Classical and Bayesian Estimation of Parameter of $SS_E(\epsilon)$-distribution Under Type-II Censored Data

P. Kumar[1], D. Kumar[2], P. Kumar[3] & U. Singh[4]

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[1]Department of Statistics, Faculty of Science and Technology,
Mahatma Gandhi Kashi Vidyapith, Varanasi, India-221002
[2,4]Department of Statistics, Banaras Hindu University, Varanasi, India-221005
[3]Department of Statistics, Udai Pratap Autonomous College, Varanasi, India-221003
pawanchauhanstranger@gmail.com

Abstract

In this present piece of work, we have considered a lifetime distribution based on trigonometric function called $SS_E(\epsilon)$-distribution and discuss its various properties which have not been added previously by host as well as any other authors. This distribution is useful and a good contribution in research under trigonometric function. We are deriving some more useful properties such as moments, conditional moments, mean deviation about mean, mean deviation about median, order statistics etc. Estimation of parameter has been done for both classical and Bayesian paradigms under Type-II censored sample. Simulation study has also been carried out to know the progress of the estimators in the sense of having smallest risk (over the sample space) at the long-run use.

Keywords: $SS_E(\epsilon)$-distribution, Type-II censoring, Bayes estimator, MLE, Gauss-Laguerre method, risk function

1. Introduction

In statistical literature, there are several lifetime distributions available, for example exponential, gamma, Weibull, Lindley distribution etc. In past studies, calculations can only be handled when the expressions corresponding to various properties obtained in the nice closed form and when this was not achieved then rarely preferred. But in this modern era due to the advancement of computational facilities this problem have been resolved almost. Mostly, algebraic and exponential functions have been used to develop the new transformation and sometimes authors see gap in trigonometric, inverse and logarithmic type transformations. Keeping this in mind, the considered transformation is the good contribution in support of filling such gap. As we aware that the use of a single model is not found suitable in every aspect, therefore to adopt a suitable baseline model is also a quite tedious job. Study explores that exponential distribution is preferably used as a lifetime distribution but the extensive use of it is restrictive in the sense of its constant hazard rate. For simplicity and flexibility, we are also using here exponential distribution as a baseline distribution In these days, many authors are introducing transformation techniques to get a new lifetime distribution with the help of available baseline distributions some of which are popular as power transformation proposed by [6], sine square distribution by [1], [20] introduced quadratic rank transmutation map (QRTM), sinofarm distribution by [23], DUS transformation proposed by [10], minimum-guarantee distribution proposed by [11], CS transformation by [3], new Sine-G family based on [13] proposed by [16], new extension of Lindley distribution given by [17], PCM transformation by [12] and many more. In such continuation, [13] have proposed a new transformation known as SS-transformation by using sine function which is given by

$$F(x) = \sin\left(\frac{\pi}{2}G(x)\right) \tag{1}$$

Where $G(x)$ is the baseline's cumulative distribution function (cdf) and the accompanying probability density function (pdf) are

$$f(x) = \frac{\pi}{2}g(x)\cos\left(\frac{\pi}{2}G(x)\right) \tag{2}$$

They have utilized baseline distribution as exponential distribution and named as SS exponential ($SS_E(\epsilon)$)-distribution and having the following form of its pdf is

$$f(x) = \frac{\pi}{2}\epsilon \times e^{-\epsilon x}\sin\left(\frac{\pi}{2}e^{-\epsilon x}\right) \quad ; \ (x,\epsilon) > 0 \tag{3}$$

and its cdf in compact form is

$$F(x) = \cos\left(\frac{\pi}{2}e^{-\epsilon x}\right) \quad ; \ (x,\epsilon) > 0 \tag{4}$$

The reported compact forms of reliability function and hazard rate function respectively are

$$R(x) = 1 - \cos\left(\frac{\pi}{2}e^{-\epsilon x}\right) \tag{5}$$

and

$$h(x) = \frac{\pi}{2}\epsilon \times e^{-\epsilon x}\cot\left(\frac{\pi}{2}e^{-\epsilon x}\right) \tag{6}$$

Figures 1, 2 and 3 presents the shape of pdf, cdf and hazard rate function of $SS_E(\epsilon)$-distribution. And Figure 3, claims that the nature of hazard rate function of the $SS_E(\epsilon)$-distribution is increasing which is different from baseline distribution.
The article is constructed as follows, introductory part have been shown in Section (1), statistical properties discussed in Section (2), estimation of parameter presented in Section (3), comparison of estimators in Section (4) and concluding remarks regarding the work quoted in Section (5).

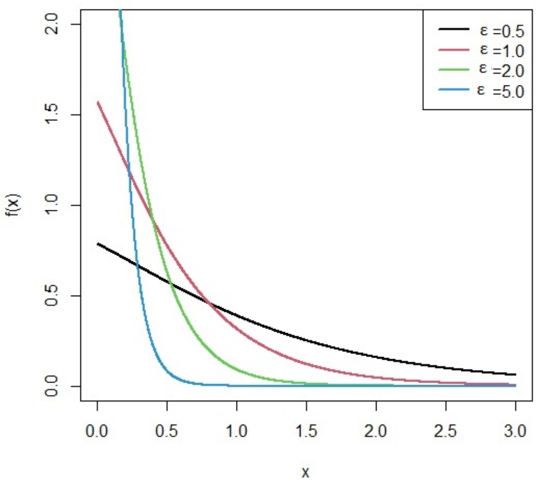

Figure 1: *Plots of pdf of $SS_E(\epsilon)$-distribution for various choices of parameter ϵ.*

