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A METHOD FOR GENERATING LIFETIME MODELS AND ITS APPLICATION TO REAL DATA

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Abstract

In the present work, we are going to propose a new transformation called Beta transformation. The new model includes the exponential distribution as a special case and it is known as Beta transformed exponential(BTE) distribution. We have been obtained its various statistical properties such as moments, moment generating function, median, hazard rate function, entropies, and order statistics. Parameters of BTE distribution are estimated by the method of maximum likelihood, Cramer-von-Mises and method of least square. Monte Carlo simulation is performed in order to investigate the performance of these estimates. Finally, two data sets have been analyzed to show how the proposed model works in practice.

Keywords: Cramer-von-Mises method, Exponential distribution, Hazard rate function, Method of maximum likelihood, , Method of least squares.

1. Introduction

The development of new methods of expanding the existing distributions is quite rich in the literature of distribution theory. There are several methods to propose new distributions by the use of some baseline distribution in statistical literature. This has been done through different approaches.
In Statistical literature no. of transformations are available to produce new cumulative distribution function (cdf) corresponding to a given cdf. Suppose, we have a cdf F(x), then the associated proposed cdf will be $G_i(x)$.

- The most popular among them is the power transformation initiated by Gupta et al. (1998) having the form
$$G_1(x) = [F(x)]^{\alpha}; \alpha > 0$$
- Quadratic rank transformation map (QRTM) proposed by Shaw and Buckley (2007) having the form
$$G_2(x) = (1+\lambda)F(x) - \lambda F^2(x); \mid \lambda \mid \leq 1$$
- DUS transformation proposed by Kumar et al. (2015) having the form
$$G_3(x) = \frac{e^{F(x)} - 1}{e - 1}; e = exp(1)$$
- SS-transformation proposed by Kumar et al. (2015) having the form
$$G_4(x) = sin(\frac{\pi}{2}F(x))$$

- Minimum Guarantee (MG)-distribution proposed by Kumar et al. (2017) having the form

$$G_5(x) = e^{1-\frac{1}{F(x)}}$$

- Log-transformation proposed by Maurya et al. (2016) and having the form

$$G_6(x) = 1 - \frac{\ln(2 - F(x))}{\ln 2}$$

- Transformation based on the generalization of Kumar et al. (2015) called GDUS transformation proposed by Maurya et al. (2017) having the form

$$G_7(x) = \frac{e^{F^{\alpha}(x)} - 1}{e - 1}; \alpha > 0$$

- New transformation initiated by Kyurkchiev (2017) to develop a sigmoid family of functions for Verhulst Logistic function is

$$G_9(x) = \frac{2F(x)}{1 + F(x)}$$

- New trigonometry based transformation called PCM proposed by Kumar et al. (2021) and having the form

$$G_{10}(x) = tan(\frac{\pi}{4}F(x))$$

The lifetime of a system can be modeled with statistical distributions that can be used in modeling lifetime data; among them, the most popular are gamma and weibull distributions. The proposed model contains several lifetime distributions as its special cases that are very flexible and able to accommodate different types of data sets since the probability density function and hazard rate can take on different forms such as increasing, decreasing, and constant shapes, and the potentiality of this model has been tested statistically by using it to model some real life data set.
In this article, We have decided to propose a new transformation known as beta transformation for $x \in \Re$ is given below

$$G(x) = \begin{cases} \frac{\beta}{\beta-1}[1 - \beta^{-F(x)}] & \text{if } \beta > 0, \beta \neq 1 \\ F(x) & \text{if } \beta = 1 \end{cases} \tag{1}$$

Where, $G(x)$ and $F(x)$ are the cdfs of the proposed transformation and baseline distribution. On differentiating (1) w.r.t. x, we get the probabilty density function (pdf) $g(x)$ and is given by

$$g(x) = \begin{cases} \frac{\beta \log \beta}{\beta-1} f(x)\beta^{-F(x)} & \text{if } \beta > 0, \beta \neq 1 \\ f(x) & \text{if } \beta = 1 \end{cases} \tag{2}$$

For $\beta \neq 1$, $g(x)$ is a weighted version of $f(x)$, where the weight function

$$w(x) = \beta^{-F(x)},$$

and g(x) can be written as

$$g(x) = \frac{f(x)w(x)}{c}.$$

where constant $c = E(w(X))$,
Here $c = \frac{\beta-1}{\beta \log \beta}$.

