Statistical properties and estimation procedures for a new flexible two parameter lifetime distribution

S. K. Singh, Suraj Yadav* and Abhimanyu Singh Yadav

•

Department of Statistics, Banaras Hindu University, Varanasi, India.
singhsk64@gmail.com & asybhu10@gmail.com
*Corresponding Author's Email: surajyadav9693@gmail.com

Abstract

In this article, a new transformation technique based on the cumulative distribution function is proposed, the proposed transformation technique is very useful to generate a class of lifetime distribution. The various statistical properties of the proposed transformation method are studied. Further, the proposed technique is illustrated by considering exponential distribution as a baseline distribution. Various statistical properties such as survival and hazard rate, moments, mean deviation about mean and median, order statistics, moment generating function (MGF), Bonferroni's, and Lorenz curves, entropy, stress-strength reliability have been discussed. Different classical estimation methods are used to estimate the unknown parameters. Finally, two real data sets are considered to justify the use of the proposed distribution in real scenario.

Keywords: Transformation technique, statistical properties, classical method of estimation, and application.

1. Introduction

In lifetime analysis, various transformation techniques are used to propose the new probability distribution by adding an additional parameter to the baseline probability distribution. The significance of these probability distributions is categorized according to their use and appropriateness of different hazard rates viz. increasing, decreasing, constant, bathtub, and upside-down bathtub (UBT). Modeling of the real-life data set is based on the nature of the hazard rate function. For example, the exponential distribution is the most suitable choice whenever data exhibits the pattern of constant hazard rate. However, underlying data exhibits a non-constant hazard rate then other generalized lifetime distributions such as Weibull, Gamma, Extended Exponential, Generalized Exponential, Lindley distributions, and many others are frequently used to desirable data. To know more about monotone and non-monotone hazard rates, see [17], [32], [4], [5], [14], and [8], etc.

In statistical literature, various method has been suggested by the several authors to generate a new flexible model, viz. [26], [24], [19], and [30]. The beta generated model is used by [15] who uses the beta distribution to develop the beta generated distributions. [11] propose the Kumarswamy-G family of distributions.[12] propose a new class of distribution by adding two more parameters. [2] introduce a new method for generating families of distributions called the T-X family. Recently, the quantile function is used to generate the T-X family of distributions by [1]. For another development in the family of distributions see, [18], [23], [20], and [25], etc. These methods are most popular to propose flexible and appropriate models. Here notable thing is that all the methods of transformation discussed above introduce one additional parameter. Unquestionably, the addition of an extra parameter increases the flexibility but at the same time, it also increases complexity in further statistical inference.

Motivated by the above-mentioned literature, this article aims to propose a new transformation technique to generate the class of distributions. The proposed transformation is illustrated with an exponential baseline model and named a new two-parameter lifetime model. The proposed model through this transformation will be considered as an alternative to the Gamma, Weibull, and Extension of Exponential distributions by using the transformation method.

Let G(x) is the cumulative distribution function (CDF) of any baseline distribution, then the CDF of new distribution is proposed by,

$$F(x) = \frac{G(x)}{G(x) + (1 + G(x))^{\alpha}} \qquad for \;\; x \in \Re \quad and\; \alpha \geq 0 \tag{1}$$

Clearly, $F(x)$ is the distribution function as it satisfy the condition to be a CDF.

$(i) \quad \lim_{x \to -\infty} F(x) = 0 \qquad (ii) \quad \lim_{x \to \infty} F(x) = 1 \qquad (iii) \quad F'(x) = f(x)$

where $f(x)$ and $g(x)$ are the probability distribution function (PDF) of proposed and baseline distribution function respectively.

(iv) It is well known that $0 \leq G(x) \leq 1$, which implies that $0 \leq F(x) \leq 1$.
(v) Clearly, $F(x)$ is a continuous function.

Now, the associated probability distribution function (PDF) $f(x)$ for (1) is,

$$f(x) = \frac{g(x)\{(1 - G(x) + \alpha G(x))\}\left\{(1 - G(x))^{(\alpha-1)}\right\}}{\left\{G(x) + (1 - G(x))^{\alpha}\right\}^2} \tag{2}$$

The survival and hazard rates are:

$$S(x) = \frac{(1-G(x))^{\alpha}}{G(x) + (1-G(x))^{\alpha}} \tag{3}$$

$$h(x) = \frac{g(x)\{1-G(x) + \alpha G(x)\}}{(1-G(x))G(x) + (1-G(x))^{\alpha+1}} \tag{4}$$

To illustrate the above transformation, let us assume exponential distribution as the base line distribution. The PDF of the exponential distribution is given by,

$$g(x) = \begin{cases} \lambda e^{-\lambda x} & for \;\; x \geq 0,\, \lambda > 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and the associated CDF is:

$$G(x) = 1 - e^{-\lambda x} \qquad\qquad for\; x \geq 0,\, \lambda > 0 \tag{6}$$

here, λ is rate parameter of exponential distribution.

Then by transformation (1), the CDF of the new flexible distribution is,

$$F(x) = \frac{1-e^{-\lambda x}}{1-e^{-\lambda x} + e^{-\lambda \alpha x}} \qquad for \quad x \geq 0, \alpha \geq 0, \lambda > 0 \tag{7}$$

The PDF, survival and hazard rate function of proposed distribution are given as:

$$f(x) = \frac{\lambda e^{-\lambda \alpha x}\left(\alpha + e^{-\lambda x} - \alpha e^{-\lambda x}\right)}{\left(1 - e^{-\lambda x} + e^{-\lambda \alpha x}\right)^2} \qquad for \quad x \geq 0, \alpha \geq 0, \lambda > 0 \tag{8}$$

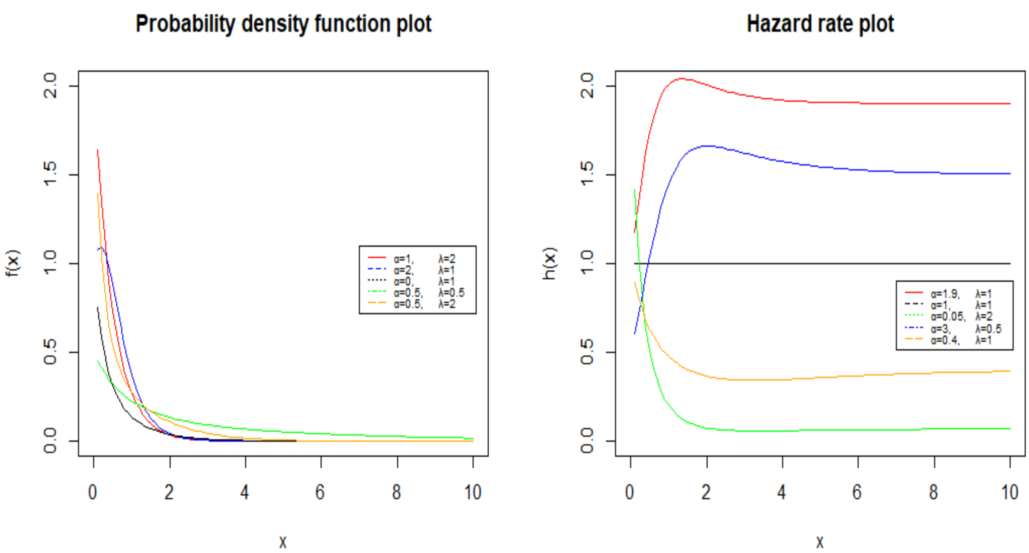
$$S(x) = \frac{e^{-\lambda \alpha x}}{\left(1 - e^{-\lambda x}\right) + e^{-\lambda \alpha x}} \tag{9}$$

and

$$h(x) = \frac{\lambda\left[\alpha + e^{-\lambda x} - \alpha e^{-\lambda x}\right]}{\left[1 - e^{-\lambda x} + e^{-\lambda \alpha x}\right]} \tag{10}$$

respectively.

The principal objective of this paper is to propose a new transformation method and derive its various statistical properties. Specifically, we substitute (6) into (1) to get the CDF of the proposed distribution and the corresponding PDF is obtained by substituting (5) and (6) into (2). Our motivation to construct a new model is: (i) it is applicable for modeling increasing, decreasing, and constant hazard shape which provides a good fit for real data sets; (ii) In our proposed model if we put $\alpha = 1$ then our proposed model reduced to baseline model; (iii) The proposed model can be considered as a good alternative model for fitting the positive data with a longer tail and (iv) The proposed model provides a better fit than some well-known lifetime models to real data sets. As, the proposed model is an alternative to Weibull, Gamma, and Extended Exponential, thus the proposed model might be a good choice for the researcher. Also, we have considered different methods of estimation to estimate the unknown value of the parameter. To check the applicability, AIC and BIC's are also constructed for the parameters of the proposed model. A simulation study has been performed to appraise the performance of the proposed estimation methods. Further, we have considered two real data set to illustrate the superiority of the proposed model and study. The plots of pdf and hazard rate function of new flexible two-parameter lifetime distribution for various values of α and λ are shown in Figure 1(a) and 1(b) respectively. From the Figure 1(b), the proposed distribution has an increasing, decreasing, and constant hazard rate.



