Comparison of Bridge Systems with Multiple Types of Components

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Abstract

This paper aims to compare some bridge systems with multiple types of components in stochastic, hazard rate, and likelihood ratio order. Such systems are generally used in the designing and production industries. These systems are supported by a buffer store that balances the fluctuation in two production lines during the production process. The survival signature tool and distortion function technique are employed to compare the performance of four different bridge systems. Survival signature and henceforth survival function is computed for each considered system. The findings of comparisons are facilitated with the help of tables and figures. The comparison of large size coherent systems based on the structure-function approach is quite challenging. As this study is based on survival signature, so it is not so complex and has future scope.

Keywords: survival signature; bridge system; survival function; distortion function.

1. Introduction

In today's competitive and technology-driven world, it has become consequential to develop safe, reliable and long-lasting systems. The accurate reliability assessment of components and systems is crucial, and hence the branch of reliability engineering is in very much demand. In reliability theory, the stochastic comparison of systems is an imperative concept and has been explored by many researchers. It is quite challenging to compare complex systems, and most realistic cases generally have complex structures. Birnbaum *et al.* [2] and Barlow and Proschan [1] compared the same order coherent systems based on component lifetime using the structure-function approach. But these methods involve analytical complexities while comparing complex manufacturing systems. Recently, system signature and survival signature have emerged as advanced and promising tools in reliability analysis. These tools have suitable applications in studying system reliability and comparing various coherent systems.

A system having monotonic structure function with each of its components being relevant is known as coherent system. Samaniego [12] introduced the concept of system signature for the systems having independent and identically distributed (*iid*) components, with common distribution function F. For such coherent systems, Samaniego [12] derived an explicit expression of the failure rate in terms of components' failure rate and *F*. The IFR closure theorem for k-out-of-n system is also discussed by researcher. Kochar *et al.* [6] further derived the expression of system signature for k-out-of-n systems with component-wise and system-wise redundancy. Samaniego

[11] extended the concept of signature for preservation, characterisation, and system reliability. The applications of network reliability and economical reliability to systems having shared components are also presented. Navarro *et al.* [9] defined a joint signature for coherent systems with shared components. They discussed the sufficient condition for bivariate stochastic ordering between the joint lifetimes of two pairs of the systems.

Coolen and Coolen-Maturi [3] extended the concept of system signature to systems with multiple types of components, and they coined the new term 'survival signature'. The survival function of the coherent systems having *iid* and exchangeable components is evaluated using the survival signature tool. Coolen *et al.* [4] further adopted this technique and developed non parametric predictive inference for studying the reliability of systems. Krpelık *et al.* [8] introduced the formula for computing system survival signatures by means of merging survival signatures of multiple subsystems. They also introduced a decomposition method that allows decoupling the dependencies among subsystems. Huang *et al.* [5] analysed the reliability of the phased mission systems having identical components in each phase using survival signature.

Several authors have worked on the stochastic comparison of coherent systems. Kochar *et al.* [6] compared various systems on the basis of stochastic, hazard rate, and likelihood ordering using the notion of system signature. Authors derived an important theorem on hazard rate ordering of the system based on its components' hazard rate ordering. Coolen and Coolen-Maturi [3] compared some coherent systems with *iid* and non-*iid* components based on a novel technique of survival signature. Koutras *et al.* [7] stochastically compared two systems having exchangeable components. They further provided a necessary and sufficient condition for examining hazard rate ordering and reverse hazard rate ordering. Samaniego and Navarro [13] presented the methodology to compare some systems having heterogeneous components in different modes (stochastic, hazard rate, and likelihood ratio ordering) using survival signature and distortion function.

The bridge systems are broadly used in system designing in addition to the series and the parallel systems. Such systems are found in the production process in various industries. The production system having two parallel production lines connected by a buffer store to balance their productivity variation is investigated as a bridge structure system [10]. The analytical evaluation of the lifetime of the bridge system is too dense. Therefore, the comparison among such systems becomes more complicated. The present study compares the lifetimes of the bridge systems having multiple types of components at different positions. The survival signature technique is used to compare these complex systems. This paper investigates some bridge systems having two/three types of components shown in Figure 1, Figure 2, and Figure 3. The comparative analysis of considered systems is done using the survival signature approach [13].

2. Definitions and Notations

The present section includes prevalent concepts, definitions, and theorems. For 'm' components system, the state vector $x = (x_1, x_2, \dots . x_m) \in \{0,1\}^m$, where

$$x_i = \begin{cases} 1, & when\ i^{th}\ component\ of\ system\ is\ working \\ 0, & when\ i^{th} component\ of\ system\ is\ not\ working \end{cases}$$

for all $i = 1, 2, 3, \dots, m$. Thus, the set $\{0,1\}^m$ represents all the possible state vectors of m-order binary coherent system. Barlow and Proschan [1] defined the structure function ϕ mapped from the set $\{0,1\}^m$ to $\{0,1\}$ as follows

$$\phi(x_1, x_2, .., x_m) = \begin{cases} 1, if\ systems\ works \\ 0, if\ systems\ fails. \end{cases}$$

As compared to structure function, system signature [12] is less general but more significant. For the coherent system of order 'm', the system signature is a probability vector such that some i^{th} component causes system failure. Mathematically, the i^{th} element 's_i', of the system signature $s = (s_1, s_2, .., s_m)$ is expressed as

$$s_i = P\ (T = X_{i:m}) = \frac{m_i}{m!}$$

where T denotes the lifetime of the system, $X_{i:m}$ represents the i^{th} order statistic of the failure time of the m-components and m_i is number of those orderings corresponding to which system fails on failure of i^{th} component. It is evident that $\forall\ i,\ s_i \geq 0$ and $\sum_{i=1}^{m} s_i = 1$.

For a coherent system with m *iid* components having a continuous lifetime distribution, the survival signature $\Phi(l)$ for $l = 0,1,2, \dots, m$ is defined as the probability of functioning of system, provided that its exactly l components are working [3]. Mathematically, the survival signature of coherent system is given by

$$\Phi(l) = \frac{\sum_{x \epsilon s_l} \phi\ (x)}{|s_l|} = \binom{m}{l}^{-1} \sum_{x \epsilon s_l} \phi(x)$$

where s_l is the set of all such state vectors whose exactly l components (x_i) are 1 and remaining are 0. The system reliability $\bar{F}_T(t)$ in terms of survival signature for iid components is

$$\bar{F}_T(t) = P\ (T > t) = \sum_{l=0}^{m} \Phi(l) \binom{m}{l} [F(t)]^{m-l}\ [\bar{F}(t)]^l$$

where $F(t)$, $\bar{F}(t)$ be the distribution and survival function respectively of components.