The survival reliability function(sf) $S(x)$ and the hazard rate function(hrf) $h(x)$ are obtained as

$$S(x) = \begin{cases} \frac{\beta^{1-F(x)}-1}{\beta-1} & \text{if } \beta \neq 1 \\ 1-F(x) & \text{if } \beta = 1 \end{cases} \tag{3}$$

and

$$h(x) = \begin{cases} f(x)\frac{\log\beta\beta^{1-F(x)}}{\beta^{1-F(x)}-1} & \text{if } \beta \neq 1 \\ \frac{f(x)}{S(x)} & \text{if } \beta = 1 \end{cases} \tag{4}$$

Lifetime models are used to explain the life of a system or device. These models are used in reliability, engineering, biological field, insurance, etc. The motivations for introducing our beta transformation model is that it is efficient to analyze lifetime data and very easy method of inducting an additional parameter to a family of distributions functions. It improve the characteristics, bring more flexibility to the given family and provide better fits than the other models having the same or higher number of parameters. The proposed method is very interesting with a closed form for the cdf and capable of modeling heavy tailed data sets.
The aim of this article is to introduce a transformation that yields new distributions by using a given baseline distribution. It contains only one new parameter other than the parameters involved in the baseline distribution. To illustrate the usefulness of this new transformation, We choose exponential as the baseline distributions in the present work.
The rest of this work is as follows. In Section 2, We introduce a special sub-case of (1), called a beta transformed exponential(BTE) distribution by considering exponential model as a parent distribution. Some mathematical properties are derived in Section 3. Certain characterizations of the proposed distribution are provided in Section 4. Estimation of parameter has been carried out in Section 5, Simulation study have been discussed in Section 5. Illustrate the flexibility of models using two real-life data sets discussed in Section 7. Finally, the article is concluded in Section 8.

2. Beta transformed exponential distribution

In this section, a sub model of the beta transformed family, called the beta transformed exponential (BTE) distribution is introduced. Let $G(x;\theta)$ be cdf of the exponential random variable given by $G(x;\theta) = 1 - e^{-\theta x}; x, \theta > 0$. Using this in equation(1), then the cdf of the BTE for $x > 0$ with the shape and scale parameters as $\beta > 0$ and $\theta > 0$ has the following form

$$G(x) = \begin{cases} \frac{\beta}{\beta-1}[1-\beta^{e^{-\theta x-1}}] & \text{if } \beta \neq 1 \\ 1-e^{-\theta x} & \text{if } \beta = 1 \end{cases} \tag{5}$$

The pdf g(x) is given by

$$g(x) = \begin{cases} \frac{\theta\log\beta}{\beta-1}e^{-\theta x}\beta^{e^{-\theta x}} & \text{if } \beta \neq 1 \\ \theta e^{-\theta x} & \text{if } \beta = 1 \end{cases} \tag{6}$$

The survival reliability function $S(x)$ and the hazard rate function(hrf) $h(x)$ are obtained as

$$S(x) = \begin{cases} \frac{\beta^{e^{-\theta x}}-1}{\beta-1} & \text{if } \beta \neq 1 \\ e^{-\theta x} & \text{if } \beta = 1 \end{cases} \tag{7}$$

and

$$h(x) = \begin{cases} \frac{\theta\log\beta}{\beta^{e^{-\theta x}}-1}e^{-\theta x}\beta^{e^{-\theta x}} & \text{if } \beta \neq 1 \\ \theta & \text{if } \beta = 1 \end{cases} \tag{8}$$

We have the following results for a general distribution function $F(x)$.