(a) (b)

Figure 1: *(a) PDF of proposed distribution. (b) Hazard function of proposed distribution.*

The content of the rest of the paper is organized as follows: In section 2, we have discussed some statistical properties such as raw moments, moment generating function (MGF), mean deviations, Bonferroni and Lorenz curves, Rényi entropy, s- entropy, cumulative residual entropy, order statistics and reliability. In section 3, different method of estimation of the proposed distribution is studied. Simulation studies are carried out, in section 4, to compare the behaviour and performance of the different estimators. In section 5, the proposed model is fitted with some competing models using two real data sets and finally, the conclusions are summarised in section 6.

2. Statistical properties of new flexible two parameter lifetime distribution

In this section, we have discussed various statistical properties of our proposed new two parameter lifetime distribution like moments, moment generating function (MGF), mean deviation about mean and median, order statistics, reliability, Renyi entropy and Shannon entropy.

2.1. Raw Moments:

The r^{th} moment about origin of the distribution with PDF (8) is obtained by,

$$\mu_r^{'}=E\left(X^r\right)=\int_0^\infty x^r f(x)dx=\int_0^\infty x^r\frac{\lambda e^{-\lambda\alpha x}\left(\alpha+e^{-\lambda x}-\alpha e^{-\lambda x}\right)}{\left(1-e^{-\lambda x}+e^{-\lambda\alpha x}\right)^2}dx$$

After simplification the above integral, we get,

$$\mu_r^{'}=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left[\frac{\alpha\Gamma(r+1)}{\lambda^r(\alpha+i-j+\alpha j)^{r+1}}+\frac{(1-\alpha)\Gamma(r+1)}{\lambda^r(1+\alpha+i-j+\alpha j)^{r+1}}\right] \quad (11)$$

The respective four moments about origin can be obtained by putting $r = 1, 2, 3$, and 4. For $r = 1$ we get mean (μ) of the distribution and is given by the following expression,

$$\mu=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left[\frac{\alpha}{\lambda(\alpha+i-j+\alpha j)^2}+\frac{(1-\alpha)}{\lambda(1+\alpha+i-j+\alpha j)^2}\right]$$

For, $r = 2, 3$, and 4 we can compute $\mu_2^{'}$, $\mu_3^{'}$, and $\mu_4^{'}$ by putting these values in equation number (11). The variance of the proposed model can be obtained using the expression,

$$V(X) = E\left(X^2\right) - \left(E(X)\right)^2$$

Similarly, we can find other moment based, skewness, and the kurtosis of the distribution.

2.2. Moment generating function (MGF):

The moment generating function (MGF) for the proposed distribution with PDF (8) is given by;

$$M_X(t)=\int_0^\infty\frac{e^{tx}\lambda e^{-\lambda\alpha x}\left[\alpha+e^{-\lambda x}-\alpha e^{-\lambda x}\right]}{\left[1-e^{-\lambda x}+e^{-\lambda\alpha x}\right]^2}dx$$

After simplification,

$$=\alpha\sum_{r=0}^{\infty}\sum_{J=0}^{\infty}\sum_{i=0}^{J}(-1)^{i}\binom{J}{i}(J+1)\frac{(t-\alpha\lambda)^{r}}{r!}\frac{\Gamma(r+1)}{\lambda^{r}(J-i+i\alpha)^{r+1}}+$$
$$\lambda(1-\alpha)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{k}(n+1)\binom{n}{k}\frac{(t-\lambda-\alpha\lambda)^{m}}{m!}\frac{\Gamma(m+1)}{((n-k+k\alpha)\lambda)^{m+1}} \quad (12)$$

2.3. Mean Deviation (MD)

The mean deviation is the mean of the deviations. It can be calculated from the mean, median, and mode. It shows how far all the observations from the middle, on average are. The mean deviation about mean and mean deviation about median is defined as,

$$\delta_1(x)=\int_0^{\infty}|\mu-x|f(x)\,dx$$

and,

$$\delta_2(x)=\int_0^{\infty}|x-M|f(x)dx$$

Respectively, here $\mu=E(X)$ is the mean and $M=median(X)$ is the median of the distribution. After simplification the mean deviation about mean and the mean deviation about median are given as;

$$\delta_1(x)=2\mu F(\mu)-2\mu+2\int_{\mu}^{\infty}xf(x)\,dx \quad (13)$$

Now, the integral is computed as,

$$\int_{\mu}^{\infty}xf(x)\,dx=\int_{\mu}^{\infty}x\frac{\lambda e^{-\lambda\alpha x}\left[\alpha+e^{-\lambda x}-\alpha e^{-\lambda x}\right]}{\left(1-e^{-\lambda x}+e^{-\lambda\alpha x}\right)^{2}}dx$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^{j}\binom{i}{j}(i+1)\times$$
$$\left[\lambda\alpha\int_{\mu}^{\infty}xe^{-(\alpha+i-j+\alpha j)\lambda x}dx+\lambda(1-\alpha)\int_{\mu}^{\infty}xe^{-(1+\alpha+i-j+\alpha j)\lambda x}dx\right]$$

By the definition of complementary incomplete gamma function and for any integer n,

$$\Gamma(n,x)=\int_x^{\infty}t^{n-1}e^{-t}dt$$
$$\Gamma(n,ax)=\int_x^{\infty}t^{n-1}e^{-at}dt$$

$$\Gamma(n,x)=\int_x^{\infty}t^{n-1}e^{-t}dt=(n-1)!e^{-x}\sum_{k=0}^{n-1}\frac{x^{k}}{k!}=(n-1)!e^{-x}e_{n-1}(x)$$

here, $e_n(x)$ is the exponential sum function. The above expression can be expressed as;

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^{j}\binom{i}{j}(i+1)\times$$
$$\left[\frac{\alpha e^{-(\alpha+i-j+\alpha j)\lambda\mu}}{(\alpha+i-j+\alpha j)^{2}\lambda}(1+(\alpha+i-j+\alpha j)\lambda\mu)\right.$$
$$\left.+\frac{(1-\alpha))e^{-(1+\alpha+i-j+\alpha j)\lambda\mu}}{(1+\alpha+i-j+\alpha j)^{2}\lambda}(1+(\alpha+i-j+\alpha j)\lambda\mu)\right] \quad (14)$$

By using equation (13) and (14), the mean deviation $\delta_1(x)$ about mean is;

$$\delta_1(x) = 2\mu\,F(\mu) - 2\mu + 2\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left[\frac{\alpha e^{-(\alpha+i-j+\alpha j)\lambda\mu}}{(\alpha+i-j+\alpha j)^2\lambda}\left(1+(\alpha+i-j+\alpha j)\lambda\mu\right) + \frac{(1-\alpha)\,e^{-(1+\alpha+i-j+\alpha j)\lambda\mu}}{(1+\alpha+i-j+\alpha j)^2\lambda}\left(1+(1+\alpha+i-j+\alpha j)\lambda\mu\right)\right] \quad (15)$$

Similarly, the mean deviation $\delta_2(x)$ about median is given by,

$$\delta_2(x) = -\mu + 2\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left[\frac{\alpha e^{-(\alpha+i-j+\alpha j)\lambda\mu}}{(\alpha+i-j+\alpha j)^2\lambda}\left(1+(\alpha+i-j+\alpha j)\lambda\mu\right) + \frac{(1-\alpha)\,e^{-(1+\alpha+i-j+\alpha j)\lambda\mu}}{(1+\alpha+i-j+\alpha j)^2\lambda}\left(1+(1+\alpha+i-j+\alpha j)\lambda\mu\right)\right] \quad (16)$$

2.4. Bonferroni and Lorenz curves

The Bonferroni [7] and Lorenz curves [21] is used to measure the inequality in the distribution of quantity in the area of economics as in term of income and wealth. The Bonferroni and Lorenz curves have various applications not only in the area of economics to study income and poverty but also in other areas like demography, medicine, insurance and reliability. Lorenz curves cannot be defined if the mean of the distribution is zero or infinite. The Bonferroni and Lorenz curves are given by,

$$B(P) = \frac{1}{P\mu}\int_0^q x f(x)dx \quad (17)$$

and

$$L(x) = \frac{1}{\mu}\int_0^q x f(x)dx \quad (18)$$

where $\mu = E(x)$ and $q = F^{-1}(p)$ respectively. Now, the integral quantity in RHS is simplified as;

$$\int_q^{\infty} x f(x)\,dx = \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\times \left[\frac{\alpha e^{-(\alpha+i-j+\alpha j)\lambda q}}{(\alpha+i-j+\alpha j)^2\lambda}(1+(\alpha+i-j+\alpha j)\lambda q) + \frac{(1-\alpha))e^{-(1+\alpha+i-j+\alpha j)\lambda q}}{(1+\alpha+i-j+\alpha j)^2\lambda}(1+(\alpha+i-j+\alpha j)\lambda q)\right]$$

Hence, the Bonferroni and Lorenz curves for the new distribution are obtained as;

$$B(P) = \frac{1}{P} - \frac{1}{P\mu}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left\{\frac{\alpha e^{-(\alpha+i-j+\alpha j)\lambda q}}{(\alpha+i-j+\alpha j)^2\lambda}(1+(\alpha+i-j+\alpha j)\lambda q) + \frac{(1-\alpha)e^{-(1+\alpha+i-j+\alpha j)\lambda q}}{(1+\alpha+i-j+\alpha j)^2\lambda}(1+(1+\alpha+i-j+\alpha j)\lambda q)\right\}\right] \quad (19)$$

and

$$L(p) = 1 - \frac{1}{\mu}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left\{\frac{\alpha e^{-(\alpha+i-j+\alpha j)\lambda q}}{(\alpha+i-j+\alpha j)^2\lambda}(1+(\alpha+i-j+\alpha j)\lambda q)\right.\right.$$

$$+\frac{(1-\alpha)e^{-(1+\alpha+i-j+\alpha j)\lambda q}}{(1+\alpha+i-j+\alpha j)^2\lambda}(1+(1+\alpha+i-j+\alpha j)\lambda q)\Bigg\}\Bigg] \tag{20}$$

respectively.