Coolen and Coolen-Maturi [3] considered the coherent system of order m, with $K > 1$ types of independent components. All the components of certain type are assumed to be identically distributed. Considering m_k components of type k, the survival signature $\Phi(l_1, l_2, \dots., l_K)$ is given by

$$\Phi(l_1, l_2, \dots., l_K) = \left[\prod_{k=1}^{K} \binom{m_k}{l_k}^{-1}\right] \sum_{x \epsilon s_{l_1, l_2, \dots, l_k}} \phi(x)$$

where l_k $(k = 1,2, \dots, K)$ is the number of functioning units of type k. In the above expression, x is a state vector given by $x = (x^1, x^2, \dots., x^K)$, where $x^k = (x_1^k, x_2^k, \dots, x_{m_k}^k)$. In case l_k $(k = 1,2, \dots, K)$ units of type k are working, then the vector x^k has precisely its l_k components (x_i^k) as 1 and remaining are 0. The set of all such state vectors is denoted by $s_{l_1, l_2, \dots, l_k}$. The reliability function $\bar{F}_T(t)$ of such systems in terms of survival signature as given by Coolen and Coolen-Maturi [3] is

$$\bar{F}_T(t) = P(T > t) = \sum_{l_1=0}^{m_1} \dots. \sum_{l_K=0}^{m_k} \left[\Phi(l_1, l_2, \dots., l_K) \prod_{i=1}^{K} \binom{m_i}{l_i}\ F_i(t)^{m_i - l_i} \bar{F}_i(t)^{l_i}\right]$$

where $F_i(t)$, $\bar{F}_i(t)$ be the distribution and survival function of the i^{th} component.

Some results on stochastic order properties which appeared in [14] are discussed below. Let T_1, T_2 be the random variables with the distribution functions $F_1(t), F_2(t)$ and reliability functions $\overline{F_1}(t), \overline{F_2}(t)$ respectively, then

- T_1 is smaller than T_2 in usual stochastic order, i.e. $T_1 \leq_{ST} T_2$ if $\overline{F_1}(t) \leq \overline{F_2}(t)$for all t;
- T_1 is smaller than T_2 in the hazard rate order, i.e. $T_1 \leq_{HR} T_2$ if $\overline{F_2}(t)/\overline{F_1}(t)$ is increasing in t;
- T_1 is smaller than T_2 in the likelihood ratio order, i.e. $T_1 \leq_{LR} T_2$ if $f_2(t)/f_1(t)$ is increasing in t; where $f_1(t)$ and $f_2(t)$ are probability density functions (pdfs) of T_1 and T_2 respectively.

Samaniego and Navarro [13] also derived a result for the comparison of two systems having m_k independent type k components with distribution function F_k for $k \in \{1,2,..,r\}$. The following theorem appeared as Theorem 2.1. in Samaniego and Navarro [13] .

Theorem 1. If T_1, T_2 be the lifetimes and Φ_1, Φ_2 be survival signatures of two systems A and B respectively and if for all vectors $(l_1, \dots, l_r)$, with $l_k = 0, \dots, m_k$ and $k = 1, \dots., r$, the inequality

$$\Phi_1(l_1, \dots., l_r) \leq \Phi_2(l_1, \dots, l_r)$$

holds, then it follows that $T_1 \leq_{ST} T_2$ for all distribution functions $F_1, \dots, F_r$.

Samaniego and Navarro [13] further proved a theorem, which aids in the comparison of two systems having different orders. For such comparisons, some irrelevant components are considered and added to the systems. The following proved result appeared as Theorem 3.1 in Samaniego and Navarro [13] .

Theorem 2. Let Φ be the survival signature of *m*-order coherent system, having *r* types of components and suppose it has to be compared with some system of order *m*+1. An irrelevant component of type-*k* is added to *m*-order coherent system, and let Φ^* be the survival signature of resulting new *m*+1 order system. Considering m_j components of type *j*, Samaniego and Navarro [13] established following relations for survival signatures Φ and Φ^*

(i) $For\ 0 \leq l_j \leq m_j,\ j = 1,2,,..,k-1,k,k+1,\dots,r,$

$$\Phi^*(l_1, \dots l_{k-1}, 0, l_{k+1}, \dots, l_r) = \Phi(l_1, \dots l_{k-1}, 0, l_{k+1}, \dots, l_r)$$

(ii) $For\ 0 \leq l_j \leq m_j, j = 1,2,,..,k-1,k,\dots,r, and\ for\ 1 \leq l_k \leq m_k,$

$$\Phi^*(l_1, \dots l_{k-1}, l_k, \dots, l_r) = \left(\frac{l_k}{m_k+1}\right) \Phi(l_1, \dots l_{k-1}, l_k - 1, \dots, l_r) + \left(\frac{m_k - l_k + 1}{m_k + 1}\right) \Phi(l_1, \dots l_{k-1}, l_k, \dots, l_r)$$

(iii) $For\ 0 \leq l_j \leq m_j,\ j = 1,2,,..,k-1,k,k+1 \dots,r,$

$$\Phi^*(l_1, \dots l_{k-1}, m_k + 1, l_{k+1}, \dots, l_r) = \Phi(l_1, \dots l_{k-1}, m_k, l_{k+1}, \dots, l_r).$$

Samaniego and Navarro [13] also adopted a generalized distorted distribution technique for comparing two systems. They employed a dual distortion function, $\bar{Q}(u_1, u_2, \dots, u_r)$ and distortion function $Q(u_1, u_2, \dots, u_r)$ in this technique. These functions satisfy the following properties:

(i) $\bar{Q}(u_1, u_2, \dots, u_r)$ is a continuous increasing function

(ii) $\bar{Q}(u_1, u_2, \dots, u_r) = 0$ if $u_i = 0\ \forall\ i \in \{1,2,..,r\}$

(iii) $\bar{Q}(u_1, u_2, \dots, u_r) = 1$ if $u_i = 1\ \forall\ i \in \{1,2,..,r\}$.