Table 1: *Behavior of the hazard functions of the three distributions.*

Parameter	Gamma	Weibull	BTE
$\beta = 1$	θ	θ	θ
$\beta > 1$	Increasing from 0 toθ	Increasing from 0 to ∞	Decreasing from $\frac{\beta\theta\log\beta}{\beta-1}$ to θ
$\beta < 1$	Decreasing from ∞ toθ	Decreasing from ∞ to 0	Increasing from $\frac{\beta\theta\log\beta}{\beta-1}$ to θ

- If $f(x)$ is a decreasing function, and $\beta \geq 1$, then $g(x)$ is a decreasing function.
- If $f(x)$ is a decreasing function, and $f(x)$ is log-convex, then for $\beta \geq 1$, the hazard rate function $h(x)$ is a decreasing function.

It can be easily seen that $f(x;\beta,\theta)$ is a unimodal function with mode at $\frac{(log(log\beta))}{\theta}$.
Here note that,
$\lim_{x\to 0} h(x) = \frac{\beta\theta\log\beta}{\beta-1}$,and
$\lim_{x\to\infty} h(x) = \theta$.
We have the following cases:

- When $\beta < 1$, $h(x)$ is an increasing function increases from $\frac{\beta\theta\log\beta}{\beta-1}$ to θ
- When $\beta > 1$, $h(x)$ is an decreasing function decreases from $\frac{\beta\theta\log\beta}{\beta-1}$ to θ
- When $\beta = 1$, $h(x)$ is a constant function.

By taking the second derivative of $f(x;\beta,\theta)$, it easily follows that the pdf of $BTE(\beta,\theta)$ is log-convex if $\beta > 1$ and log-concave if $\beta < 1$;
Table 1 provides the comparison of the hazard function of the BTE distribution with the corresponding hazard functions of Weibull and Gamma distributions. In all these cases the shape and scale parameters are assumed to be β and θ, respectively. It is clear from Table 1 that the hazard function of the BTE distribution is a decreasing or an increasing function depending on the shape parameter similarly as the Gamma and Weibull distributions, the ranges are quite different.

Figure 1 and 2 provides the plots of the pdf and hrf of the model for different values of β when $\theta = 1$

BTE distribution for $\beta > 1$, $\frac{\beta\log\beta}{\beta-1}$ is a decreasing function from 1 to 0, as β varies from 1 to ∞. If $X \sim BTE(\beta,\theta)$, then BTE distribution has the following mixture representation:

$$X = \begin{cases} X_1 \; with \; probability & \text{if } \frac{\log\beta}{\beta-1} \\ X_2 \; with \; probability & \text{if } 1 - \frac{\log\beta}{\beta-1}, \end{cases} \tag{9}$$

where X_1 and X_2 have the following pdfs:

$$f_{X_1(x)} = \theta e^{-\theta x}; x > 0 \tag{10}$$

$$f_{X_2(x)} = \frac{\log\beta}{\beta - 1 - \log\beta}\theta e^{-\theta x}(\beta^{e^{-\theta x}} - 1); x > 0, \tag{11}$$

respectively. From (9), as β approaches 1, X behaves like an exponential distribution, and as β increases, it behaves like X_2.

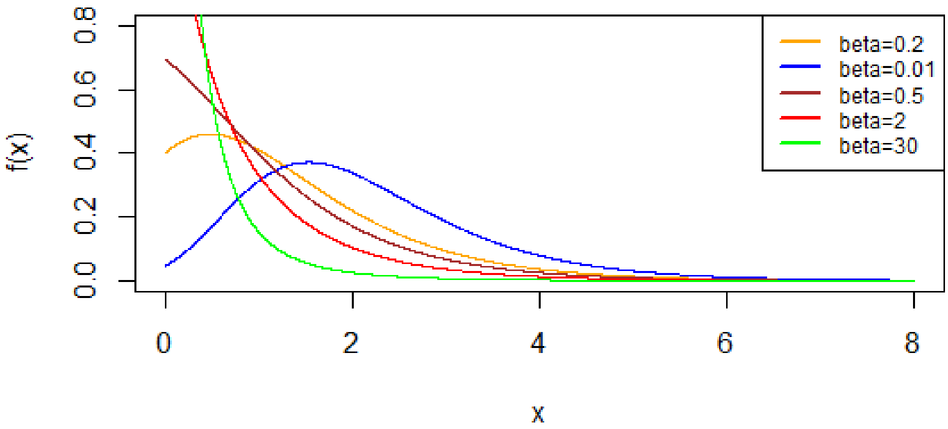


Figure 1: *plot of pdf of distribution*

3. The basic mathematical properties

This section provides some mathematical properties of proposed distribution.