2.5. Renyi entropy

Rényi entropy [28] is a most popular measure of average amount of uncertainty of a random variable X. If X is a random variable with probability distribution function $f(x)$ then the Rényi entropy is defined as

$$\mathcal{J}_R(\gamma)=\frac{1}{1-\gamma}\, log\left\{\int f^{\gamma}(x)\,dx\right\} \tag{21}$$

where, $\gamma>0$ and $\gamma\neq 1$. Now, from equation (8) we get,

$$\int_0^\infty f^{\gamma}(x)\,dx=\int_0^\infty \left\{\frac{\lambda e^{-\lambda\alpha x}\left[\alpha+e^{-\lambda x}-\alpha e^{-\lambda x}\right]}{(1-e^{-\lambda x}+e^{-\lambda\alpha x})^2}\right\}^{\gamma}dx$$

$$=\lambda^{\gamma}\alpha^{\gamma+l-j}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{j+k+l}\binom{-2\gamma}{i}\binom{\gamma}{J}\binom{i}{k}\binom{j}{l}\frac{1}{(\gamma\alpha+i-k+\alpha k+j)\lambda}$$

By putting the above value in the equation (21) we get,

$$\mathcal{J}_R(\gamma)=\frac{\gamma}{1-\gamma}log\lambda+\left(\frac{\gamma+l-j}{1-\gamma}\right)log\alpha$$
$$+\frac{1}{1-\gamma}log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{j+k+l}\binom{-2\gamma}{i}\binom{\gamma}{J}\binom{i}{k}\binom{j}{l}\frac{1}{(\gamma\alpha+i-k+\alpha k+j)\lambda}\right] \tag{22}$$

2.6. s-Entropy

Shannon entropy was proposed by [29], and is a particular case of Rényi entropy as $\gamma\to 1$. It can be defined as $E\left[-logf(X)\right]$.

$$logf(X)=log\lambda-\alpha\lambda x+log\alpha+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\alpha^n n}\left(e^{-\lambda x}-\alpha e^{-\lambda x}\right)^n+2\sum_{m=1}^{\infty}\frac{\left(e^{-\lambda x}-e^{-\lambda\alpha x}\right)^m}{m}$$

After using the result based on series expansion of $log(1+x)$ and $log(1-x)$ in the above expression, the expression for Shannon entropy is given by,

$$E\left[-logf(X)\right]=-log\lambda+\lambda\alpha E(X)-log\alpha-\sum_{n=1}^{\infty}\sum_{l=0}^{n}\frac{(-1)^{n+l+1}}{\alpha^n n}\binom{n}{l}E\left(e^{-\lambda nx}\right)$$
$$-2\sum_{m=1}^{\infty}\sum_{s=0}^{m}\frac{(-1)^s}{m}\binom{m}{s}E\left(e^{-\lambda mx+\lambda sx-\lambda\alpha sx}\right) \tag{23}$$

The equation number (23) can be computed with the help of following results;

$$E\left(e^{-\lambda nx}\right)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left[\frac{\lambda\alpha}{(n+\alpha+i-j+\alpha j)\lambda}+\frac{\lambda(1-\alpha)}{(n+\alpha+1+i-j+\alpha j)\lambda}\right] \tag{24}$$

and

$$E\left(e^{-\lambda mx+\lambda sx-\lambda\alpha sx}\right)=\sum_{k=0}^{\infty}\sum_{t=0}^{k}(-1)^t\binom{k}{t}(k+1)\times \left[\frac{\alpha}{(m-s+\alpha s+k+\alpha-t+\alpha t)}+\frac{1-\alpha}{(m-s+\alpha s+\alpha+1+k-t+\alpha t)}\right] \quad (25)$$

respectively.
Corr: 1. The cumulative residual entropy [27] defined as,

$$\mathcal{J}_C=-\int \Pr(X>x)\log\left(\Pr(X>x)\right)dx$$

$$=-\int_0^{\infty}\frac{e^{-\lambda\alpha x}}{\left(1-e^{-\lambda x}+e^{-\lambda\alpha x}\right)}\left(-\lambda\alpha x-log\left(1-e^{-\lambda x}+e^{-\lambda\alpha x}\right)\right)dx$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\binom{i}{j}\left[\frac{\lambda\alpha}{\left((\alpha+i-j+\alpha j)\lambda\right)^2}-\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{(-1)^l}{k}\binom{k}{l}\frac{1}{(\alpha+i-j+\alpha j+k-l+\alpha l)}\right]$$

2.7. Order Statistics

Suppose that $X_1, X_{2,}\ldots\ldots..X_n$ is a random sample of size n from the proposed continuous probability distribution function (PDF), $f(x)$. Let $X_{1:n}<X_{2:n}<\cdots<X_{n:n}$ denote the corresponding order statistics. We know that the probability density function of r^{th} order statistics $X_{r:n}$, say $f_r(x)$, where the population PDF and CDF are $f(X)$ and $F(x)$ respectively, is given as follows,

$$f_r(x)=\frac{n!}{(r-1)!\,(n-r)!}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}F^{r+i-1}(x)\,f(x) \quad (26)$$

Consequently, using the equation (7) and (8) in (26) we get,

$$f_r(x)=\frac{n!}{(r-1)!\,(n-r)!}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\left[\frac{1-e^{-\lambda x}}{1-e^{-\lambda x}+e^{-\lambda\alpha x}}\right]^{r+i-1}\times\left[\frac{\lambda e^{-\lambda\alpha x}\left(\alpha+e^{-\lambda x}-\alpha e^{-\lambda x}\right)}{\left(1-e^{-\lambda x}+e^{-\lambda\alpha x}\right)^2}\right] \quad (27)$$

and corresponding r^{th} order statistics of CDF $F_r(x)$ is,

$$F_r(x)=\sum_{j=r}^{n}\sum_{m=0}^{n-j}\binom{n}{j}\binom{n-j}{m}(-1)^m F^{j+m}(x) \quad (28)$$

Hence from the equation (7), equation (28) can be written as,

$$F_r(x)=\sum_{j=r}^{n}\sum_{m=0}^{n-j}(-1)^m\binom{n}{j}\binom{n-j}{m}\left(\frac{1-e^{-\lambda x}}{1-e^{-\lambda x}+e^{-\lambda\alpha x}}\right)^{j-m} \quad (29)$$

2.8. Reliability

In this section, we have discussed about reliability of a component. In the context of reliability, the stress-strength model explains the life of a component which has a random strength X that is subjected to random stress Y. The component will fail if the stress applied to it exceeds the strength and the component will work properly whenever $X>Y$. So, $P[X>Y]$ is a measure of component reliability. It has various number of applications in many areas such as in science, engineering etc. In the field of stress-strength model there has been number of works as regarded

estimation of reliability R when X and Y are independent random variable belonging to the same univariate family of distribution.
Here, we derive the reliability R when X and Y are independent random variables from the proposed distribution with parameter (α_1, λ_1) and (α_2, λ_2) respectively. So, the reliability R is defined as bellow,

$$R=P\left(X>Y\right)=\int_0^\infty f_X\left(x,\alpha_1,\lambda_1\right)\ F_Y(x,\alpha_2,\lambda_2)dx$$

$$=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{i}\sum_{l=0}^{k}(-1)^{j+1}\binom{i}{j}\binom{k}{l}(i+1)\left[\lambda_1\alpha_1\left\{\frac{1}{(\alpha_1+i-j+\alpha_1 j)\,\lambda_1+(k-l+\alpha_2 l)\,\lambda_2}-\frac{1}{(\alpha_1+i-j+\alpha_1 j)\,\lambda_1+(k-l+\alpha_2 l+1)\,\lambda_2}\right\}+\lambda_1\left(1-\alpha_1\right)\times\left\{\frac{1}{(1+\alpha_1+i-j+\alpha_1 j)\,\lambda_1+(k-l+\alpha_2 l)\,\lambda_2}-\frac{1}{(1+\alpha_1+i-j+\alpha_1 j)\,\lambda_1+(k-l+\alpha_2 l+1)\,\lambda_2}\right\}\right] \quad (30)$$

3. Methods of estimation

In this section, we will discuss different method of estimation namely, maximum likelihood estimation (MLE), maximum product spacing estimation (MPS), least square estimation (LSE) and weighted least square estimation (WLSE), Cramer-von-Mises estimation (CVME), Anderson Darling estimation (ADE) to estimate the unknown parameter of the considered model.