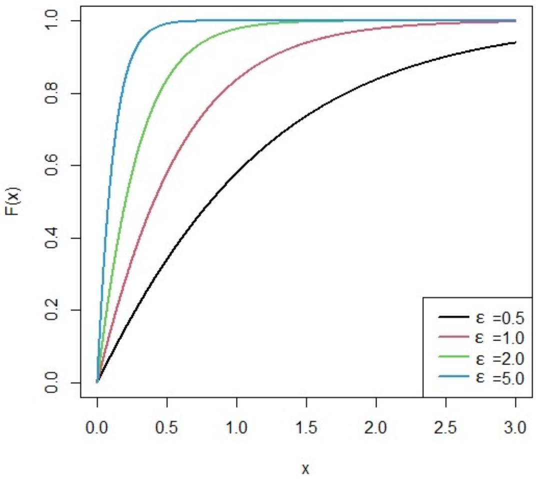

Figure 2: *Plots of cdf of $SS_E(\epsilon)$-distribution for various choices of parameter ϵ.*

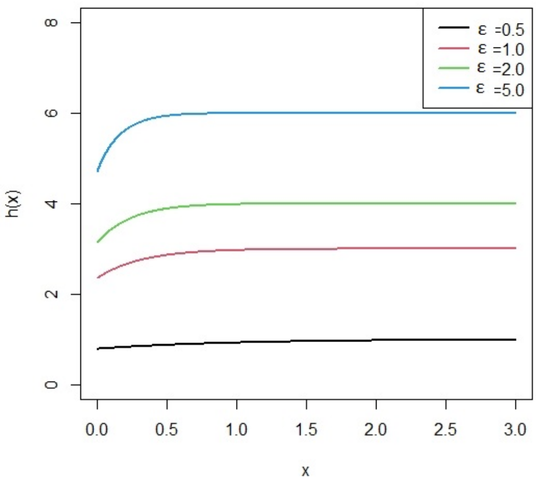

Figure 3: *Plots of hazard rate function of $SS_E(\epsilon)$-distribution for various choices of parameter ϵ.*

2. Statistical Properties

In this section, we are discussing some statistical properties of $SS_E(\epsilon)$-distribution which have not derived yet namely mean deviation about mean, mean deviation about median, order statistics etc. Firstly, we have discussed two lemma which are-

Statement (Lemma-1)

$$\xi_1(\epsilon, r, \zeta) = \int_0^\infty x^r e^{-\zeta x} \times \sin\left(\frac{\pi}{2} e^{-\epsilon x}\right) dx = \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!} \times \left(\frac{\pi}{2}\right)^{2k+1} \left[\frac{r!}{((2k+1)\epsilon + \zeta)^{r+1}}\right]$$

Proof:

$$\xi_1(\epsilon, r, \zeta) = \int_0^\infty x^r e^{-\zeta x} \sin\left(\frac{\pi}{2} e^{-\epsilon x}\right) dx = \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \left[\int_0^\infty x^r e^{-((2k+1)\epsilon + \zeta)x} dx\right]$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!} \times \left(\frac{\pi}{2}\right)^{2k+1} \left[\frac{r!}{((2k+1)\epsilon + \zeta)^{r+1}}\right]$$

The r^{th} order moment about origin of $SS_E(\epsilon)$-distribution have already obtained by [13]. Here, we obtain the same by using lemma 1 and is

$$E(X^r) = \frac{\pi}{2} \epsilon \times \xi_1(\epsilon, r, \epsilon) \tag{7}$$

on putting $r = 1, 2, 3, 4$ in (7), we get the first four raw moments of $SS_E(\epsilon)$-distribution and are

$$E(X) = \frac{\pi}{2} \epsilon \times \xi_1(\epsilon, 1, \epsilon) \ ; \ E(X^2) = \frac{\pi}{2} \epsilon \times \xi_1(\epsilon, 2, \epsilon)$$
$$E(X^3) = \frac{\pi}{2} \epsilon \times \xi_1(\epsilon, 3, \epsilon) \ ; \ E(X^4) = \frac{\pi}{2} \epsilon \times \xi_1(\epsilon, 4, \epsilon)$$

And, first four central moments are calculated by the following relations,

$$\mu_2 = \mu_2' - \mu_1'^2$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

On using above relations of central moments, we can obtain the measures of skewness and kurtosis, viz., β_1, γ_1 and β_2, γ_2 respectively by the following expressions

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \implies \gamma_1 = \sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}}$$
$$\beta_2 = \frac{\mu_4}{\mu_2^2} \implies \gamma_2 = \beta_2 - 3 = \frac{\mu_4}{\mu_2^2} - 3$$

Statement (Lemma-2)

$$\xi_2(\epsilon, r, \zeta, t) = \int_t^\infty x^r e^{-\zeta x} \times \left[\sin\left(\frac{\pi}{2} e^{-\epsilon x}\right)\right] dx$$
$$= \sum_{k=0}^{\infty} \sum_{l=0}^{r} \frac{(-1)^{2k+1}}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \times \left[\frac{r! \times e^{-((2k+1)\epsilon + \zeta)t} \times (((2k+1)\epsilon + \zeta)t)^l}{l!((2k+1)\epsilon + \zeta)^{r+1}}\right]$$