3.1. Quantile function

The q^{th} quantile x_q of the BTE random variable is given by

$$x_q = -\frac{1}{\theta} \log \left[1 + \frac{\log(1 - \frac{q(\beta-1)}{\beta})}{\log \beta} \right]. \tag{12}$$

3.2. Moments

In this subsection, we intend to derive the moments and the moment generating function of the BTE distribution. Let X follow (6), then, the r^{th} moment of X is derived as

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x;\beta,\theta) dx, \tag{13}$$

using(6)in(13), we get

$$\mu_r' = \frac{r!}{\theta^r(\beta-1)} \sum_{k=1}^{\infty} \frac{(\ln \beta)^k}{k! k^n} \tag{14}$$

Furthermore, a general expression for the moment generating function (mgf) of the BTE random variable X is given by

$$M_X(t) = \frac{\theta}{(\beta-1)} \sum_{k=0}^{\infty} \frac{(\ln \beta)^{k+1}}{k!} \left[\frac{1}{\theta + \theta k - t} \right]; t < \theta \tag{15}$$

3.3. Sample Generation

The method to generate a sample is the inverse CDF transformation method. If X is $U(0,1)$ with CDF $F(x)$, then by the transformation, we generate the sample from the equation $G(x) = U$

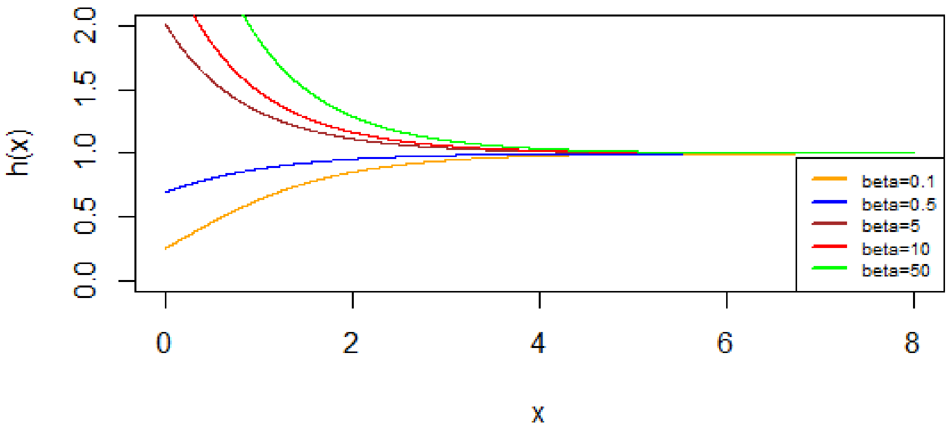


Figure 2: *plot of hrf of distribution*

implies $x = G^{-1}(U)$ of BTE distribution

$$x = -\frac{1}{\theta}\log\left[1 + \frac{\log(1 - \frac{U(\beta-1)}{\beta})}{\log\beta}\right]. \tag{16}$$

3.4. Order Statistics

Order statistics are used in applied fields of statistics such as reliability and lifetime testing. Let $X_1, X_2, ..., X_n$ be a random sample from $BTE(\beta, \theta)$. Also, let $X_{(1)}, X_{(2)}, ..., X_{(n)}$, denote the corresponding order statistics. Then the pdf and cdf of k^{th} order statistics, are given by

$$f_X(x) = \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k} f(x)$$

$$= \frac{n!}{(k-1)!(n-k)!}\frac{\theta\log\beta}{\beta-1}e^{-\theta x}\beta^{e^{-\theta x}}[\frac{\beta}{\beta-1}[1-\beta^{-F(x)}]]^{k-1}$$

$$\left[1 - [\frac{\beta}{\beta-1}[1-\beta^{-F(x)}]]\right]^{n-k} \tag{17}$$

and

$$F_X(x) = \sum_{j=k}^{n}\binom{n}{j}[F(x)]^j[1-F(x)]^{n-j}$$

$$= \sum_{j=k}^{n}\binom{n}{j}\left[\frac{\beta}{\beta-1}[1-\beta^{-F(x)}]\right]^j$$

$$\left[1 - [\frac{\beta}{\beta-1}[1-\beta^{-F(x)}]]\right]^{n-j} \tag{18}$$

respectively.
The pdf of the minimum and maximum of order statistics are obtained by putting $X = X_1$ and $X = X_n$ respectively in equation (6).