3.1. Maximum likelihood estimation (MLE)

Let $x_1, x_2, \ldots\ldots\ldots x_n$ be the random samples of size n from the proposed distribution. Then the log-likelihood function of the proposed distribution is given as,

$$logL = \text{nlog}\,\lambda - \alpha\lambda\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}log\left(\alpha+e^{-\lambda x_i}-\alpha e^{-\lambda x_i}\right)-2\sum_{i=1}^{n}log\left(1-e^{-\lambda x_i}+e^{-\lambda\alpha x_i}\right) \quad (31)$$

Now, differentiating equation (31) with respect to parameters α and λ we get,

$$\frac{\partial logL}{\partial\alpha}=-\lambda\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{1-e^{-\lambda x_i}}{\left(\alpha+e^{-\lambda x_i}-\alpha e^{-\lambda x_i}\right)}+2\sum_{i=1}^{n}\frac{\lambda x_i e^{-\lambda\alpha x_i}}{\left(1-e^{-\lambda x_i}+e^{-\lambda\alpha x_i}\right)} \quad (32)$$

and

$$\frac{\partial logL}{\partial\lambda}=\frac{n}{\lambda}-\alpha\sum_{i=1}^{n}x_i+\sum_{i}^{n}\frac{(\alpha-1)\,x_i e^{-\lambda x_i}}{\left(\alpha+e^{-\lambda x_i}-\alpha e^{-\lambda x_i}\right)}-2\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}-\alpha x_i e^{-\lambda\alpha x_i}}{\left(1-e^{-\lambda x_i}+e^{-\lambda\alpha x_i}\right)} \quad (33)$$

Now, putting the equation (32) and (33) equal to zero, we have two non-liner likelihood equations. After solving these equations, we get MLEs $\widehat{\alpha}$ and $\widehat{\lambda}$ of parameters α and λ. These equations are not in closed form consequently it cannot be solved analytically. Therefore, Newton-Raphson method is used to get the MLE's of the parameters α and λ.

3.2. Maximum Product Spacing Estimation (MPSE)

This method is one of the most popular method of estimation and it was developed by [9]. Let $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ be the ordered sample of size n and we define spacing as,

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x,\alpha,\lambda)dx \qquad ; i = 1,2,3\ldots\ldots\ldots(n+1)$$

$$= F\left(x_{(i)},\alpha,\lambda\right) - F\left(x_{(i-1)},\alpha,\lambda\right)$$

Where, initial conditions are $F\left(x_{(0)}\alpha,\lambda\right) = 0$, $F\left(x_{(n+1)}\alpha,\lambda\right)$ and sum of all the spacing will be zero.
We are taking the observation from the proposed distribution, now from the equation (7) the $D_i's$ are defined as,

$$D_i = \frac{1-e^{-\lambda x_{(i)}}}{1-e^{-\lambda x_{(i)}}+e^{-\lambda\alpha x_{(i)}}} - \frac{1-e^{-\lambda x_{(i-1)}}}{1-e^{-\lambda x_{(i-1)}}+e^{-\lambda\alpha x_{(i-1)}}} \qquad ; for\ all\ i = 1,2,\ldots.n \tag{34}$$

For, i = $2,3,\ldots.$n. The MPS estimator $\widehat{\alpha}_{mps}$ and $\widehat{\lambda}_{mps}$ of α and λ are obtained by maximising the geometric mean of the differences,

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{1/(n+1)}$$

after taking logarithm of G we get,

$$logG = \left(\frac{1}{n+1}\right)\sum_{i=1}^{n+1} logD_i \tag{35}$$

substituting the value of D_i from the equation (34) in equation (35) we get,

$$logG = \left(\frac{1}{n+1}\right)\sum_{i=1}^{n+1} log\left[\frac{1-e^{-\lambda x_{(i)}}}{1-e^{-\lambda x_{(i)}}+e^{-\lambda\alpha x_{(i)}}} - \frac{1-e^{-\lambda x_{(i-1)}}}{1-e^{-\lambda x_{(i-1)}}+e^{-\lambda\alpha x_{(i-1)}}}\right] \tag{36}$$

It may be noted that from the equation (36) we can get the derivatives $\frac{\partial logL}{\partial\alpha}$, $\frac{\partial logL}{\partial\lambda}$ and set it equal to zero, the equation, thus obtained, cannot solved analytically, therefore, the same numerical technique may be used to obtain the solution.

3.3. Least Squares Estimation (LSE)

This method is most popular method [31]. Let $x_{(1)} < x_{(2)} \ldots\cdots < x_{(n)}$ be ordered sample of size n from proposed distribution. LSEs $\widehat{\alpha}_{ls}$ and $\widehat{\lambda}_{ls}$ of parameters α and λ are obtained by minimizing

$$Z(\alpha,\lambda) = \sum_{i=1}^{n}\left(F\left(x_{(i)},\alpha,\lambda\right) - E\left[F\left(x_i\right)\right]\right)^2$$

where, $E\left[F\left(x_i\right)\right] = \frac{i}{n+1} \quad ; i = 1,2,\ldots\ldots(n+1)$
then,

$$Z(\alpha,\lambda) = \sum_{i=1}^{n}\left(\frac{1-e^{-\lambda x_{(i)}}}{1-e^{-\lambda x_{(i)}}+e^{-\lambda\alpha x_{(i)}}} - \frac{i}{n+1}\right)^2 \tag{37}$$

In order to minimize $Z(\alpha,\lambda)$given in (37), we differentiate equation (37) with respect to α and λ

and equating to zero, which results of the following equations,

$$\frac{\partial Z(\alpha,\lambda)}{\partial(\alpha)}=\sum_{i=1}^{n}F_{\alpha}'\left(x_{(i)},\alpha,\lambda\right)\left(F\left(x_{(i)},\alpha,\lambda\right)-\frac{i}{n+1}\right)=0 \tag{38}$$

$$\frac{\partial Z(\alpha,\lambda)}{\partial(\lambda)}=\sum_{i=1}^{n}F_{\lambda}'\left(x_{(i)},\alpha,\lambda\right)\left(F\left(x_{(i)},\alpha,\lambda\right)-\frac{i}{n+1}\right)=0 \tag{39}$$

The above two non-linear equations cannot be solved analytically, therefore, numerical technique is used for solution.

3.4. Weighted Least Squares Estimation (WLSE)

The estimation procedure to obtain the estimates of the parameters through WLSE is quite similar to the LSE with a slight change that it minimizes the weighted sum of squared deviation between true and expected CDF at observed ordered sample points, where weights are inversely proportional to the $var\left[F\left(x_{(i)}\right)\right]$. Thus, WLSE is obtained by minimizing

$$W(\alpha,\lambda)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i)},\alpha,\lambda\right)-\frac{i}{n+1}\right]^2$$

$$W(\alpha,\lambda)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{1-e^{-\lambda x_{(i)}}}{1-e^{-\lambda x_{(i)}}+e^{-\lambda\alpha x_{(i)}}}-\frac{i}{n+1}\right]^2 \tag{40}$$

To get the WLSE estimates $\widehat{\alpha}_{WLS}$ and $\widehat{\lambda}_{WLS}$ of parameters α and λ, differentiate equation (40) with respect to α and λ and equating to zero, which results of the following equations

$$\frac{\partial W(\alpha,\lambda)}{\partial\alpha}=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}F_{\alpha}'\left(x_{(i)},\alpha,\lambda\right)\left[F\left(x_{(i)},\alpha,\lambda\right)-\frac{i}{n+1}\right]^2=0 \tag{41}$$

$$\frac{\partial W(\alpha,\lambda)}{\partial\lambda}=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}F_{\lambda}'\left(x_{(i)},\alpha,\lambda\right)\left[F\left(x_{(i)},\alpha,\lambda\right)-\frac{i}{n+1}\right]^2=0 \tag{42}$$

Again, equation (41) and (42) cannot be solved analytically, therefore, numerical technique is used to secure the solution.

3.5. Cramer-von-Mises Estimation (CVME)

This method of estimation is proposed by [22], the method is the minimum distance method based on the difference between empirical and cumulative distribution functions. See, [10, 13] for more detail about this method. The CVM estimator of the parameters are obtained by minimizing

$$C(\alpha,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F\left(x_{(i)},\alpha,\lambda\right)-\frac{2i-1}{2n}\right)^2 \tag{43}$$

To get the CVM estimates $\widehat{\alpha}_{CVM}$ and $\widehat{\lambda}_{CVM}$ of parameters α and λ, differentiate equation (43) with respect to α and λ and equating to zero, which results of the following equations

$$\frac{\partial C(\alpha,\lambda)}{\partial\alpha}=\sum_{i=1}^{n}F_{\alpha}'\left(x_{(i)},\alpha,\lambda\right)\left(F\left(x_{(i)},\alpha,\lambda\right)-\frac{2i-1}{2n}\right) \tag{44}$$

$$\frac{\partial C(\alpha,\lambda)}{\partial\lambda}=\sum_{i=1}^{n}F_{\lambda}'\left(x_{(i)},\alpha,\lambda\right)\left(F\left(x_{(i)},\alpha,\lambda\right)-\frac{2i-1}{2n}\right) \tag{45}$$

Again, equation (44) and (45) cannot be solved analytically, therefore, same numerical technique is used to obtain the solution.