(iv) $Q(u_1, u_2, \dots, u_r) = 1 - \bar{Q}(1-u_1, 1-u_2, \dots, 1-u_r)$

The survival function $\bar{F}_T(t)$ of coherent system having *r* types of components can be expressed as-

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t), \dots, \bar{F}_r(t)),$$

where $\overline{F_l}$ is the reliability function of components of type *l*. The lifetimes T_1 and T_2 of the two coherent systems with *r* types of components can be compared using the distortion function as discussed below. The following proved result appeared as Theorem 4.1. in Samaniego and Navarro [13].

Theorem 3. Let $F_1, F_2, \dots, F_r$ be the distribution functions of the components of type 1, type 2,…, type r respectively. Samaniego and Navarro [13] proved that if $\bar{Q}_1$ and $\bar{Q}_2$ be the dual distortion functions of two considered systems, then

(i) $T_1 \leq_{ST} T_2$ holds for all $F_1, \dots, F_r$ if and only if $\bar{Q}_1 \leq \bar{Q}_2$ in $(0,1)^r$;

(ii) $T_1 \leq_{HR} T_2$ holds for all $F_1, \dots, F_r$ if and only if $\bar{Q}_2/\bar{Q}_1$is decreasing in $(0,1)^r$;

(iii) $T_1 \leq_{LR} T_2$ holds for all $F_1, \dots, F_r$, if the distributions of T_1 and T_2 are absolutely continuous, and if $\gamma(u_1, u_2, \dots, u_r, v_2, \dots, v_r)$ is decreasing in $u_1, u_2, \dots, u_r$ and increasing (decreasing) in v_r in $(0,1)^r \times (0,\infty)^{r-1}$ and $F_1 \leq_{LR} F_i$ $(\geq_{LR})$ for i = 2, …, r where

$$\gamma(u_1, u_2, \dots, u_r, v_2, \dots, v_r) = \frac{D_1\bar{Q}_2(u_1,u_2,\dots,u_r)+\sum_{i=2}^{r} v_i D_i \bar{Q}_2(u_1,u_2,\dots,u_r)}{D_1\bar{Q}_1(u_1,u_2,\dots,u_r)+\sum_{i=2}^{r} v_i D_i \bar{Q}_1(u_1,u_2,\dots,u_r)},$$

$D_i\bar{Q}_j$ denotes the partial derivatives of $\bar{Q}_j$ about i^{th} component for $i \in \{1, \dots, r\}$ and $j \in \{1,2\}$ and u_r denotes components' reliability function of type r and v_r denotes the ratio of pdfs of components of type r to the type 1.

3. Analysis and Discussion

The purpose of this article is to compare the bridge systems having multiple types of components. The survival signature tool is used to compare the considered systems in three different senses (stochastic, hazard rate, and likelihood ratio ordering). The bridge system as shown in Figure 1 has two units x_{11}, x_{21} of type 1 and three components namely x_{12}, x_{22}, x_{32} of type 2. The second considered system as shown in Figure 2 has again two components of type 1 and three components type 2, but at different positions. The bridge system (Figure 3) having three types of components is also investigated in this study.

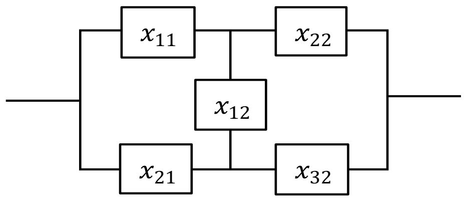

Figure 1: *System A (five-component bridge system)*

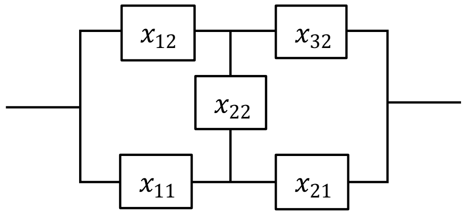

Figure 2: *System B (five-component bridge system with changed positions of components)*

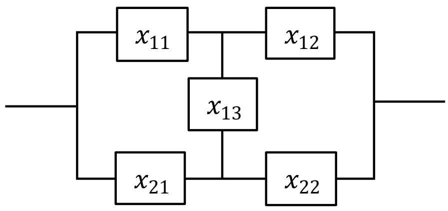

Figure 3: *System C (five-component bridge system containing three types of components)*

3.1. Comparison of two bridge systems with two types of components at different positions

Theorem 4. Consider two bridge systems of order five with two types of components at different positions. Let T_1, T_2 be the lifetimes of bridge systems A and B (Figure 1 and Figure 2) respectively. Then, T_1 is smaller than T_2 in usual stochastic order i.e., $T_1 \leq_{ST} T_2$.

Proof. Let $\Phi_1(l_1, l_2)$ and $\Phi_2(l_1, l_2)$ be the survival signatures of the systems A and B respectively. These systems have two and three components of type 1 and type 2 respectively. The general expression of the survival signature $\Phi(l_1, l_2)$ for considered systems is as follows

$$\Phi(l_1, l_2) = \binom{2}{l_1}^{-1} \binom{3}{l_2}^{-1} \sum_{x \epsilon s_{l_1, l_2}} \phi(x)$$

where s_{l_1, l_2} is set of all state vectors of the system.