3.5. Entropy

The entropy of a random variable measures the variation of the uncertainity. A large value of entropy indicates the greater uncertainty in the data. The concept of entropy is important in different areas such as physics, probability and statistics, communication theory, and economics, etc. Several measures of entropy have been studied and compared in the literature.
If X is an absolute continuous random variable with $RE_X(\rho)$ for $\rho > 0$ and $\rho \neq 1$, is defined as

$$RE_X(\rho) = \frac{1}{1-\rho} \log \left[\int_{-\infty}^{\infty} f(x)^{\rho} dx \right] \tag{19}$$

From equation(19), we get

$$RE_X(\rho) = \frac{\rho}{1-\rho} \log \left(\frac{\theta \log \beta}{\beta - 1} \right) + \frac{1}{1-\rho} \log \left(\sum_{k=0}^{\infty} \frac{\ln(\beta)^k}{k!\rho(k+1)} \right). \tag{20}$$

4. Characterization of Beta transformed exponential distribution

In this section, we present certain characterizations of the BTE distribution based on a simple relationship between two truncated moments. This characterization result employs a theorem due to Glanzel (1987), which stated as follows:

Theorem 1. Let (Ω,F, $\mathbf{P}$) be a given probability space and let $H = [a, b]$ be an interval for some $a < b$ ($a = -\infty$, $b = \infty$ might as well be allowed). Let $X : \Omega \to \mathbf{H}$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on $\mathbf{H}$ such that

$$E[q_2(X)|X \geq x] = E[q_1(X)|X \geq x]\eta(x), x\epsilon\mathbf{H},$$

is defined with some real function η. Assume that q_1, q_2 are continuous functions, η has continuous derivative and F is twice continuously differentiable and strictly monotone function on the set $\mathbf{H}$. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of $\mathbf{H}$. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(x) = \int_a^x \mathcal{C} \mid \frac{\eta'(\mu)}{\eta(\mu)q_1(u) - q_2(u)} \mid \exp(-s(u))du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' q_1}{\eta q_1 - q_2}$ and $\mathcal{C}$ is a constant, chosen to make $\int_{\mathbf{H}} dF = 1$.

Proposition 1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \beta^{-e^{-\theta x}}$ and $q_2(x) = q_1(x)e^{-\theta x}$ for $x > 0$. The random variable X has pdf (6) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{\theta + 1}{\theta} e^{-x}, x > 0$$

Proof. Let X be a random variable with pdf (6), then
$(1 - F(x))E[q_1(X) \mid X \geq x] = \frac{\log \beta}{\beta - 1} e^{-\theta x}, x > 0$
and
$(1 - F(x))E[q_2(X) \mid X \geq x] = \frac{\theta \log \beta}{\beta - 1} \frac{e^{-x(\theta+1)}}{\theta+1}, x > 0$
and finally
$\eta(x)q_1(x) - q_2(x) = \frac{q_1 e^{-x}}{\theta} > 0, for x > 0.$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta q_1(x) - q_2(x)} = -(1+\theta), x > 0$$

and hence
$s(x) = -(1+\theta)x, x > 0$, or $e^{-s(x)} = e^{(1+\theta)x}, x > 0$. Now, in view of Theorem 1, X has density(6). ■

Corollary 1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 1. The pdf of X is (6) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta q_1(x) - q_2(x)} = -(1+\theta), x > 0$$

Remark 1. The general solution of the differential equation in Corollary 1 is

$$\eta(x) = e^{-(1+\theta)x}\left[\int (1+\theta)[q_1(x)]^{-1} q_2(x) e^{(1+\theta)x} dx + D\right]$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 1 with $D = 0$. However, it should be also noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

5. Estimation and simulation

In this section, we use the method of maximum likelihood, method of Cramer-von-Mises and ordinary least square method for estimation of parameters of BTE distributions.