3.6. Anderson-Darling Method of Estimation (ADE)

This method is based on the minimization criteria of Anderson-Darling statistic [3] . The ADE estimate can be obtained by minimizing the following equation,

$$A(\alpha,\lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[log\left(F\left(x_{(i)},\alpha,\lambda\right)\right) + log\left(\overline{F}\left(x_{(n+1-i)},\alpha,\lambda\right)\right)\right] \tag{46}$$

where, $\overline{F}\left(x_{(n+1-i)},\alpha,\lambda\right) = 1 - F\left(x_{(i)},\alpha,\lambda\right)$. Therefore, the AD estimates $\widehat{\alpha}_{ADE}$ and $\widehat{\lambda}_{ADE}$ of the parameters α and λ can be obtained as the solutions of the partial differentiation based on equation (46) using same iterative procedure.

4. Simulation Study

In this section, the Monte Carlo simulation study has been performed to assess the performance of the different estimators obtained via different method of estimation viz., MLE, MPS, LSE, WLSE, CVME, and ADE.

In order to perform simulation, the random sample for the for the different variation of the sample size, and parameters. In particular, $n = 10, 20, \ldots, 100$ and $\alpha = (0.75, 1, 1.5, 2.5)$, $\lambda = 0.5$ are chosen. The estimators obtained via considered methods are not assumed any explicit mathematical form and not yield closed form solution, therefore N-R method is used to secure estimates of the parameters. The average estimate, MSE of the parameters using the above methods reported in Table 1-4 based on $N = 5000$ replication using the following formula.
Average estimates:

$$\widehat{\alpha}_{AE} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\alpha}_i \quad , \quad \widehat{\lambda}_{AE} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\lambda}_i$$

Mean square error:

$$\widehat{\alpha}_{MSE} = \frac{1}{N}\sum_{i=1}^{N}(\alpha_i - \widehat{\alpha})^2 \quad , \quad \widehat{\lambda}_{MSE} = \frac{1}{N}\sum_{i=1}^{N}\left(\lambda_i - \widehat{\lambda}\right)^2$$

From Table 1,2,3 and 4 it has been overserved that the MSE of the parameter decreases as the sample size increases, which ensures the consistency of the proposed estimators. It is important to mention that ML and MPS methods are based on likelihood and others are based on a distance measure. Further, if we fixed $\lambda = 0.5$ and varied $\alpha = (0.5, 1.0,\ 1.5,\ 2.0, 4.0)$ the Table 1 and 2 show that, in likelihood-based methods for shape parameter α and scale parameter λ, MPS and MLE perform well respectively. Furthermore, in the considered distance measure there is no trend for the shape parameter however there is a specific trend for the scale parameter viz.

$$CVME < ADE < WLSE < LSE$$

Next, if we fixed $\alpha = 0.5$ and varied $\lambda = (0.5, 1.0,\ 1.5,\ 2.0, 4.0)$ the Table 3 and 4 show that in likelihood-based methods MPS is better than MLE for shape parameter α and MLE is better than MPS for scale parameter λ. Further, in the considered distance measure there is the same conclusion as in the previous case. Also, we have calculated coverage probability and average length of 95% confidence interval. The MSEs of point estimates and average lengths of interval estimates decreases with increasing sample sizes.

Table 1: *Average estimate, corresponding MSE (in brackets), coverage probability (CP), and average length (AL) of the parameter α.*

			MLE	MPS	LSE	WLSE	CVME	ADE		
n	α	λ	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	CP	AL
	0.5		0.7480(0.4486)	0.4206(0.1025)	0.5093(0.1580)	0.5280(0.1919)	0.8315(0.6497)	0.6123(0.2148)	0.9021	2.7740
	1		1.2587(0.9402)	0.6711(0.3066)	0.7418(0.3047)	0.7731(0.3167)	1.2324(0.8992)	0.9776(0.4341)	0.8964	4.9517
10	1.5	0.5	1.6001(1.1336)	0.8334(0.6915)	0.9000(0.6562)	0.9409(0.6356)	1.4972(1.0294)	1.2206(0.6386)	0.8746	6.4410
	2		1.8439(1.3678)	0.9495(1.2598)	1.0172(1.2853)	1.0650(1.2287)	1.7136(1.2342)	1.3941(0.9913)	0.8578	7.4948
	4		3.7018(7.0880)	1.7501(6.3543)	1.7955(6.0933)	1.9273(5.8014)	3.1482(5.4229)	2.6206(4.9758)	0.8352	16.6406
	0.5		0.6214(0.1700)	0.4564(0.0725)	0.5348(0.1216)	0.5457(0.1189)	0.6761(0.2575)	0.5729(0.1198)	0.9226	1.5587
	1		1.1372(0.5215)	0.7987(0.2494)	0.9064(0.3177)	0.9336(0.3121)	1.1634(0.6126)	1.0094(0.3567)	0.9032	3.0567
20	1.5	0.5	1.5862(0.8163)	1.0880(0.4970)	1.1987(0.5313)	1.2552(0.5276)	1.5510(0.8429)	1.3760(0.5689)	0.8882	4.4341
	2		1.8794(0.9814)	1.2730(0.9129)	1.3964(0.8732)	1.4651(0.8341)	1.8158(1.0075)	1.6155(0.8123)	0.8668	5.3555
	4		3.7750(5.0184)	2.4191(4.3008)	2.6459(3.9490)	2.8126(3.8230)	3.5429(4.4899)	3.1448(3.8077)	0.8588	11.8705
	0.5		0.5635(0.0856)	0.4568(0.0491)	0.5153(0.0726)	0.5248(0.0710)	0.5975(0.1170)	0.5386(0.0694)	0.9206	1.1230
	1		1.0698(0.3216)	0.8346(0.1949)	0.9365(0.2681)	0.9641(0.2614)	1.1028(0.4134)	0.9939(0.2584)	0.9148	2.2968
30	1.5	0.5	1.5375(0.5993)	1.1729(0.4137)	1.2994(0.4845)	1.3492(0.4785)	1.5419(0.6836)	1.4047(0.4811)	0.8964	3.4581
	2		1.9344(0.8011)	1.4569(0.7000)	1.5892(0.7153)	1.6585(0.6753)	1.8931(0.8565)	1.7430(0.6775)	0.8848	4.4816
	4		3.7371(3.9179)	2.7059(3.5130)	2.9295(3.3280)	3.1099(3.2099)	3.5529(3.6898)	3.2870(3.2222)	3.2870	9.5240
	0.5		0.5431(0.0457)	0.4764(0.0316)	0.5157(0.0420)	0.5238(0.0399)	0.5615(0.0558)	0.5288(0.0395)	0.9352	0.8225
	1		1.0231(0.1800)	0.8725(0.1339)	0.9481(0.1800)	0.9704(0.1757)	1.0426(0.2285)	0.9806(0.1635)	0.9112	1.6688
50	1.5	0.5	1.5300(0.4326)	1.2806(0.3240)	1.3832(0.3893)	1.4262(0.3804)	1.5297(0.4835)	1.4483(0.3800)	0.9000	2.6398
	2		1.9594(0.5990)	1.6229(0.5192)	1.7557(0.6009)	1.8135(0.5593)	1.9498(0.7049)	1.8444(0.5491)	0.8974	3.4954
	4		3.7675(2.7696)	3.0364(2.5694)	3.2692(2.6738)	3.4104(2.5143)	3.6703(2.9441)	3.4859(2.9441)	0.8862	7.3997
	0.5		0.5165(0.0202)	0.4820(0.0168)	0.5029(0.0207)	0.5095(0.0197)	0.5241(0.0236)	0.5102(0.0193)	0.9428	0.5450
	1		1.0021(0.0787)	0.9188(0.0689)	0.9607(0.0833)	0.9770(0.0788)	1.0061(0.0919)	0.9800(0.0760)	0.932	1.1384
100	1.5	0.5	1.5070(0.2048)	1.3649(0.1768)	1.4444(0.2228)	1.4698(0.2080)	1.5181(0.2491)	1.4720(0.2020)	0.9288	1.8126
	2		1.9760(0.3487)	1.7767(0.3169)	1.8749(0.3911)	1.9127(0.3543)	1.9750(0.4278)	1.9167(0.3448)	0.9174	2.4772
	4		3.8139(1.6466)	3.3752(1.6005)	3.5492(1.8477)	3.6490(1.6550)	3.7598(1.9505)	3.6633(1.6357)	0.8970	5.2742