Table 1: *Survival signature Φ_1 of the system A*

$\Phi_1(l_1, l_2)$	$l_2 = 0$	$l_2 = 1$	$l_2 = 2$	$l_3 = 3$
$l_1 = 0$	0	0	0	0
$l_1 = 1$	0	1/3	1	1
$l_1 = 2$	0	2/3	1	1

Table 2: *Survival signature Φ_2 of the system B*

$\Phi_2(l_1, l_2)$	$l_2 = 0$	$l_2 = 1$	$l_2 = 2$	$l_3 = 3$
$l_1 = 0$	0	0	1/3	1
$l_1 = 1$	0	0	2/3	1
$l_1 = 2$	1	1	1	1

As discussed in Theorem 1, the survival signatures $\Phi_1(l_1, l_2)$ and $\Phi_2(l_1, l_2)$ given in Table 1 and Table 2 are non-comparable because $\Phi_1(0,2) < \Phi_2(0,2)$ and $\Phi_1(1,2) > \Phi_1(1,2)$. Thus, the domination of survival signature is not possible for the considered systems. To compare these systems, we need to do further analysis. Let $\bar{F}_{T_1}(t), \bar{F}_{T_2}(t)$ be the survival functions of the bridge systems A and B with components distribution function $F_1(t)$ and $F_2(t)$. The difference between survival function $\bar{F}_{T_2}(t)$ and $\bar{F}_{T_1}(t)$ is given by

$$\bar{F}_{T_2}(t) - \bar{F}_{T_1}(t) = \sum_{l_1=0}^{2} \sum_{l_2=0}^{3} (\Phi_2(l_1, l_2) - \Phi_1(l_1, l_2)) \binom{2}{l_1}\binom{3}{l_2} F_1(t)^{2-l_1} \bar{F}_1(t)^{l_1} F_2(t)^{3-l_2} \bar{F}_2(t)^{l_2}. \quad (1)$$

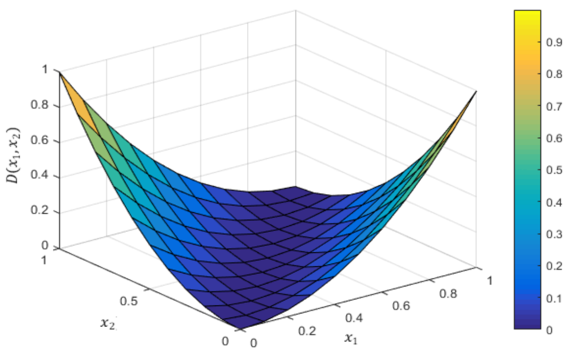

Figure 4: *The difference function $D(x_1, x_2)$.*

To simplify the system's comparison, we consider the variable $\bar{F}_1(t) = 1 - F_1(t)$ as x_1 and $\bar{F}_2(\mathrm{t}) = 1 - F_2(t)$ as x_2. So, the pair (x_1, x_2) belongs to unit square as $t \in [0, \infty)$. The above difference in Equation (1), is taken as $D(x_1, x_2)$ and can be represented as-

$$D(x_1,x_2) = {x_1}^2(1-x_2)^3 - 2x_1x_2(1-x_1)x_2(1-x_2)^2 + {x_1}^2x_2(1-x_2)^2 + {x_2}^2(1-x_1)^2(1-x_2) - 2x_1{x_2}^2(1-x_1)(1-x_2) + {x_2}^3(1-x_1)^2 .$$

The difference function $D(x_1, x_2)$ illustrated in Figure 4 has clearly non-negative values for each value of x_1 and x_2. i.e., $D(x_1, x_2) \geq 0,\ \forall\, x_1, x_2 \in [0,1]$. This implies that $\bar{F}_{T_2}(t) \geq \bar{F}_{T_1}(t)$. Hence the lifetime T_1 is smaller than lifetime T_2 in usual stochastic order i.e., $T_1 \leq_{ST} T_2$ holds for all $\bar{F}_1(t), \bar{F}_2(t)$.

3.2. Comparison of bridge systems using distortion functions

In this part, the systems A and B are compared as per stochastic, hazard rate and likelihood ratio ordering, by using their distortion functions.

Theorem 5. Let T_1, T_2 be the lifetimes of the bridge systems A and B (Figure 1 and Figure 2) respectively. These systems have two types of components with the distribution functions $F_1(t), F_2(t)$ and reliability function $\bar{F}_1(t), \bar{F}_2(t)$. The lifetime of system A is smaller than the lifetime of system B in usual stochastic order but not in hazard rate and likelihood ratio order.

Proof. Let $\bar{Q}_1$ and $\bar{Q}_2$ be dual distortion functions of systems A and B respectively. We have,

$$\bar{Q}_2(x_1,x_2) - \bar{Q}_1(x_1,x_2) = (1-x_2)^3{x_1}^2 - 2x_1x_2(1-x_2)^2(1-x_1) + {x_1}^2x_2(1-x_2)^2 + (1-x_1)^2{x_2}^2(1-x_2) - 2x_1{x_2}^2(1-x_1)(1-x_2) + {x_2}^3(1-x_1)^2.$$

Figure 4 indicates that $\bar{Q}_2(x_1, x_2) \geq \bar{Q}_1(x_1, x_2)\ \forall\, x_1, x_2 \in [0,1]$. Using Theorem 3, we can say that the system lifetime T_1 is smaller than system lifetime T_2 in usual stochastic order. i.e., $T_1 \leq_{ST} T_2$ hold for all $\bar{F}_1(t), \bar{F}_2(t)$.

Let R be the ratio of $\bar{Q}_2$ to $\bar{Q}_1$ i.e.,

$$R(x_1, x_2) = \frac{\bar{Q}_2}{\bar{Q}_1}$$

Figure 5 exhibits that the ratio $R(x_1, x_2)$ is neither increasing nor decreasing in x_1, x_2 in $(0,1)^2$. Data presented in Table 3 confirms the same. Using Theorem 3, we can say the system lifetime T_1 is not smaller than system lifetime T_2 in hazard rate order i.e., $T_1 \nleq_{HR} T_2$. Figure 6, further shows that these two bridge systems are not hazard rate ordered when the components of type-1 and type-2 follow exponential and Weibull distribution respectively.