5.1. Method of Maximum Likelihood Estimation

This is an extensively used method initiated by C.F. Gauss and elaborative study initiated by Prof. R. A. Fisher to obtain the estimator of the unknown parameter of the distribution. If $X_1, X_2 ..., X_n$ be a set of random observations from the population $BTE(\beta, \theta)$ distribution having pdf $g(x; \beta, \theta)$, then its log likelihood function will be as follows

$$\log L = n \log \theta + n \log\left(\frac{\log \beta}{\beta - 1}\right) - \theta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} e^{-\theta x_i} \log(\beta). \tag{21}$$

The likelihood equations are,

$$\frac{\partial \log L}{\partial \beta} = \frac{n(\beta - 1 - \beta \log \beta)}{\beta(\beta - 1)\log \beta} + \frac{1}{\beta} e^{-\theta x_i} = 0, \tag{22}$$

and

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{-\theta x_i} = 0. \tag{23}$$

The MLE of β and θ can be obtained by solving this nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as the Newton-Raphson algorithm.

5.2. Method of Cramer-von Mises

Cramer-von-Mises type minimum distance estimators are based on minimizing the distance between the theoretical and empirical cumulative distribution functions. Macdonald(1971) provided empirical evidence that the bias of these estimators is smaller than the bias of other minimum distance estimators. The Cramer-von-Mises estimators, $\hat{\beta}_{CME}$ and $\hat{\theta}_{CME}$ are the values of β and θ minimizing

$$C(\beta,\theta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(t_i \mid \beta,\theta) - \frac{2i-1}{2n} \right]^2 .$$

Differentiating the above equation partially, with respect to the parameters β and θ respectively and equating them to zero, we get the normal equations. Since the normal equations are non-linear, we can use iterative method to obtain the solution.

5.3. Method of Least-Square Estimation

The least square estimators were proposed by Swain et al. (1988) to estimate the parameters of Beta distributions. Here, we apply the same technique for the BTE distribution. The least square estimators of the unknown parameters β and θ of BTE distribution can be obtained by minimizing

$$\sum_{i=1}^{n} \left[F(t_i \mid \beta,\theta) - \frac{i}{n+1} \right]^2 .$$

with respect to unknown parameters β and θ.

5.4. Simulation study

We conduct Monte Carlo simulation studies to compare the performance of the estimators discussed in the previous sections and the process is repeated 1000 times. We evaluate the performance of the estimators based on bias and mean squared error. Methods are compared for sample sizes $n = 500, 700$ and 1000.

For each estimate we calculate the mean-squared error. The statistics are obtained using the following formulae.

$MSE(\hat{\beta}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\beta}-\beta)^2$ $\qquad MSE(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\theta}-\theta)^2$

The estimates, and the mean square errors (MSE) of the parameter estimates for the Maximum likelihood estimation procedure, method of Cramer-von-Mises and method of least squares are presented in Tables 1-3.

From Tables , we note that the maximum likelihood method performs well for estimating the model parameters. Also, as the sample size increases, the MSEs of the average estimates of maximum likelihood estimates decrease as expected.

The following observations can be drawn from the Tables 1-3.

1. All the estimators show the property of consistency, i.e. the MSE decreases as the sample size increases.
2. The MSE of $\hat{\beta}$ decreases with an increasing n for all the method of estimations.
3. The MSE of $\hat{\theta}$ decreases with an increasing n for all the method of estimations.
4. The MSE of $\hat{\beta}$ and $\hat{\theta}$ generally increases with an increasing beta and theta for any given n in all methods of estimation.
5. In terms of MSE, all the methods of estimation produce smaller MSE for $\hat{\beta}$ compared to that of $\hat{\theta}$.