Table 2: *Average estimate, corresponding MSE (in brackets), coverage probability (CP), and average length (AL) of the parameter λ.*

n	α	λ	MLE $\hat{\lambda}$	MPS $\hat{\lambda}$	LSE $\hat{\lambda}$	WLSE $\hat{\lambda}$	CVME $\hat{\lambda}$	ADE $\hat{\lambda}$	CP	AL
10	0.5	0.5	0.6673(0.3265)	0.8401(0.5701)	0.7561(0.3530)	0.7459(0.3447)	0.6104(0.2124)	0.6804(0.2809)	0.9219	2.0061
	1		0.6841(0.2847)	0.9104(0.5775)	0.8337(0.3811)	0.8189(0.3685)	0.6635(0.2175)	0.7269(0.2686)	0.9680	2.1378
	1.5		0.7356(0.3258)	1.0049(0.7196)	0.9328(0.4947)	0.9116(0.4639)	0.7372(0.2748)	0.7988(0.3227)	0.8746	6.4410
	2		0.7680(0.3239)	1.0590(0.7175)	1.0069(0.5788)	0.9860(0.5528)	0.7879(0.3119)	0.8585(0.3748)	0.9986	2.3944
	4		0.8259(0.6516)	1.3265(8.4690)	1.1403(0.9113)	1.1075(0.8641)	0.8682(0.4780)	0.9547(0.6153)	0.9970	2.7113
20	0.5	0.5	0.5911(0.1410)	0.6685(0.2076)	0.6183(0.1171)	0.6086(0.1128)	0.5557(0.0896)	0.5900(0.1039)	0.9344	1.3010
	1		0.6122(0.1256)	0.7151(0.1965)	0.6716(0.1326)	0.6586(0.1255)	0.5976(0.0963)	0.6321(0.1106)	0.9628	1.3512
	1.5		0.6344(0.1332)	0.7579(0.2170)	0.7201(0.1628)	0.7022(0.1527)	0.6357(0.1144)	0.6691(0.1322)	0.9812	1.3977
	2		0.6649(0.1426)	0.8052(0.2433)	0.7676(0.1929)	0.7475(0.1792)	0.6739(0.1319)	0.7096(0.1506)	0.9944	1.4626
	4		0.6755(0.1574)	0.8551(0.2959)	0.8163(0.2518)	0.7882(0.2262)	0.7021(0.1670)	0.7426(0.1912)	0.9944	1.5231
30	0.5	0.5	0.5717(0.0816)	0.6169(0.1016)	0.5886(0.0696)	0.5813(0.0685)	0.5487(0.0573)	0.5721(0.0656)	0.9416	1.0205
	1		0.5866(0.0731)	0.6539(0.1015)	0.6241(0.0777)	0.6119(0.0731)	0.5766(0.0613)	0.6009(0.0687)	0.9608	1.0440
	1.5		0.6022(0.0815)	0.6845(0.1172)	0.6557(0.0961)	0.6403(0.0882)	0.6023(0.0746)	0.6254(0.0813)	0.9756	1.0649
	2		0.6132(0.0725)	0.7068(0.1125)	0.6810(0.1005)	0.6630(0.0905)	0.6227(0.0755)	0.6457(0.0823)	0.9922	1.0844
	4		0.6352(0.0870)	0.7579(0.1514)	0.7297(0.1385)	0.7029(0.1204)	0.6580(0.1005)	0.6818(0.1090)	0.9916	1.1660
50	0.5	0.5	0.5393(0.0373)	0.5636(0.0414)	0.5522(0.0379)	0.5457(0.0368)	0.5297(0.0338)	0.5428(0.0362)	0.9502	0.7331
	1		0.5622(0.0387)	0.6023(0.0484)	0.5852(0.0420)	0.5759(0.0390)	0.5581(0.0359)	0.5717(0.0375)	0.9634	0.7580
	1.5		0.5658(0.0381)	0.6163(0.0498)	0.5989(0.0469)	0.5866(0.0423)	0.5687(0.0395)	0.5816(0.0410)	0.9702	0.7622
	2		0.5742(0.0390)	0.6324(0.0543)	0.6132(0.0504)	0.5995(0.0454)	0.5802(0.0415)	0.5934(0.0433)	0.9840	0.7795
	4		0.5918(0.0438)	0.6686(0.0695)	0.6471(0.0646)	0.6280(0.0558)	0.6066(0.0510)	0.6199(0.0528)	0.9880	0.8447
100	0.5	0.5	0.5203(0.0167)	0.5316(0.0174)	0.5262(0.0167)	0.5221(0.0164)	0.5154(0.0157)	0.5217(0.0164)	0.9472	0.4975
	1		0.5350(0.0165)	0.5558(0.0186)	0.5471(0.0177)	0.5413(0.0168)	0.5342(0.0161)	0.5404(0.0165)	0.9632	0.5027
	1.5		0.5340(0.0158)	0.5609(0.0189)	0.5487(0.0180)	0.5418(0.0168)	0.5343(0.0163)	0.5413(0.0166)	0.9696	0.5028
	2		0.5404(0.0170)	0.5717(0.0213)	0.5594(0.0207)	0.5512(0.0188)	0.5438(0.0185)	0.5504(0.0185)	0.9716	0.5169
	4		0.5577(0.0217)	0.5990(0.0299)	0.5866(0.0294)	0.5739(0.0258)	0.5674(0.0250)	0.5725(0.0255)	0.9776	0.5664

Table 3: *Average estimate, corresponding MSE (in brackets), coverage probability (CP), and average length (AL) of the parameter α.*

			MLE	MPS	LSE	WLSE	CVME	ADE		
n	α	λ	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	CP	AL
10	0.5	0.5	0.7480(0.4486)	0.4206(0.1025)	0.5093(0.1580)	0.5280(0.1919)	0.8315(0.6497)	0.6123(0.2148)	0.9021	2.7740
		1	0.8849(0.7309)	0.4884(0.1380)	0.5704(0.1908)	0.5877(0.2157)	0.9441(0.8618)	0.7115(0.3390)	0.9412	3.3487
		1.5	0.9648(0.8586)	0.5286(0.1485)	0.6263(0.2364)	0.6443(0.2600)	1.0448(1.0950)	0.7730(0.4007)	0.9656	3.6730
		2	1.0314(0.9580)	0.5621(0.1575)	0.6512(0.2421)	0.6718(0.2728)	1.0759(1.1039)	0.8143(0.4252)	0.9838	3.9668
		4	1.2500(2.3613)	0.6568(0.3859)	0.8041(0.6648)	0.8251(0.7489)	1.3927(3.0527)	0.9708(1.0589)	0.9870	5.0005
20	0.5	0.5	0.6214(0.1700)	0.4564(0.0725)	0.5348(0.1216)	0.5457(0.1189)	0.6761(0.2575)	0.5729(0.1198)	0.9226	1.5587
		1	0.6754(0.2220)	0.4931(0.0856)	0.5840(0.1686)	0.5932(0.1650)	0.7406(0.3659)	0.6208(0.1553)	0.9380	1.7072
		1.5	0.7226(0.2984)	0.5248(0.1091)	0.6145(0.2036)	0.6300(0.2045)	0.7796(0.4392)	0.6630(0.2110)	0.9516	1.8383
		2	0.7451(0.3109)	0.5400(0.1096)	0.6327(0.2048)	0.6442(0.1985)	0.8047(0.4506)	0.6790(0.2101)	0.9734	1.9020
		4	0.7940(0.4595)	0.5710(0.1594)	0.6835(0.3378)	0.6970(0.3382)	0.8773(0.7492)	0.7230(0.3191)	0.9782	2.0525
30	0.5	0.5	0.5635(0.0856)	0.4568(0.0491)	0.5153(0.0726)	0.5248(0.0710)	0.5975(0.1170)	0.5386(0.0694)	0.9206	1.1230
		1	0.6090(0.1140)	0.4915(0.0579)	0.5558(0.0986)	0.5649(0.0946)	0.6459(0.1643)	0.5806(0.0926)	0.9366	1.2204
		1.5	0.6233(0.1264)	0.5021(0.0626)	0.5699(0.1114)	0.5811(0.1096)	0.6631(0.1879)	0.5934(0.1025)	0.9462	1.2543
		2	0.6472(0.1460)	0.5200(0.0696)	0.5951(0.1299)	0.6052(0.1288)	0.6929(0.2193)	0.6162(0.1179)	0.9616	1.3059
		4	0.6558(0.1619)	0.5262(0.0768)	0.6018(0.1511)	0.6120(0.1465)	0.7013(0.2553)	0.6237(0.1299)	0.9624	1.3278
50	0.5	0.5	0.5431(0.0457)	0.4764(0.0316)	0.5157(0.0420)	0.5238(0.0399)	0.5615(0.0558)	0.5288(0.0395)	0.9352	0.8225
		1	0.5649(0.0533)	0.4946(0.0344)	0.5370(0.0496)	0.5460(0.0485)	0.5851(0.0674)	0.5512(0.0475)	0.9448	0.8580
		1.5	0.5708(0.0528)	0.4996(0.0335)	0.5442(0.0527)	0.5530(0.0504)	0.5932(0.0724)	0.5570(0.0476)	0.9534	0.8675
		2	0.5871(0.0639)	0.5130(0.0391)	0.5587(0.0616)	0.5666(0.0582)	0.6091(0.0848)	0.5716(0.0569)	0.9600	0.8971
		4	0.5911(0.0676)	0.5163(0.0413)	0.5650(0.0671)	0.5721(0.0632)	0.6164(0.0926)	0.5770(0.0607)	0.9542	0.9009
100	0.5	0.5	0.5165(0.0202)	0.4820(0.0168)	0.5029(0.0207)	0.5095(0.0197)	0.5241(0.0236)	0.5102(0.0193)	0.9428	0.5450
		1	0.5237(0.0203)	0.4886(0.0165)	0.5133(0.0211)	0.5179(0.0199)	0.5349(0.0245)	0.5184(0.0194)	0.9504	0.5511
		1.5	0.5341(0.0223)	0.4979(0.0174)	0.5226(0.0227)	0.5279(0.0217)	0.5447(0.0267)	0.5283(0.0212)	0.9526	0.5642
		2	0.5384(0.0231)	0.5020(0.0179)	0.5250(0.0230)	0.5312(0.0222)	0.5472(0.0270)	0.5319(0.0215)	0.9584	0.5689
		4	0.5401(0.0245)	0.5035(0.0189)	0.5267(0.0243)	0.5333(0.0235)	0.5490(0.0286)	0.5339(0.0231)	0.9542	0.5704