Table 3: *The ratio* $R(x_1, x_2)$

$x_1 \rightarrow$ $x_2 \downarrow$	0.00010	0.09999	0.19988	0.29977	0.39966	0.49955	0.59944	0.69933
0.00010	1.000	499.908	999.330	1498.767	1998.220	2497.687	2997.169	3496.667
0.09999	458.761	1.000	1.235	1.635	2.086	2.566	3.068	3.588
0.19988	861.762	1.220	1.000	1.076	1.236	1.436	1.663	1.912
0.29977	1239.023	1.567	1.073	1.000	1.039	1.129	1.251	1.397

0.39966	1611.871	1.939	1.216	1.037	1.000	1.024	1.085	1.172
0.49955	1998.480	2.331	1.390	1.121	1.023	1.000	1.017	1.063
0.59944	2417.177	2.754	1.588	1.231	1.081	1.017	1.000	1.013
0.69933	2889.461	3.225	1.812	1.365	1.162	1.061	1.013	1.000

To compare the hazard rate ordering of system A and component of type 1, the ratio $R^1_{x_1}(x_1, x_2)$ is computed. We get

$$R^1_{x_1} = \frac{\bar{Q}_1(x_1, x_2)}{x_1} = 2x_2(1-x_2)^2(1-x_1) + 6{x_2}^2(1-x_1)(1-x_2) + 2{x_2}^3(1-x_1) + 2x_1x_2(1-x_2)^2 + 3x_1{x_2}^2(1-x_2) + x_1{x_2}^3.$$

Figure 7 indicates that the ratio $R^1_{x_1}(x_1, x_2)$ increases with increase in x_2, but it decreases with increase in x_1 in $(0,1)^2$. Therefore, the lifetimes of system A and type 1 components are not comparable in the hazard rate order, i.e., $T_1 \nleq_{HR} X_1$ where X_1 indicates the type 1 component's lifetime. Similarly, for hazard rate order comparison of system A and the type 2 components, the ratio $R^1_{x_2}$ is evaluated. We obtain

$$R^1_{x_2} = \frac{\bar{Q}_1(x_1, x_2)}{x_2} = 2x_1(1-x_2)^2(1-x_1) + 6x_1x_2(1-x_1)(1-x_2) + 2x_1{x_2}^2(1-x_1) + 2{x_1}^2(1-x_2)^2 + 3{x_1}^2x_2(1-x_2) + {x_1}^2{x_2}^2.$$

Here, the ratio $R^1_{x_2}(x_1, x_2)$ decreases with increase in x_1, but it neither increases nor decreases with increase in x_2 in $(0,1)^2$. Therefore, $T_1 \nleq_{HR} X_2$, where X_2 is type 2 component's lifetime. In the same manner, system B is compared with type 1 and type 2 components in hazard rate order by evaluating the ratios $R^2_{x_1}$ and $R^2_{x_2}$ respectively. We have

$$R^2_{x_1} = \frac{\bar{Q}_2(x_1, x_2)}{x_1} = \frac{{x_2}^2}{x_1}(1-x_2)(1-x_1)^2 + \frac{{x_2}^3}{x_1}(1-x_1)^2 + 4(1-x_1){x_2}^2(1-x_2) + 2{x_2}^3(1-x_1) + x_1(1-x_2)^3 + 3x_1x_2(1-x_2)^2 + 3x_1(1-x_2){x_2}^2 + x_1{x_2}^3$$

and

$$R^2_{x_2} = \frac{\bar{Q}_2(x_1, x_2)}{x_2} = x_2(1-x_2)(1-x_1)^2 + (1-x_1)^2{x_2}^2 + 4x_1x_2(1-x_1)(1-x_2) + 2x_1{x_2}^2(1-x_1) + \frac{{x_1}^2(1-x_2)^3}{x_2} + 3{x_1}^2(1-x_2)^2 + 3{x_1}^2x_2(1-x_2) + {x_1}^2{x_2}^2.$$

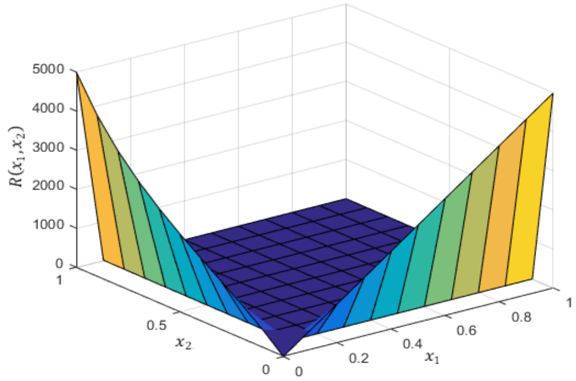

Figure 5: *The Graphical interpretation of the function* $R(x_1, x_2)$

Here, the ratio $R^2_{x_1}(x_1, x_2)$ increases with x_2 but it is not monotonic in x_1 in (0,1)². Thus, $T_2 \nleq_{HR} X_1$. The ratio $R^2_{x_2}(x_1, x_2)$ increases with increase in x_1 but decreases with increase in x_2 in (0,1)². Therefore, $T_2 \nleq_{HR} X_2$.

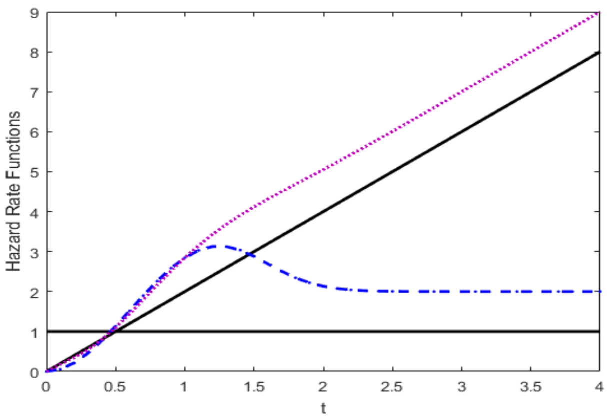

Figure 6: *Hazard rate functions of the bridge systems (A (dash), B (dot)) and their components (dark lines). Type 1 and Type 2 components follow exponential and Weibull distribution* $(a = 2, b = 1)$ *respectively for* $t > 0$

Let X_1, X_2 be the lifetimes of the components of type 1 and type 2 with respective pdfs $f_1(t)$, $f_2(t)$. The components of Type 1 and type-2 are assumed to be exponentially (mean = 1) and Weibull $(a = 2, b = 1)$ distributed respectively. The ratio $\frac{f_2(t)}{f_1(t)}$ is increasing in t as shown in Figure 8. Hence, we get that X_1 is smaller than X_2 in likelihood ratio ordering i.e., $X_1 \leq_{LR} X_2$. For likelihood ratio ordering comparison of systems A and B, as per Theorem 3, we have function $\Upsilon\left(x_1, x_2, \frac{f_2(t)}{f_1(t)}\right)$ as-

$$\Upsilon\left(x_1, x_2, \frac{f_2(t)}{f_1(t)}\right) = \frac{\begin{array}{c} v(2{x_2}^2(x_1-1)^2 - 2x_1{x_2}^2(x_1-1) - 2x_2(x_2-1)(x_1-1)^2 + 8x_1x_2(x_1-1)(x_2-1)) \\ -2x_1(x_2-1)^3 + 6x_1x_2(x_2-1)^2 - 2x_1{x_2}^2(x_2-1) + 2{x_2}^2(x_1-1)(x_2-1) \end{array}}{\begin{array}{c} v(2{x_1}^2(x_2-1)^2 - 2{x_1}^2x_2(x_2-1) - 2x_1(x_1-1)(x_2-1)^2 + 8x_1x_2(x_1-1)(x_2-1)) \\ -2{x_2}^3(x_1-1) + 2x_1x_2(x_2-1)^2 - 2x_2(x_1-1)(x_2-1)^2 + 6{x_2}^2(x_1-1)(x_2-1) \end{array}}$$

where $\frac{f_2(t)}{f_1(t)} = v$.