Table 2: *Simulation result for $\beta = 0.5$ and $\theta = 0.1$.*

n	$Est.$	MLE	CVM	LSE
500	$\hat{\beta}$	0.5173	0.5087	0.5245
	$\hat{\theta}$	0.1004	0.1004	0.1021
	$MSE(\hat{\beta})$	5.247×10^{-5}	7.3129×10^{-5}	7.4119×10^{-5}
	$MSE(\hat{\theta})$	0.0209	0.0244	0.0311
700	$\hat{\beta}$	0.5171	0.5222	0.5306
	$\hat{\theta}$	0.1000	0.1003	0.0994
	$MSE(\hat{\beta})$	3.702×10^{-5}	7.0427×10^{-5}	5.4085×10^{-5}
	$MSE(\hat{\theta})$	0.0150	0.0220	0.0260
1000	$\hat{\beta}$	0.5013	0.5062	0.5845
	$\hat{\theta}$	0.1004	0.1005	0.0972
	$MSE(\hat{\beta})$	2.495×10^{-5}	4.0516×10^{-5}	3.9591×10^{-5}
	$MSE(\hat{\theta})$	0.0093	0.0129	0.0187

Table 3: *Simulation result for $\beta = 0.9$ and $\theta = 0.5$.*

n	$Est.$	MLE	CVM	LSE
500	$\hat{\beta}$	0.9315	0.9527	0.9941
	$\hat{\theta}$	0.5044	0.5049	0.4966
	$MSE(\hat{\beta})$	0.0019	0.0029	0.0025
	$MSE(\hat{\theta})$	0.0943	0.1899	0.1144
700	$\hat{\beta}$	0.9196	0.9489	0.9046
	$\hat{\theta}$	0.5018	0.5044	0.5011
	$MSE(\hat{\beta})$	0.0013	0.0025	0.0016
	$MSE(\hat{\theta})$	0.0664	0.1047	0.0556
1000	$\hat{\beta}$	0.9260	0.9310	0.9120
	$\hat{\theta}$	0.5004	0.4948	0.5034
	$MSE(\hat{\beta})$	0.0009	0.0014	0.0014
	$MSE(\hat{\theta})$	0.0436	0.0503	0.0484

6. Applications

In this section, we consider two real life data sets to illustrate the importance of the proposed distribution. The model parameters are estimated by the method of maximum likelihood and compare the fit of the BTE distribution with the following distributions: KuE,EW,W and E models.
(a) Kumaraswamy Exponential (KuE) distribution having pdf

$$f(x;\theta,\beta,c) = \theta\beta c e^{-cx}(1-e^{cx})^{\theta-1}[1-(1-e^{-cx})^{\theta}]^{\beta-1}; x>0, \theta,\beta,c>0. \quad (24)$$

(b) Exponentiated Weibull (EW) distribution having pdf

$$f(x;\theta,\beta,c) = \theta\beta^{\theta} c x^{\theta-1} e^{-}(\beta x)^{\theta}(1-e^{-(\beta x)^{\theta}})^{c-1}; x>0, \theta,\beta,c>0. \quad (25)$$

(c)Weibull (W) distribution having pdf

$$f(x;\theta,\beta) = \beta\theta^{\beta} x^{\beta-1} e^{(-\theta x)^{\beta}}; x>0, \theta,\beta>0. \quad (26)$$

Table 4: *Simulation result for $\beta = 1.5$ and $\theta = 1$.*

n	$Est.$	MLE	CVM	LSE
500	$\hat{\beta}$	1.5861	1.7673	1.6562
	$\hat{\theta}$	1.004	0.9999	1.0142
	$MSE(\hat{\beta})$	0.0101	0.0235	0.0218
	$MSE(\hat{\theta})$	0.3336	0.9141	0.8282
700	$\hat{\beta}$	1.5470	1.5931	1.5058
	$\hat{\theta}$	1.005	1.0114	1.0263
	$MSE(\hat{\beta})$	0.0075	0.0128	0.0174
	$MSE(\hat{\theta})$	0.2346	0.4939	0.2132
1000	$\hat{\beta}$	1.5278	1.5810	1.5941
	$\hat{\theta}$	1.003	0.9929	1.0107
	$MSE(\hat{\beta})$	0.0048	0.0052	0.0152
	$MSE(\hat{\theta})$	0.1388	0.1505	0.3750