Table 4: ***Average estimate, corresponding MSE (in brackets), coverage probability (CP), and average length (AL) of the parameter λ.***

			MLE	MPS	LSE	WLSE	CVME	ADE		
n	α	λ	$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\lambda}$	CP	AL
		0.5	0.6673(0.3265)	0.8401(0.5701)	0.7561(0.3530)	0.7459(0.3447)	0.6104(0.2124)	0.6804(0.2809)	0.9219	2.0061
		1	1.0886(0.5368)	1.3732(0.9025)	1.2920(0.7307)	1.2710(0.7011)	1.0419(0.4598)	1.1508(0.5634)	0.8852	3.2606
10	0.5	1.5	1.4210(0.7104)	1.7915(1.0438)	1.7105(0.9571)	1.6764(0.9080)	1.3752(0.6680)	1.5178(0.7507)	0.8530	4.2231
		2	1.6901(0.8701)	2.1557(1.0643)	2.0771(1.0523)	2.0381(1.0064)	1.6672(0.8659)	1.8324(0.8789)	0.8380	5.0404
		4	3.2462(3.9636)	4.1322(4.5452)	3.9455(4.4978)	3.8684(4.2918)	3.1487(3.8866)	3.5183(3.9629)	0.7940	9.5658
		0.5	0.5911(0.1410)	0.6685(0.2076)	0.6183(0.1171)	0.6086(0.1128)	0.5557(0.0896)	0.5900(0.1039)	0.9344	1.3010
		1	1.0851(0.3558)	1.2222(0.4673)	1.1608(0.3942)	1.1436(0.3778)	1.0436(0.3130)	1.1061(0.3461)	0.9110	2.3593
20	0.5	1.5	1.5361(0.6136)	1.7292(0.7719)	1.6564(0.6916)	1.6260(0.6685)	1.4886(0.5637)	1.5723(0.6165)	0.8800	3.3108
		2	1.9234(0.7698)	2.1675(0.9012)	2.0903(0.8724)	2.0557(0.8501)	1.8759(0.7469)	1.9870(0.7958)	0.8742	4.1137
		4	3.7479(3.0296)	4.2350(3.4680)	4.0701(3.4265)	4.0019(3.3306)	3.6457(3.0171)	3.8748(3.1226)	0.8592	8.0157
		0.5	0.5717(0.0816)	0.6169(0.1016)	0.5886(0.0696)	0.5813(0.0685)	0.5487(0.0573)	0.5721(0.0656)	0.9416	1.0205
		1	1.0743(0.2605)	1.1607(0.3147)	1.1155(0.2511)	1.1002(0.2447)	1.0393(0.2135)	1.0819(0.2347)	0.9234	1.9081
30	0.5	1.5	1.5641(0.4701)	1.6897(0.5472)	1.6331(0.4934)	1.6088(0.4780)	1.5219(0.4264)	1.5833(0.4543)	0.9028	2.7595
		2	1.9950(0.6302)	2.1572(0.7043)	2.0966(0.7009)	2.0667(0.6790)	1.9525(0.6221)	2.0341(0.6486)	0.8992	3.4823
		4	3.9617(2.5757)	4.2871(2.8736)	4.1582(2.7998)	4.0980(2.7071)	3.8725(2.5033)	4.0339(2.6208)	0.8880	6.9524
		0.5	0.5393(0.0373)	0.5636(0.0414)	0.5522(0.0379)	0.5457(0.0368)	0.5297(0.0338)	0.5428(0.0362)	0.9502	0.7331
		1	1.0420(0.1413)	1.0903(0.1542)	1.0657(0.1376)	1.0532(0.1358)	1.0219(0.1250)	1.0470(0.1329)	0.9324	1.4151
50	0.5	1.5	1.5299(0.2733)	1.6014(0.2941)	1.5685(0.2765)	1.5505(0.2730)	1.5038(0.2540)	1.5416(0.2676)	0.9228	2.0756
		2	2.0063(0.4690)	2.1016(0.4961)	2.0582(0.4837)	2.0352(0.4692)	1.9732(0.4503)	2.0240(0.4639)	0.9120	2.7205
		4	4.0068(1.8972)	4.1955(2.0007)	4.1190(1.9678)	4.0726(1.9180)	3.9486(1.8315)	4.0461(1.8783)	0.9094	5.3994
		0.5	0.5203(0.0167)	0.5316(0.0174)	0.5262(0.0167)	0.5221(0.0164)	0.5154(0.0157)	0.5217(0.0164)	0.9472	0.4975
		1	1.0314(0.0626)	1.0541(0.0651)	1.0440(0.0639)	1.0368(0.0626)	1.0226(0.0605)	1.0359(0.0620)	0.9350	0.9803
100	0.5	1.5	1.5243(0.1433)	1.5584(0.1472)	1.5423(0.1442)	1.5319(0.1424)	1.5105(0.1379)	1.5308(0.1413)	0.9526	0.5642
		2	2.0210(0.2438)	2.0665(0.2494)	2.0475(0.2470)	2.0310(0.2421)	2.0053(0.2368)	2.0293(0.2416)	0.9316	1.9212
		4	4.0084(0.9497)	4.0983(0.9621)	4.0687(0.9915)	4.0342(0.9895)	3.9851(0.9550)	4.0317(0.9820)	0.9282	3.7997

5. Real data application

In this section, the applicability of the proposed flexible extension has been discussed based on two survival data sets. The data set-I is taken from [6] which represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. The second data set has been obtained from [16] which consists of survival times, in week, of 33 patients suffering from acute Myelogenous Leukaemia. The summary of both data sets is given in Table 5.
Further, to show the superiority of the proposed model, the following well-known lifetime models are taken.

1. Extension of exponential distribution with pdf

$$f(x) = \alpha\lambda(1+\lambda x)^{\alpha-1}e^{\left(1-(1+\lambda x)^{\alpha}\right)} \qquad ; x > 0,\ \alpha > 0,\ \lambda > 0$$

2. Weibull distribution with pdf

$$f(x) = \frac{\alpha}{\lambda^{\alpha}}x^{\alpha-1}e^{-\left(\frac{x}{\lambda}\right)^{\alpha}} \qquad ; x \geq 0,\ \alpha > 0,\ \lambda > 0$$

3. Gamma distribution with pdf

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x} \qquad ; x > 0,\ \alpha > 0, \lambda > 0$$

where, α is shape and λ is the scale parameter.
The superiority of the proposed extension with the above considered families of the distribution are shown with the help of model criterion tools. Hence, the criterion like p-value, Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), and Kolmogorov-Smirnov (KS) test are taken.
To compare the models, Table 6-7, contains the values of the parameters estimated by the maximum likelihood, AIC, BIC, and KS statistics with the p-value for fitted data sets. From the K-S test statistics or associated p-value, it may be seen that the proposed model provides better fit than Weibull, Gamma, and Extension of Exponential models. Also, a similar result is concluded on the basis of negative of Log-likelihood which is higher than other three. Also based on the relative model selection criteria it is observed that the proposed model has smaller AIC and BIC in comparison to other thee considered models. Hence, the proposed mode is more suitable for considered real phenomena.

Further, the plots of the empirical cumulative distribution function (ECDF) and the fitted CDF for the considered two data set are shown in Figure 2. From Figure 2, it is concluded that the proposed model fits better to considered real data in comparison to other competitive models. Hence it may be taken as the alternative to the several lifetime models.

