	0.053	0.053	0.053
V9	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	1	0.053	0.053
V10	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	1	0.053
V11	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	1

Jennrich test: χ^2= 30.1976, p-value= 0.9973857 (H_0: all the correlations are equal)

Table 6. *Mean Vector of data sets*

Data Set	Mean Vector										
	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11
Adelaide	87821.85	87785.72	88185.84	87680.8	88436.53	87564.24	88660.64	88424.98	87703.94	89191.15	88471.47
Perth	88115.64	86299.28	87054.98	87490.71	87172.86	87227.25	87479.91	87259.51	86110.12	88026.43	87450.23

Table 7: *Confidence Intervals for tests of the "Wave Energy Converters Data Set"*

Confidence Intervals	90%			95%			99%		
	L	U	LB	L	U	LB	L	U	LB
Asymptotic	0.5567	0.5587	0.5577	0.5566	0.5588	0.5587	0.5565	0.5589	0.5588
Bootstrap	0.5563	0.5612	0.5568	0.5558	0.5617	0.5563	0.5549	0.5626	0.5553

4. Conclusions

In this article, we proposed a method to estimate the stress–strength reliability and all unknown parameters under the equi-corelated multivariate normal setup. We provide MOM method to estimate these unknown parameters and use them to estimate of δ and R. We also obtain the asymptotic distribution of estimated δ. The simulation results indicate that performance than MOM in terms of MSE, RB and MAE for different choices of the parameters. Simulation studies illustrate that, the MSE, RB and MAE of this estimator reduce as the sample size increases and they almost achieved the true value of R. Also, the simulation studies illustrate that the proposed method has the best coverage probability and also produces the shortest band of confidence intervals. The stress-strength reliability of the given data set is $\hat{R}$=0.55872. The performance of method of moments estimator (MOM) of R is consistent for different sample size and quite good for small sample size.

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