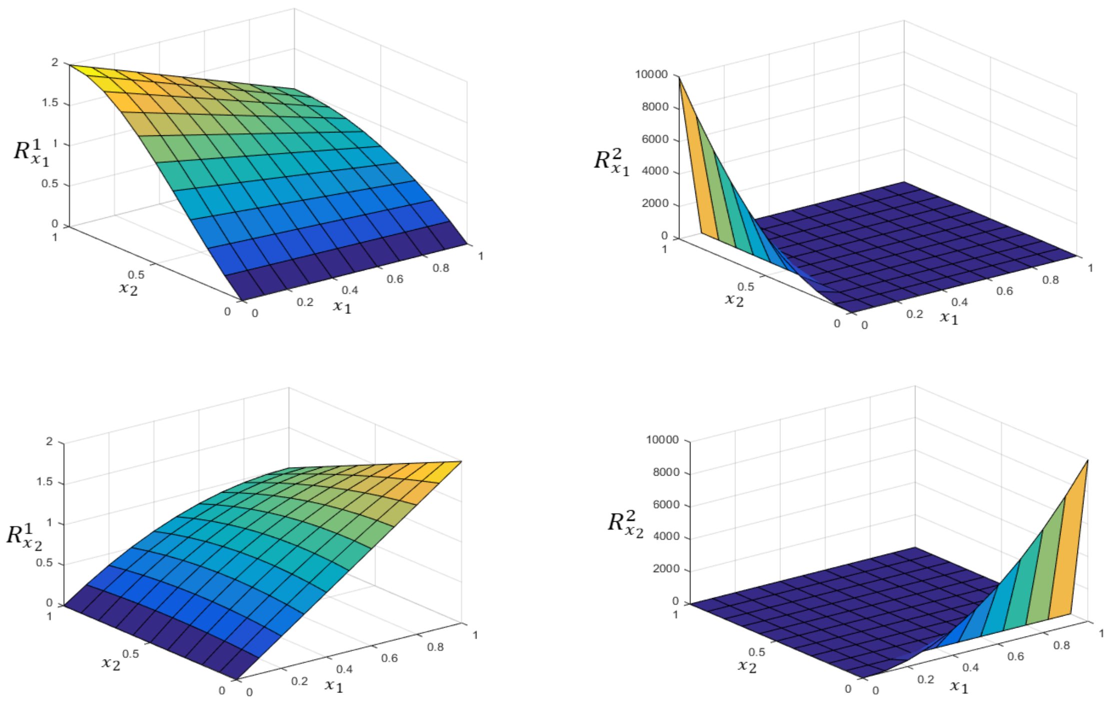

Figure 7: *The Ratio* $R^1_{x_1}$, $R^2_{x_1}$, $R^1_{x_2}$, $R^2_{x_2}$

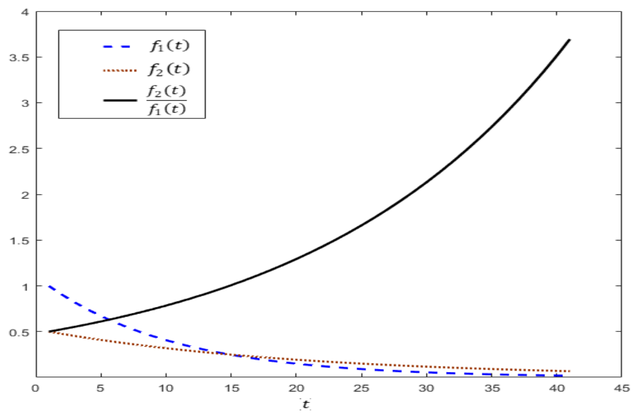

Figure 8: *Likelihood ratio ordering of the components of type 1 and type 2*

Table 4 indicates that the function $\Upsilon(x_1, x_2, 0.0001)$ is increasing in x_1 for the particular value of x_2. But we can see the function is neither increasing nor decreasing in x_2 for any particular values of x_1. In Table 5, the function $\Upsilon(x_1, 0.09999, v)$ is increasing in x_1 for the particular values of v. Table 5 further shows that the function is increasing in v for $x_1 = 0.0001$, but it is decreasing in v for $x_1 = 0.09999, 0.19998, 0.29977$. Hence, we get that the function $\Upsilon\left(x_1, x_2, \frac{f_2(t)}{f_1(t)}\right)$ is increasing in x but not monotonic in $\frac{f_2(t)}{f_1(t)}$ in set $(0,1)^2 \times (0, \infty)$. Therefore, these considered bridge systems are not likelihood ratio ordered. i.e., $T_1 \nleq_{LR} T_2$.