Proof:

$$\xi_2(\epsilon, r, \zeta, t) = \int_t^\infty x^r e^{-\zeta x} \left[\sin\left(\frac{\pi}{2} e^{-\epsilon x}\right)\right] dx = \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \times \int_t^\infty x^r e^{-((2k+1)\epsilon + \zeta)x} dx$$
$$= \sum_{k=0}^{\infty} \sum_{l=0}^{r} \frac{(-1)^{2k+1}}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \times \left[\frac{r! \times e^{-((2k+1)\epsilon + \zeta)t} \times [((2k+1)\epsilon + \zeta)t]^l}{l! \left[(2k+1)\epsilon + \zeta\right]^{r+1}}\right]$$

2.1. Conditional Moments

The conditional moment of r^{th} order is represented by $E(X^r|X > r)$ then by using lemma 2, we get

$$E(X^r|X > x) = \frac{\pi}{2} \times \frac{\epsilon \times \xi_2(\epsilon, r, \epsilon, x)}{\left[1 - cos\left(\frac{\pi}{2}e^{-\epsilon x}\right)\right]} \tag{8}$$

2.2. Quantile Function

Using (4), we get the quantile function of order p $(Q(p))$ is

$$\cos\left(\frac{\pi}{2}e^{-\epsilon Q(p)}\right) = p \implies e^{-\epsilon Q(p)} = \frac{2}{\pi}\cos^{-1}(p)$$

Therefore,

$$Q(p) = -\frac{1}{\epsilon}\ln\left(\frac{2}{\pi}\cos^{-1}(p)\right) \tag{9}$$

2.3. Median

On putting $p = \frac{1}{2}$ in equation (9) we will easily get the median of $SS_E(\epsilon)$-distribution and if M_d be the median of $SS_E(\epsilon)$-distribution then the expression is

$$M_d = -\frac{1}{\epsilon}\ln\left(\frac{2}{3}\right) \tag{10}$$

which is same expression as obtained by [13].

Table 1: *Mean, median, variance, skewness and kurtosis of $SS_E(\epsilon)$-distribution for different values of ϵ.*

ϵ	Mean	Median	Variance	Skewness(γ_1)	Kurtosis(γ_2)
0.2	0.25814	2.02733	0.25695	4.13316	19.53054
0.7	0.25379	0.57924	0.16079	2.73800	6.85106
1	0.23603	0.40547	0.11153	2.53429	5.44002
2	0.17377	0.20273	0.04342	2.28171	3.84498
5	0.09108	0.08109	0.00919	2.12107	2.91886
10	0.05023	0.04055	0.00253	2.06265	2.60720
15	0.03446	0.02703	0.00118	2.04642	2.45738

Table 1 shows that the mean, median, variance, skewness (γ_1) and kurtosis (γ_2) of the $SS_E(\epsilon)$-distribution with pdf (3) for different choices of parameter ϵ. The values of mean, median and variance of $SS_E(\epsilon)$-distribution are decreases as values of parameter ϵ increases this shows that mean, median and variance of the $SS_E(\epsilon)$-distribution inversely related to parameter ϵ. Since, $\gamma_1 > 0$ and $\gamma_2 > 0$ so the nature of $SS_E(\epsilon)$-distribution has positively skewed and leptokurtic distribution for considered choices of parameter.

2.4. Mean deviation about mean and median

The mean deviation (MD) about mean is another measure of dispersion and is defined as,

$$\phi_1(x) = \int_0^\infty |x - \mu| f(x)dx$$

where μ is the mean of $SS_E(\epsilon)$-distribution, then

$$\phi_1(x) = \int_0^\mu (\mu - x)f(x)dx + \int_\mu^\infty (x - \mu)f(x)dx$$

$$\implies \phi_1(x) = 2\mu \times F(\mu) - 2\mu + 2\left[\int_\mu^\infty xf(x)dx\right]$$

where notations have their usual meanings then by using lemma 2, we get

$$\int_{\mu}^{\infty} xf(x)dx = \frac{\pi}{2}\epsilon \times \ \xi_2(\epsilon, 1, \epsilon, \mu)$$

Therefore,

$$\phi_1(x) = 2\mu \times F(\mu) - 2\mu + \pi\epsilon \times \xi_2(\epsilon, 1, \epsilon, \mu) \tag{11}$$

In the similar way, MD about median is

$$\begin{aligned}\phi_2(x) &= \int_0^{\infty} |x - M| f(x)dx = \int_0^{M} (M - x) f(x)dx + \left[\int_M^{\infty} (x - M) f(x)dx\right] \\ &= -\mu + 2\int_M^{\infty} xf(x)dx\end{aligned}$$

Now, by lemma 2, we have

$$\int_{M}^{\infty} xf(x)dx = \frac{\pi}{2}\epsilon \times \xi_2(\epsilon, 1, \epsilon, M)$$

Finally,

$$\phi_2(x) = -\mu + \pi\epsilon \times \xi_2(\epsilon, 1, \epsilon, M) \tag{12}$$

2.5. Order Statistics

Let us take random sample of size n from the $SS_E(\epsilon)$-distribution say, $\{X_1, X_2, \ldots, X_n\}$ and associated order statistics is $X_{(1)} < X_{(2)} < ... < X_{(r)}$, then pdf of r^{th} order statistics is

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!} \times F^{r-1}(x) \times f(x) \times [1 - F(x)]^{n-r}$$

$$\Longrightarrow f_r(x) = \frac{n!}{(r-1)!(n-r)!} \times \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} F^{r+i+1}(x) \times f(x) \tag{13}$$