Table 5: *Summary of the considered data sets.*

Data	Min.	Q1	Median	Mean	Q3	SD	Skewness	Kurtosis	Max.
I	0.080	1.080	1.560	1.837	2.303	1.215	1.754	7.151	7.000
II	1.00	4.00	22.00	40.88	65.00	46.703	1.164	3.122	156.00

6. Conclusion

In this article, we have introduced a new transformation technique to generate the class of lifetime distributions. Further, the purposed transformation technique is illustrated via exponential distribution as baseline distribution and named a new flexible two-parameter lifetime distribution. The

Table 6: *MLE, AIC, BIC and KS statistics with the p-value for the data set-I.*

	ML Estimates			KS-Test			
Distributions	$\hat{\alpha}$	$\hat{\lambda}$	-LogL	Statistics	p-value	AIC	BIC
Proposed	8.2801	0.1257	102.4141	0.09658	0.5127	208.8283	213.3816
Weibull	1.6172	2.0558	104.0168	0.11346	0.3121	212.0336	216.5869
Gamma	2.4379	1.3273	102.9648	0.10372	0.4207	209.9296	214.4829
EE	5.0262	0.0731	109.6485	0.19617	0.0078	223.2970	227.8504

Table 7: *MLE, AIC, BIC and KS statistics with the p-value for the data set-II.*

	ML Estimates			KS-Test			
Distributions	$\hat{\alpha}$	$\hat{\lambda}$	-LogL	Statistics	p-value	AIC	BIC
Proposed	0.2659	0.0763	151.7045	0.10004	0.8959	307.4091	310.4021
Weibull	0.7764	35.3613	153.5868	0.13668	0.5684	311.1737	314.1667
Gamma	0.6877	0.0168	153.6737	0.13900	0.5466	311.3473	314.3403
EE	0.4897	0.0998	153.7430	0.13920	0.5440	311.4860	314.4790

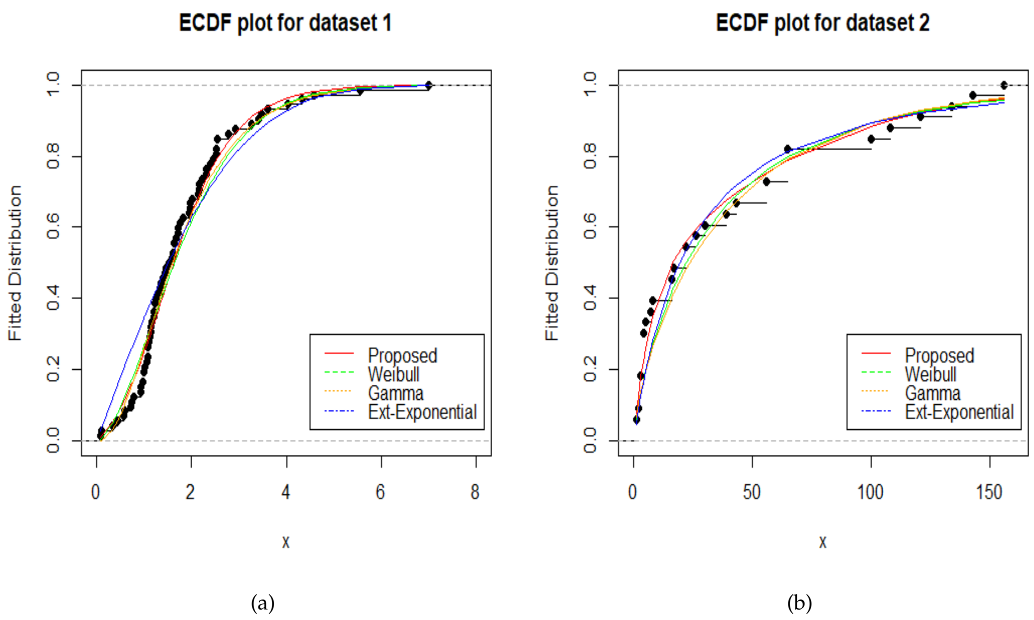


Figure 2: *(a) ECDF and fitted CDF plot of proposed distribution for data 1. (b) ECDF and fitted CDF plot of proposed distribution for data 2.*

proposed distribution has an increasing, decreasing, and constant hazard nature see Figure 1(b). Next, the different distributional properties are derived viz. mean, moments, moment generating function (MGF), mean deviation about mean and median, Bonferroni and Lorenz curves, Renyi entropy, s-entropy, cumulative residual entropy, *rth* order statistics, and reliability. The unknown parameter of the proposed model is estimated by different methods of estimation namely MLE, MPS, LSE, WLSE, CVME, and ADE. To compare the performances of different estimators obtained via different estimation methods Monte Carlo simulation study has been performed in Table 1,2,3 and 4. The superiority of the present study and model has been illustrated by constructing two real data sets. From the Table 6,7 and Figure 2, It is observed that the proposed new flexible two-parameter lifetime distribution provides better fits to the considered data sets among the most popular distributions viz., Weibull, Gamma, and extension of exponential distribution. Therefore, we may conclude that the proposed model might be considered an alternative to other considered models.

References

[1] Aljarrah, M. A., Lee, C., and Famoye, F. (2014). On generating TX family of distributions using quantile functions. *Journal of Statistical Distributions and Applications*, 1(1), 1-17.

[2] Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71(1), 63-79.

[3] Anderson, T. W., Darling, D. A. (1952). "Asymptotic Theory of Certain "Goodness of Fit" Criteria Based on Stochastic Processes." *The Annals of Mathematical Statistics, Ann. Math. Statist.* 23(2), 193-212.

[4] Barlow, R. E., Marshall, A. W., and Proschan, F. (1963). Properties of probability distributions with monotone hazard rate. *The Annals of Mathematical Statistics*, 34(2): 375-389.

[5] Barlow, R. E. and Proschan, F. (1975). Statistical theory of reliability and life testing: probability models. *Technical report, Florida State Univ Tallahassee*.

[6] Bjerkedal, T. (1960). Acquisition of resistance in Guinea Pies infected with different doses of Virulent Tubercle Bacilli. *American Journal of Hygiene*, 72, 130-48.

[7] Bonferroni, C. E. (1930). *Elementi di statistica generale*.

[8] Carrasco, J.M.F., Ortega, E.M.M., Cordeiro G.M. (2008) A generalized modified Weibull distribution for life time modeling. *Comput Stat Data Anal* 53:450-462.

[9] Cheng, R. C. H. and Amin, N. A. K. (1983). Estimating parameters in continuous univariate distributions with a shifted origin. *Journal of the Royal Statistical Society. Series B (Methodological)*, 45(3): 394-403.

[10] Choi, K. and Bulgren, W.G. (1968). An Estimation Procedure for Mixtures of Distributions. *Journal of the Royal Statistical Society: Series B (Methodological)*, 30: 444-460.

[11] Cordeiro, G. M., and de Castro, M. (2011). A new family of generalized distributions. *Journal of statistical computation and simulation*, 81(7), 883-898

[12] Cordeiro, G. M., Ortega, E. M., and da Cunha, D. C. (2013). The exponentiated generalized class of distributions. *Journal of data science*, 11(1), 1-27.

[13] Dennis D., Boos (1981) Minimum Distance Estimators for Location and Goodness of Fit, *Journal of the American Statistical Association*, 76:375, 663-670.

[14] Deshpande, J. V. and Suresh, R. P. (1990). Non-monotonic ageing. *Scandinavian journal of statistics*, 17(3): 257-262.

[15] Eugene, N., Lee, C., & Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and methods*, 31(4), 497-512.

[16] Feigl, P., and Zelen, M. (1965). Estimation of Exponential Survival Probabilities with Concomitant Information, *Biometrics*, 21, 826-838.

[17] Gompertz, B. (1825). On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. *Philosophical transactions of the Royal Society of London*, 115: 513-583.

[18] Granzotto, D. C. T., Louzada, F., & Balakrishnan, N. (2017). Cubic rank transmuted distributions: inferential issues and applications. *Journal of statistical Computation and Simulation*, 87(14), 2760-2778.

[19] Gupta, R. C., Gupta, P. L., and Gupta, R. D. (1998). Modeling failure time data by lehmann alternatives. *Communications in Statistics-Theory and Methods*, 27(4): 887-904.

[20] Kumar, D., Singh, U., & Singh, S. K. (2015). A method of proposing new distribution and its application to Bladder cancer patient data. *J. Stat. Appl. Pro. Lett*, 2(3), 235-245.

[21] Lorenz, M. O. (1905). Methods of measuring the concentration of wealth. *Publications of the American statistical association*, 90 (70) 209-219.

[22] MacDonald, P. D. M. (1971). Comment on an estimation procedure for mixtures of distributions by Choi and Bulgren. *Journal of Royal Statistical Society: Series B*, 33(2), 326-329.

[23] Mahdavi, A., & Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. *Communications in Statistics-Theory and Methods*, 46(13), 6543-6557.

[24] Marshall, A. W., & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84(3), 641-652.

[25] Maurya, S. K., Kumar, D., Singh, S. K., & Singh, U. (2018). One parameter decreasing failure rate distribution. *International Journal of Statistics & Economics*, 19(1), 120-138.

[26] Mudholkar, G. S. and Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, 42(2): 299-302.

[27] Rao, M., Chen, Y., Vemuri, B.C., Wang F. (2004) Cumulative residual entropy: a new measure of information. IEEE Trans Inf Theory 50:1220–1228

[28] Renyi, A. (1961). On measures of entropy and information. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*; Vol. 1, p. 547-561.

[29] Shannon, C.E. (1951). Prediction and entropy of printed English. *The Bell System Technical Journal*, 30:0 50-64.

[30] Shaw, W. T. and Buckley, I. R. C. (2007). The alchemy of probability distributions: beyond gram-charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. *arXiv preprint arXiv*: 0901.0434.

[31] Swain, J., Venkatraman, S. and Wilson, J. (1988). Least squares estimation of distribution function in Johnsons translation system. *Journal of Statistical Computation and Simulation*, 29, 271-297.

[32] Weibull, W. (1951). A statistical distribution of wide applicability. *Journal of applied mechanics*, 103(730): 293-297.