Table 4: *The function* $\Upsilon(x_1, x_2, 0.0001)$

$x_1 \rightarrow$ $x_2 \downarrow$	0.00010	0.09999	0.19988	0.29977	0.39966	0.49955	0.59944	0.69933
0.00010	1.000	908.936	1665.738	2306.254	2855.383	3331.387	3747.970	4115.610
0.09999	0.083	1.000	1.965	2.985	4.062	5.203	6.412	7.697
0.19988	0.138	0.549	1.000	1.497	2.049	2.664	3.355	4.135
0.29977	0.173	0.414	0.687	1.000	1.361	1.783	2.283	2.885
0.39966	0.193	0.349	0.530	0.744	1.000	1.312	1.701	2.199
0.49955	0.200	0.304	0.428	0.579	0.764	1.000	1.307	1.725
0.59944	0.193	0.264	0.349	0.454	0.587	0.762	1.000	1.343
0.69933	0.173	0.219	0.276	0.346	0.438	0.561	0.735	1.000

Table 5: *The function* $\Upsilon(x_1, 0.09999, v)$

$x_1 \rightarrow$ $x_2 \downarrow$	0.00010	0.09999	0.19988	0.29977	0.39966	0.49955	0.59944	0.69933
0.00010	0.083	1.000	1.965	2.985	4.062	5.203	6.412	7.697
0.09999	0.175	1.000	1.715	2.349	2.923	3.451	3.945	4.413
0.19988	0.266	1.000	1.540	1.963	2.312	2.611	2.876	3.116
0.29977	0.358	1.000	1.410	1.704	1.932	2.119	2.279	2.422
0.39966	0.449	1.000	1.311	1.519	1.672	1.795	1.898	1.989
0.49955	0.541	1.000	1.232	1.379	1.484	1.565	1.634	1.694
0.59944	0.633	1.000	1.168	1.270	1.340	1.395	1.440	1.479
0.69933	0.724	1.000	1.115	1.182	1.228	1.263	1.291	1.316

3.3. Comparison of two bridge systems with different number of components

Theorem 6. Suppose T_1, T_4 be the lifetimes of the bridge systems A and D shown in Figure 1 and Figure 9 respectively. The system D has six components, where type 1 components are x_{11}, x_{21} and x_{31} and type 2 components are x_{12}, x_{22} and x_{32}. Then the lifetime T_4 is smaller than T_1 in usual stochastic order. i.e., $T_4 \leq_{ST} T_1$.

Proof. Let $\Phi_4(l_1, l_2)$ be the survival signature of the system D. The survival signature $\Phi_1(l_1, l_2)$ of system A is already discussed and given in Table 1. An independent irrelevant component of type $k = 1$ is added to system A, and let us suppose that $\Phi_1^*(l_1, l_2)$ be the survival signature of new resulting system of order 6. Using Theorem 2, we have

(i) $For\ 0 \leq l_2 \leq m_2$

$$\Phi_1^*(0, l_2) = \Phi_1(0, l_2)$$

(ii) For $1 \leq l_1 \leq m_1$ and $0 \leq l_2 \leq m_2$

$$\Phi_1^*(l_1, l_2) = \left(\frac{l_1}{m_1 + 1}\right) \Phi_1(l_1 - 1, l_2) + \left(\frac{m_1 - l_1 + 1}{m_1 + 1}\right) \Phi_1(l_1, l_2)$$

(iii) For $0 \leq l_2 \leq m_2$

$$\Phi_1^*(m_1 + 1, l_2) = \Phi_1(m_1, l_2).$$

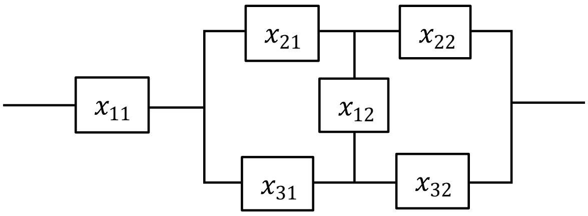

Figure 9: *System D (six-component bridge system)*

Tables 6 and 7 indicate that the survival signature Φ_1^* is greater than Φ_4 for all values of l_1, l_2 i.e., $\Phi_1^*(l_1, l_2) \geq \Phi_4(l_1, l_2)\ \forall\ l_1, l_2 \in \{0,1,2,3\}$. Thus, the lifetime T_4 is smaller than T_1 in usual stochastic order, i.e., $T_4 \leq_{ST} T_1$.

Table 6: *Survival signature Φ_1^* of the bridge system A with irrelevant component of type-1*

$\Phi_1^*(l_1, l_2)$	$l_2 = 0$	$l_2 = 1$	$l_2 = 2$	$l_2 = 3$
$l_1 = 0$	0	0	0	0
$l_1 = 1$	0	2/9	2/3	2/3
$l_1 = 2$	0	4/9	1	1
$l_1 = 3$	0	2/3	1	1

Table 7: *Survival signature Φ_4 of the bridge system D*

$\Phi_4(l_1, l_2)$	$l_2 = 0$	$l_2 = 1$	$l_2 = 2$	$l_2 = 3$
$l_1 = 0$	0	0	0	0
$l_1 = 1$	0	0	0	0
$l_1 = 2$	0	2/9	2/3	2/3
$l_1 = 3$	0	2/3	1	1

3.4. Comparison of lifetimes of two bridge systems with two and three types of components

Theorem 7. Consider two bridge systems A and C, shown in Figures 1 and 3. Let T_1 and T_3 be the respective lifetimes of systems A and C. Type 1, type 2 and type 3 components are assumed to be *iid* with reliability functions $\bar{F}_1$, $\bar{F}_2$ and $\bar{F}_3$ respectively. Then $T_1 \leq_{ST} T_3$ if $\bar{F}_2(t) \leq \bar{F}_3(t)$.

Proof. Let $\Phi_1(l_1, l_2)$ and $\Phi_3(l_1, l_2, l_3)$ be the survival signature of bridge systems A and C respectively. Here, system C contains two components of type 1, two components of type 2, and one component of type 3. The survival signature $\Phi_3(l_1, l_2, l_3)$ can be written as $\Phi_3(l_1, l_2, l_3) = \binom{2}{l_1}^{-1}\binom{2}{l_2}^{-1}\binom{1}{l_3}^{-1}\sum_{x \epsilon s_{l_1,l_2,l_3}} \phi(x)$, and is given in Table 8. For comparison of bridge systems A and C, we have added an irrelevant component of type 3 ($k = 3$) to system A. Using Theorem 2, we have survival signature $\Phi_1^*(l_1, l_2, l_3)$ of resulting 6-components system as:

(i) For $0 \leq l_j \leq m_j;\ j = 1,2$

$$\Phi_1^*(l_1, l_2, 0) = \Phi(l_1, l_2, 0)$$

(ii) For $0 \leq l_j \leq m_j\ ; j = 1,2$

$$\Phi_1^*(l_1, l_2, m_3 + 1) = \Phi(l_1, l_2, m_3).$$

Similarly, we have added one component of type 2 ($k = 2$) which is irrelevant in nature to system C. Using Theorem 2, the survival signature Φ_3^* of the resultant 6-components system is given by