Now, using (3) and (4) in 13), we have

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!} \times \frac{\pi}{4}\epsilon \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \times \sin\left(\pi e^{-\epsilon x}\right) \left[\cos\left(\frac{\pi}{2} e^{-\epsilon x}\right)\right]^{r+i} \tag{14}$$

and corresponding cdf of r^{th} order statistics is

$$F_r(x) = \sum_{i=r}^{n} \binom{n}{i} F^i(x) \times [1 - F(x)]^{n-i} = \sum_{i=r}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} (-1)^j F^{i+j}(x) \tag{15}$$

Using equation (4) in (15), we obtain the expression of cdf of r^{th} order statistic of $SS_E(\epsilon)$-distribution as follows

$$F_r(x) = \sum_{i=r}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} (-1)^j \left\{\cos\left(\frac{\pi}{2} e^{-\epsilon x}\right)\right\}^{i+j} \tag{16}$$

3. Estimation of Parameter

In this section, we have discussed the estimation of parameter ϵ of $SS_E(\epsilon)$-distribution for Type-II censored data under both Classical and Bayesian paradigms. It is observed that, it is not possible to obtain the failure times of all the test units placed on a life testing experiment because of the associated costs such as cost of per unit is high or limitations on experimental time etc. Therefore, such situations are handled by removal of test units before the actual failure occurs and are termed as censoring scheme. Since, the removal of these units can be done in various possible ways, these are further known as various type of censoring scheme. The two widely used censoring schemes are Type-I and Type-II censoring schemes. Here, we consider Type-II censoring only. Let $x_{(1)}, x_{(2)}, \ldots, x_{(r)}$ be r-ordered Type-II right censored random observations obtained from n units placed on a life testing experiment where each unit has its lifetime and follows $SS_E(\epsilon)$-distribution having pdf (3) with largest $(n-r)$ lifetimes have been censored, then the likelihood function is given by [4] is

$$L_C(\epsilon|\underline{X}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(x_{(i)};\epsilon)\left(1 - F(x_{(r)};\epsilon)\right)^{n-r} \tag{17}$$

Several authors have been done their work in this direction, [18] have been discussed Bayesian estimation of parameter under Type-II censored data, [9] worked on classical and Bayesian estimation of reliability estimation of Maxwell distribution under Type-II censored data, [22] have discussed the Bayesian estimation of exponentiated gamma parameter and reliability function under Type-II censored data for asymmetric loss function, [7] presents the statistical evidences of Type-II censored data, [19] have been derived the Bayesian estimation techniques of system reliability for Weibull distribution under Type-II censored data, [8] discussed the comparison between same Bayesian estimation methods for the parameter of exponential distribution based on Type-II censored data. [5] have been discussed the estimation procedure for new lifetime models under classical and Bayesian set-up in the presence of Type-II censored sample. [2] studied the various properties of Pareto distribution using Type-II hybrid censored sample data.

3.1. Classical Estimation

Using (17), the likelihood function of the $SS_E(\epsilon)$-distribution under Type-II censoring scheme is

$$L_C(\epsilon|\underline{X}) = \frac{n!}{(n-r)!}\epsilon^r e^{-\epsilon\sum_{i=1}^{r} x_{(i)}}\prod_{i=1}^{r}\sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1 - \cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r} \tag{18}$$

and taking logarithm on both sides of (18), we get

$$\ln L_C = \ln\frac{n!}{(n-r)!} + r\ln\epsilon + \epsilon\sum_{i=1}^{r} x_{(i)} + \sum_{i=1}^{r}\ln\left[\sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\right] + (n-r)\ln\left[1 - \cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right] \tag{19}$$

On differentiating (19) w.r.to ϵ and equate the resultant to zero, we get

$$\frac{d\ln L_C}{d\epsilon} = \frac{r}{\epsilon} + \sum_{i=1}^{r} x_{(i)} - \frac{\pi}{2}\sum_{i=1}^{r} x_{(i)}e^{-\epsilon x_{(i)}}\left[\cot\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\right] + (n-r)\left[\frac{\pi}{2}\times\frac{x_{(r)}e^{-\epsilon x_{(r)}}\sin\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)}{1 - \cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)}\right] = 0 \tag{20}$$

The above equation cannot be solved analytically. So, we use numerical approximation technique through R software to solve them numerically in terms of ϵ i.e. $\hat{\epsilon}_{MC}$ which maximizes the equation (18).

3.2. Bayesian Estimation

In Bayesian paradigm, posterior probability is an effect of two components prior probability and likelihood function, and calculated from the statistical model for the observed data. The prior distribution of the parameters is assumed before the data observed. There are different kinds of prior distribution of parameters defined as proper and improper priors. Another way to define the priors based on available advanced information is known as informative and non-informative priors.

Here, we use informative prior as a Gamma(a, b) prior for ϵ of $SS_E(\epsilon)$-distribution and having the following form

$$\pi(a,b) = \frac{b^a}{\Gamma a}\epsilon^{a-1}e^{-b\epsilon} \quad ;\epsilon > 0,\ a,b > 0 \tag{21}$$

where, hyper-parameters are a and b. If two information's which are independent in nature on ϵ (say prior mean and prior variance are known) are provided, they can be obtained, for more details see [21], [14], [15]. The mean and variance of the prior distribution (21) are $\frac{a}{b}$ and $\frac{a}{b^2}$ respectively. Thus, we take $M = \frac{a}{b}$ and $V = \frac{a}{b^2}$ giving $b = \frac{M}{V}$ and $a = \frac{M^2}{V}$. The informative gamma prior behaves like non-informative prior if hyper-parameters changes i.e. if we fixed prior mean and taking large prior variance.
The posterior density of ϵ given the sample observations $\underline{X}$ is given below

$$\psi_C(\epsilon|\underline{X}) = \frac{L_C(\epsilon|\underline{X}) \times \pi(a,b)}{\int_0^\infty L_C(\epsilon|\underline{X}) \times \pi(a,b)d\epsilon} \tag{22}$$

By using equation (21) and (18) in (22), posterior density of ϵ given $\underline{X}$ under Type-II censoring is

$$\psi_C(\epsilon|\underline{X}) = \frac{\epsilon^{r+a-1}e^{-\epsilon(\sum_{i=1}^r x_{(i)}+b)}\left\{\prod_{i=1}^r \sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r}\right\}}{\int_0^\infty \epsilon^{r+a-1}e^{-\epsilon(\sum_{i=1}^r x_{(i)}+b)}\left\{\prod_{i=1}^r \sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r}\right\}d\epsilon} \tag{23}$$

The expressions for considered loss functions namely squared error loss function (SELF) and general entropy loss function (GELF) having the following forms

$$L_S(\hat{\epsilon}_{SC},\epsilon) = (\hat{\epsilon}_{SC} - \epsilon)^2 \tag{24}$$

$$L_G(\hat{\epsilon}_{GC},\epsilon) = \left(\frac{\hat{\epsilon}_{GC}}{\epsilon}\right)^c - c\ln\left(\frac{\hat{\epsilon}_{GC}}{\epsilon}\right) - 1 \tag{25}$$

If $\hat{\epsilon}_{GC}$ is a Bayes estimator of ϵ for Type-II censoring under GELF then, we get

$$\hat{\epsilon}_{GC} = \left\{\frac{\int_0^\infty \epsilon^{r+a-c-1}e^{-\epsilon(\sum_{i=1}^r x_{(i)}+b)} \times \prod_{i=1}^r \sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r}d\epsilon}{\int_0^\infty \epsilon^{r+a-1}e^{-\epsilon(\sum_{i=1}^r x_{(i)}+b)} \times \prod_{i=1}^r \sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r}d\epsilon}\right\}^{-\frac{1}{c}} \tag{26}$$

Putting $c = -1$ in equation (26), we get the Bayes estimator $\hat{\epsilon}_{SC}$ of ϵ for Type-II censoring, we get

$$\hat{\epsilon}_{SC} = \left\{\frac{\int_0^\infty \epsilon^{r+a}e^{-\epsilon(\sum_{i=1}^r x_{(i)}+b)} \times \prod_{i=1}^r \sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r}d\epsilon}{\int_0^\infty \epsilon^{r+a-1}e^{-\epsilon(\sum_{i=1}^r x_{(i)}+b)} \times \prod_{i=1}^r \sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r}d\epsilon}\right\} \tag{27}$$

The above equations (26) and (27) are not solvable analytically. Therefore, we propose some numerical approximation technique to get the solution. Basically, we have used here Gauss-Laguerre quadrature formula to obtain the solution.

4. Comparison of Estimators

In this section, we compare the performance of the considered estimators ($\hat{\epsilon}_{MC}$, $\hat{\epsilon}_{SC}$, $\hat{\epsilon}_{GC}$) of parameter ϵ of $SS_E(\epsilon)$-distribution in the presence of Type-II censoring scheme in terms of lowest risks (expected loss over Ω) under GELF. It is clear that, the expressions of risk function are not obtained in implicit form. So, we use Gauss-Laguerre quadrature formula to obtain the estimators ($\hat{\epsilon}_{SC}$ and $\hat{\epsilon}_{GC}$) of parameter ϵ for computing the risks under Type-II censored data. To know the performance of estimator in long run use, we simulate 20,000 samples for different sample size n and different effective sample size r (for Type-II censoring) from $SS_E(\epsilon)$-distribution with different choices of values of parameter ($\epsilon = 1.5,\ 2.0,\ 3.0$) and loss parameter $c = \pm 2$.

Tables 2, 3 and 4 represents the risks for the variations in hyper-parameters (variation in prior variance ($V = 0.5,\ 1.0,\ 2.0,\ 5.0,\ 80$) for fixed prior mean ($M = 1.0,\ 2.0,\ 3.0$)) when true value of the parameter $\epsilon = 2$ for sample size $n = 30$ with different censoring schemes $r = 12,\ 18,\ 24$ and 30.

Tables 5 and 6 shows the variation in n and r with minimum prior variance (high confidence level $V = 0.5$) and prior mean ($M = 2.0$) for the true value of $\epsilon = 1.5$ and 3, respectively.

Tables 2, 3 and 4 presents the simulated risks under GELF for variation in prior variance (high to low confidence level) with fixed prior mean (M). We see that the risks under GELF for the Bayes estimators of ϵ under SELF and GELF are increases as values of prior variance increases (high to low confidence level) and if prior mean increases then the risks under GELF decreases for the Bayes estimators under SELF and GELF for Type-II censored samples. Bayes estimator under GELF $\hat{\epsilon}_{GC}$ outperforms MLEs ($\hat{\epsilon}_{MC}$) and SELF ($\hat{\epsilon}_{SC}$) under Type-II censored sample when under estimation is more serious as compared to over estimation ($c = -2$) and when over estimation is more serious as compared to under estimation ($c = +2$), then Bayes estimator under SELF ($\hat{\epsilon}_{SC}$) outperforms MLEs ($\hat{\epsilon}_{MC}$) and GELF ($\hat{\epsilon}_{GC}$) under Type-II censored sample. It is also noted that when prior variance is large (low confidence level i.e. very weak information about the parameter ϵ) then classical estimator MLEs ($\hat{\epsilon}_{MC}$) performs better than the Bayes estimators under SELF ($\hat{\epsilon}_{SC}$) and GELF ($\hat{\epsilon}_{GC}$).