(i) For $0 \le l_j \le m_j;\ j = 1,3$

$$\Phi_3^*(l_1, 0, l_3) = \Phi_3(l_1, 0, l_3)$$

(ii) For $0 \le l_j \le m_j\ ;j = 1,3$ and $1 \le l_2 \le m_2$

$$\Phi_3^*(l_1, l_2, l_3) = \frac{l_2}{3}\Phi_3(l_1, l_2 - 1, l_3) + \frac{3 - l_2}{3}\Phi_3(l_1, l_2, l_3)$$

(iii) For $0 \le l_j \le m_j\ ;j = 1,3$

$$\Phi_3^*(l_1, m_2 + 1, l_3) = \Phi_3(l_1, m_2, l_3).$$

Table 8: *The survival signature $\Phi_3(l_1, l_2, l_3)$ of the system C*

l_1	l_2	l_3	$\Phi_3(l_1, l_2, l_3)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
0	2	0	0
0	2	1	0
1	0	0	0
1	0	1	0
1	1	0	1/2
1	1	1	1
1	2	0	1
1	2	1	1
2	0	0	0
2	0	1	0
2	1	0	1
2	1	1	1
2	2	0	1
2	2	1	1

Table 9 shows that the survival signatures Φ_1^* and Φ_3^* are identical for all the combinations of l_1, l_2, l_3 except two cases. The survival signature Φ_1^* and Φ_3^* are not dominated in any sense since $\Phi_1^*(1,1,1) < \Phi_3^*(1,1,1)$ but $\Phi_1^*(1,2,0) > \Phi_3^*(1,2,0)$. So, the comparison of systems A and C needs further analysis. Let $\bar{F}_{T_1}(t), \bar{F}_{T_3}(t)$ be the respective reliability functions of systems A and C. We have

$$\bar{F}_{T_3}(t) - \bar{F}_{T_1}(t) = \sum_{l_1=0}^{2}\sum_{l_2=0}^{3}\sum_{l_3=0}^{1}[(\Phi_3^*\ (l_1, l_2, l_3) - \Phi_1^*(l_1, l_2, l_3)]\binom{2}{l_1}\binom{3}{l_2}\binom{1}{l_3}\ F_1(t)^{2-l_1}\bar{F}_1(t)^{l_1}F_2(t)^{3-l_2}\bar{F}_2(t)^{l_2}F_3(t)^{1-l_3}\bar{F}_3(t)^{l_3}$$

Using survival signature given in Table 9, we get

$$\bar{F}_{T_3}(t) - \bar{F}_{T_1}(t) = -2\ F_1(t)\bar{F}_1(t)F_2(t)\bar{F}_2(t)^2F_3(t) + 2\ F_1(t)\bar{F}_1(t)F_2(t)^2\bar{F}_2(t)\bar{F}_3(t) \quad (2)$$

Table 9: *The survival signature Φ_1^* and Φ_3^* of systems after adding an irrelevant component of type-3 and type-2 respectively to system A and system C*

l_1	l_2	l_3	$\Phi_1^*(l_1, l_2, l_3)$	$\Phi_3^*(l_1, l_2, l_3)$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
0	2	0	0	0
0	2	1	0	0
0	3	0	0	0
0	3	1	0	0
1	0	0	0	0
1	0	1	0	0
1	1	0	1/3	1/3
1	1	1	1/3	2/3
1	2	0	1	2/3
1	2	1	1	1
1	3	0	1	1
1	3	1	1	1
2	0	0	0	0
2	0	1	0	0
2	1	0	2/3	2/3
2	1	1	2/3	2/3
2	2	0	1	1
2	2	1	1	1
2	3	0	1	1
2	3	1	1	1

To simplify the comparison process, we have taken variable $\bar{F}_1(t)$ =1 $- F_1(t)$ as x_1, $\bar{F}_2(t)$ =1 $- F_2(t)$ as x_2 and $\bar{F}_3(t)$ =1 $- F_3(t)$ as x_3. The 3-tuple (x_1, x_2, x_3) lies in the unit cube as t varies from 0 to ∞. For $t \in [0, \infty)$, the difference $\bar{F}_{T_3}(t) - \bar{F}_{T_1}(t)$ given in Equation (2) can be written as the multivariable function $D(x_1, x_2, x_3)$ as

$$D(x_1, x_2, x_3) = -2x_1(1 - x_1)(1 - x_2)(1 - x_3)y^2 + 2x_1x_2x_3(1 - x_1)(1 - x_2)^2$$
$$= 2x_1x_2(1 - x_1)(1 - x_2)(x_3 - x_2).$$

If $x_2 \leq x_3$ or x_3 = 1 then $D(x_1, x_2, x_3) \geq 0$. In addition, $D(x_1, x_2, x_3) = 0 \; if \; x_1, x_2 = 1 \; or \; x_2 = x_3$. This implies that the system's lifetime T_1 is smaller than T_3 in usual stochastic order if the components lifetime of type 2 is less than the component lifetime of type 3. i.e., $T_1 \leq_{ST} T_3$ if $\bar{F}_2(t) \leq \bar{F}_3(t)$.

4. Conclusion

The bridge structures are generally used in the design and production industry. The comparative study of such systems is crucial to ensure system productivity and to distinguish the system that performs well. Comparing bridge systems having *iid* and multiple types of components without knowing their component's distribution is very challenging. In this paper, we have seen that the lifetime of bridge system A is smaller than the lifetime of bridge system B in usual stochastic order.

However, the lifetimes of these systems (Figure 1 and Figure 2) are not found to be hazard rate and likelihood ratio ordered. Further, coherent systems A (five order) and D (six order) are compared stochastically by adding irrelevant components. It is found that the lifetime of system D is smaller than A in usual stochastic order. For stochastic comparison of lifetimes of bridge systems A and C, a result has been derived by imposing some conditions on the survival function of its components. This study compares bridge systems by considering different cases with the aid of survival signature. There is further scope to analyse the reliability characteristics and compare the combination of higher-order multi-state bridge systems with different types of components.

Conflict of Interest Declaration: The authors have no conflicts of interest to declare.

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