Tables 5 and 6 shows that the variation in sample size n and corresponding different Type-II censoring schemes r for the true values of parameter $\epsilon = 1.5$ and 3. Table 5 provides simulated risks of the Bayes estimators ($\hat{\epsilon}_{SC}$) of ϵ under SELF outperforms MLEs ($\hat{\epsilon}_{MC}$) and Bayes estimators of ϵ under GELF ($\hat{\epsilon}_{GC}$) in both cases under estimation is more serious than over estimation and vice-versa. While Table 6 provides Bayes estimator under GELF ($\hat{\epsilon}_{GC}$) outperforms the Bayes estimator under SELF ($\hat{\epsilon}_{SC}$) and MLE ($\hat{\epsilon}_{MC}$) for the situation when under estimation is more serious than over estimation but in reverse case, Bayes estimator under SELF ($\hat{\epsilon}_{SC}$) outperforms MLEs ($\hat{\epsilon}_{MC}$) and Bayes estimator under GELF ($\hat{\epsilon}_{GC}$) for the true value of parameter $\epsilon = 3$. It is also observed that the risks of all estimators of ϵ for Type-II censored sample decreases with increase in the value of n and r for all considered values of the parameter ϵ.

Table 2: *Risks of the estimators of ϵ under GELF when prior variance varies for fixed $n = 30$, $r(r = 12, 18, 24)$, $\epsilon = 2.0$, $M = 1$ and $c = \pm 2$.*

V	scheme r	*c* =-2			*c*=2		
		MLE	SELF	GELF	MLE	SELF	GELF
	12	0.15180	0.13386	0.11519	0.20147	0.09545	0.13667
0.5	18	0.09915	0.09065	0.08189	0.12088	0.07236	0.07057
	24	0.07407	0.06885	0.06373	0.08397	0.05817	0.05724
	30	0.05938	0.05558	0.05214	0.06572	0.04751	0.04789
	12	0.15180	0.13616	0.11989	0.20147	0.12083	0.13550
1	18	0.09915	0.08988	0.08482	0.12088	0.08641	0.08943
	24	0.07407	0.06883	0.06587	0.08397	0.06675	0.06850
	30	0.05938	0.05605	0.05410	0.06572	0.05494	0.05612
	12	0.15180	0.14083	0.13321	0.20147	0.15280	0.16792
2	18	0.09915	0.09324	0.08994	0.12088	0.09933	0.10516
	24	0.07407	0.06992	0.06806	0.08397	0.07367	0.07693
	30	0.05938	0.05718	0.05594	0.06572	0.05952	0.06165
	12	0.15180	0.14373	0.13906	0.20147	0.17322	0.17442
10	18	0.09915	0.09650	0.09440	0.12088	0.10864	0.11646
	24	0.07407	0.07278	0.07155	0.08397	0.07953	0.08381
	30	0.05938	0.05919	0.05836	0.06572	0.06330	0.06606
	12	0.15180	0.15085	0.14829	0.20147	0.18860	0.19630
80	18	0.09915	0.09847	0.09735	0.12088	0.11827	0.12777
	24	0.07407	0.07445	0.07382	0.08397	0.08530	0.09051
	30	0.05938	0.06022	0.05979	0.06572	0.06677	0.07010

Table 3: *Risks of the estimators of ϵ under GELF when prior variance varies for fixed $n = 30$, $r(r = 12, 18, 24)$, $\epsilon = 2.0$, $M = 2$ and $c = \pm 2$.*

V	scheme r	*c* =-2			*c*=2		
		MLE	SELF	GELF	MLE	SELF	GELF
	12	0.15180	0.05785	0.05776	0.20147	0.06075	0.06133
0.5	18	0.09915	0.05112	0.05095	0.12088	0.05416	0.05772
	24	0.07407	0.04445	0.04421	0.08397	0.04712	0.04948
	30	0.05938	0.03881	0.03858	0.06572	0.04070	0.04237
	12	0.15180	0.08699	0.08659	0.20147	0.09806	0.09889
1	18	0.09915	0.06802	0.06764	0.12088	0.07527	0.08079
	24	0.07407	0.05566	0.05534	0.08397	0.06073	0.06412
	30	0.05938	0.04725	0.04700	0.06572	0.05079	0.05314
	12	0.15180	0.11490	0.11311	0.20147	0.13630	0.14083
2	18	0.09915	0.08291	0.08192	0.12088	0.09455	0.10148
	24	0.07407	0.06399	0.06330	0.08397	0.07033	0.07421
	30	0.05938	0.05327	0.05274	0.06572	0.05719	0.05974
	12	0.15180	0.13415	0.13238	0.20147	0.17091	0.17531
5	18	0.09915	0.09104	0.09008	0.12088	0.10704	0.11543
	24	0.07407	0.06982	0.06916	0.08397	0.07840	0.08300
	30	0.05938	0.05707	0.05665	0.06572	0.06254	0.06558
	12	0.15180	0.15018	0.14771	0.20147	0.19933	0.19672
80	18	0.09915	0.09963	0.09857	0.12088	0.11883	0.12845
	24	0.07407	0.07551	0.07482	0.08397	0.08598	0.09119
	30	0.05938	0.06064	0.06018	0.06572	0.06703	0.07033

Table 4: *Risks of the estimators of ϵ under GELF when prior variance varies for fixed $n = 30$ and $r(r = 12, 18, 24)$, $\epsilon = 2.0$, $M = 3$ and $c = \pm 2$.*

V	schemes r	c =-2			c=2		
		MLE	SELF	GELF	MLE	SELF	GELF
0.5	12	0.15180	0.11285	0.12353	0.20147	0.18003	0.18618
	18	0.09915	0.08463	0.09185	0.12088	0.12879	0.14174
	24	0.07407	0.06730	0.07252	0.08397	0.09833	0.10731
	30	0.05938	0.05613	0.06017	0.06572	0.07954	0.08627
1	12	0.15180	0.10204	0.11257	0.20147	0.16348	0.16861
	18	0.09915	0.07569	0.08181	0.12088	0.11215	0.12486
	24	0.07407	0.06083	0.06489	0.08397	0.08519	0.09338
	30	0.05938	0.05096	0.05389	0.06572	0.06833	0.07411
2	12	0.15180	0.10845	0.11640	0.20147	0.16802	0.16768
	18	0.09915	0.07780	0.08184	0.12088	0.10803	0.11971
	24	0.07407	0.06115	0.06362	0.08397	0.07953	0.08651
	30	0.05938	0.05096	0.05263	0.06572	0.06339	0.06810
5	12	0.15180	0.12592	0.12934	0.20147	0.17881	0.17903
	18	0.09915	0.08871	0.09023	0.12088	0.11345	0.12417
	24	0.07407	0.06864	0.06947	0.08397	0.08334	0.08942
	30	0.05938	0.05580	0.05640	0.06572	0.06568	0.06971
80	12	0.15180	0.14540	0.14786	0.20147	0.18968	0.19429
	18	0.09915	0.09905	0.09951	0.12088	0.12069	0.13089
	24	0.07407	0.07306	0.07313	0.08397	0.08471	0.09015
	30	0.05938	0.05921	0.05997	0.06572	0.06597	0.06942

Table 5: *Risks of the estimators of ϵ under GELF when $V = 0.5$, $M = 2$ and $c = \pm 2$ for true value of $\epsilon = 1.5$.*

V	scheme r	c =-2			c=2		
		MLE	SELF	GELF	MLE	SELF	GELF
15	6	0.30733	0.11180	0.12843	0.58496	0.17749	0.19775
	9	0.20172	0.09625	0.10710	0.30346	0.14726	0.17215
	12	0.14916	0.08295	0.09059	0.20110	0.12254	0.14004
	15	0.12036	0.07335	0.07906	0.15104	0.10453	0.11761
20	8	0.22597	0.09958	0.11208	0.35915	0.15420	0.15351
	12	0.14976	0.08279	0.09040	0.20231	0.12191	0.13939
	16	0.11135	0.06989	0.07502	0.13774	0.09803	0.10978
	20	0.09013	0.06117	0.06495	0.10558	0.08244	0.09104
30	12	0.15112	0.08386	0.09189	0.20662	0.12435	0.12456
	18	0.09839	0.06540	0.06998	0.11985	0.09049	0.10073
	24	0.07255	0.05290	0.05584	0.08325	0.06937	0.07586
	30	0.05884	0.04529	0.04738	0.06545	0.05727	0.06186
60	24	0.07548	0.05425	0.05723	0.08712	0.07144	0.07669
	36	0.04909	0.03924	0.04075	0.05379	0.04825	0.05158
	48	0.03676	0.03096	0.03186	0.03932	0.03655	0.03855
	60	0.02966	0.02581	0.02643	0.03120	0.02961	0.03098

Table 6: *Risks of the estimators of ϵ under GELF when $V = 0.5$, $M = 2$ and $c = \pm 2$ for true value of $\epsilon = 3.0$.*

V	scheme r	c =-2			c=2		
		MLE	SELF	GELF	MLE	SELF	GELF
	6	0.30882	0.18838	0.14822	0.57630	0.11627	0.20923
15	9	0.20002	0.14235	0.11626	0.29734	0.09092	0.07627
	12	0.14878	0.11418	0.09590	0.19835	0.07558	0.06526
	15	0.12139	0.09769	0.08385	0.14948	0.06649	0.05869
	8	0.22674	0.15471	0.12462	0.35157	0.09761	0.16869
20	12	0.14766	0.11417	0.09571	0.19338	0.07540	0.06492
	16	0.11053	0.09059	0.07813	0.13388	0.06245	0.05537
	20	0.08972	0.07659	0.06737	0.10351	0.05458	0.04936
	12	0.15287	0.11776	0.09863	0.20241	0.07752	0.12301
30	18	0.09955	0.08390	0.07307	0.11933	0.05891	0.05277
	24	0.07382	0.06505	0.05808	0.08374	0.04801	0.04407
	30	0.05927	0.05340	0.04848	0.06563	0.04099	0.03822
	24	0.07358	0.06458	0.05742	0.08556	0.04785	0.06514
60	36	0.04797	0.04366	0.04011	0.05294	0.03478	0.03278
	48	0.03608	0.03355	0.03141	0.03875	0.02795	0.02676
	60	0.02952	0.02785	0.02637	0.03116	0.02385	0.02303

5. Conclusion

In this paper, we have been consider a lifetime distribution by using sine function which has proposed by [13]. We have also discussed some statistical properties of the considered distribution such as conditional moments, mean deviation about mean, mean deviation about median and derived expressions of the pdf and cdf of r^{th} order statistics. Mean, median and variance are inversely related to the parameter ϵ of $SS_E(\epsilon)$-distribution and the distribution has positively skewed and leptokurtic nature. We have developed classical and Bayesian estimation procedure for estimation of parameter ϵ under Type-II censored data. And also check the workout of the estimators at the long-run by performing simulation study. The Bayes estimator under SELF ($\hat{\epsilon}_{SC}$) outperforms MLE ($\hat{\epsilon}_{MC}$) and Bayes estimator under GELF ($\hat{\epsilon}_{GC}$) for the true value of parameter $\epsilon = 1.5$ whatever the seriousness i.e. over estimation is more serious than under estimation and reversely. In all other considered cases, Bayes estimator under GELF ($\hat{\epsilon}_{GC}$) outperforms MLE ($\hat{\epsilon}_{MC}$) and Bayes estimator under SELF ($\hat{\epsilon}_{SC}$) when under estimation is more serious than over estimation but in reverse case Bayes estimator under SELF ($\hat{\epsilon}_{SC}$) outperform MLE ($\hat{\epsilon}_{MC}$) and Bayes estimator under GELF ($\hat{\epsilon}_{GC}$). Finally, we see that risks under GELF decreases as sample informations (n & r) increases.

References

[1] Al-Faris, R.Q. and Khan, S. (2008). Sine square distribution: a new statistical model based on the sine function. Journal of Applied Probability and Statistics, 3(1), 163-173.

[2] Cetinkaya, C. (2020). Inference Based on Type-II Hybrid Censored Data from a Pareto Distribution. Journal of Reliability and Statistical Studies, 253-264.

[3] Chesneau, C., Bakouch, H.S. and Hussain, T. (2019). A new class of probability distributions via cosine and sine functions with applications. Communications in Statistics-Simulation and Computation, 48(8), 2287-2300.

[4] Cohen, A.C. (1965). Maximum likelihood estimation in the Weibull distribution based on complete and on censored samples. Technometrics, 7(4), 579-588.

[5] Goyal, T., Rai, P.K. and Maurya, S.K. (2020). Bayesian estimation for GDUS exponential distribution under type-I progressive hybrid censoring. Annals of Data Science, 7(2), 307-345.

[6] Gupta, R.C., Gupta, P.L. and Gupta, R.D. (1998). Modeling failure time data by Lehman alternatives. Communications in Statistics-Theory and methods, 27(4), 887-904.

[7] HABIBI, R.A., Emadi, M., Arashi, M. and Arghami, N.R. (2011). Statistical evidences in type-II censored data.

[8] Hussein, N.A., Abbas, S.A., Hussain, A.H. and Salman, A.M. (2020), July. A Comparison between Some Bayesian Estimation Methods for Parameter Exponential Distribution by Using Data Type II Censoring. In Journal of Physics: Conference Series 1591(1), 012039. IOP Publishing.

[9] Krishna, H. and Malik, M. (2009). Reliability estimation in Maxwell distribution with Type?II censored data. International Journal of Quality & reliability management.

[10] Kumar, D., Singh, U. and Singh, S.K. (2015). A method of proposing new distribution and its application to Bladder cancer patients data. J. Stat. Appl. Pro. Lett, 2(3), 235-245.

[11] Kumar, D., Singh, U. and Singh, S.K. (2017). Life time distribution: derived from some minimum guarantee distribution, Sohag J. Math, 4(1), 7-11.

[12] Kumar, D., Kumar, P., Kumar, P., Singh, S.K. and Singh, U. (2021). PCM transformation: properties and their estimation. Journal of Reliability and Statistical Studies, 373-392.

[13] Kumar, D., Singh, U. and Singh, S.K. (2015). A new distribution using sine function-its application to bladder cancer patients data. Journal of Statistics Applications & Probability, 4(3), 417.

[14] Kumar, D., Kumar, P., Singh, S.K. and Singh, U. (2019). A new asymmetric loss function: estimation of parameter of exponential distribution. J Stat Appl Probab Lett, 6(1), 37-50.

[15] Kumar, D., Kumar, P., Kumar, P., Singh, U. and Chaurasia, P.K. (2020). A new asymmetric loss function for estimation of any parameter. International Journal of Agricultural and Statistical Sciences, 16(2), 489-501.

[16] Mahmood, Z. and Chesneau, C. (2019). A new sine-G family of distributions: properties and applications.

[17] Maurya, S.K., Kumar, D., Kumar, P., Singh, S.K. and Singh, U. (2020). A new extension of Lindley distribution and its application. Journal of Scientific Research, 64(2), 366-373.

[18] Moore, A.H. and Bilikam, J.E. (1978). Bayesian estimation of parameters of life distributions and reliability from type II censored samples. IEEE Transactions on Reliability, 27(1), 64-67.

[19] Okasha, H.M. (2012). E-Bayesian estimation of system reliability with Weibull distribution of components based on type-2 censoring. Journal of Advanced Research in Scientific Computing, 4(4), 34-45.

[20] Shaw, W.T. and Buckley, I.R. (2009). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. arXiv preprint arXiv:0901.0434.

[21] Singh, S.K., Singh, U. and Kumar, D. (2013). Bayes estimators of the reliability function and parameter of inverted exponential distribution using informative and non-informative priors. Journal of Statistical computation and simulation, 83(12), 2258-2269.

[22] Singh, S.K., Singh, U. and Kumar, D. (2011). Bayesian estimation of the exponentiated gamma parameter and reliability function under asymmetric loss function. REVSTAT Stat J, 9(3), 247-260.

[23] Sinha, R.K. (2012). A thought on exotic statistical distributions. International Journal of Mathematical and Computational Sciences, 6(1), 49-52.