Table 5: *The descriptive statistics of Data set.*

Min	1st Q	Median	Mean	3rd Q	Max
0.30	17.50	40.00	46.33	60.00	154.00

(d) Exponential (E) distribution having pdf

$$f(x;\theta) = \theta e^{-\theta x}; x > 0, \theta > 0 \tag{27}$$

The values of the log-likelihood functions$-\ln(L)$, AIC(Akaike Information Criterion), AICC(Akaike Information Criterion with correction) and BIC(Bayesian Information Criterion) are calculated for the five distributions in order to verify which distribution fits better to data. The better distribution corresponds to smaller $-\ln(L)$, AIC, AICC and BIC values. Here, $AIC = -2\ln(L) + 2k$, $AICC = -2\ln(L) + (\frac{2kn}{n-k-1})$ and $BIC = -2\ln(L) + k\ln(n)$; where L is the likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters and n is the sample size. The K-S distance $D_n = sup_x|F(x) - F_n(x)|$,where,$F_n(x)$ is the empirical distribution. Kolmogorov-Smirnov (K-S) statistic is computed to compare the fitted models.
The required computations are carried out in the R-language introduced by R Development Core Team (2019).

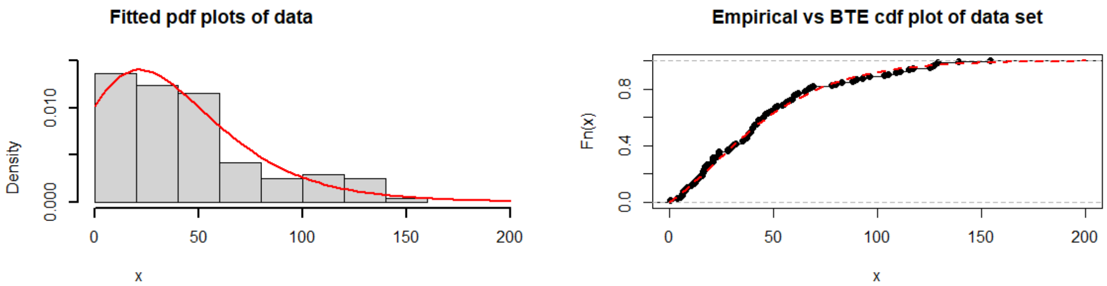
6.1. Data set 1

The first real data set represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 taken from Lee (1992). The data are:
(0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3,11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0,83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0). The data is skewed-to-the right with skewness =1.0432 and kurtosis =0.4021
The descriptive statistics of the above data set are given in Table 4. The values in Table 5 shows that the BTE distribution leads to a better fit to the other four models.

Figure 3, shows the fitted density curves, Empirical and the fitted cumulative distribution functions for the data set 1.

Table 6: *Maximum likelihood parameter estimates and goodness of fit for various models fitted for the Data set.*

Model	parameter estimates	log L	AIC	AICC	BIC	K-S	p-value
BTE	$\hat{\beta} = 0.131$ $\hat{\theta} = 0.033$	-579.155	1162.309	1162.411	1167.901	0.0534	0.8802
KuE	$\hat{\theta} = 1.651$ $\hat{\beta} = 0.098$ $\hat{c} = 0.231$	-583.314	1172.63	1172.83	1181.02	0.1152	0.0803
EW	$\hat{\theta} = 1.393$ $\hat{\beta} = 0.017$ $\hat{c} = 0.798$	-579.879	1165.76	1165.96	1174.15	0.0664	0.6606
W	$\hat{\beta} = 1.306$ $\hat{\theta} = 0.019$	-580.024	1164.05	1164.15	1169.64	0.0588	0.7967
E	$\hat{\theta} = 0.022$	-585.128	1172.26	1172.29	1175.05	0.1206	0.0594



(a) Fitted pdf plots of Data set 1

(b) Empirical and the fitted cumulative distribution functions for the data set 1

Figure 3. *Histogram with fitted pdf's (left) and Empirical cdf with fitted cdf's (right) for the data set 1.*

6.2. Data set 2

Here we consider the data set of the life of fatigue of Kelvar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed. The data sets are taken from Andrews and Herzberg (1985). The data are:
(0.0251, 0.6751, 1.0483, 1.4880, 1.8808, 2.2460, 3.4846, 0.0886, 0.6753, 1.0596, 1.5728, 1.8878, 2.2878, 3.7433, 0.0891, 0.7696, 1.0773, 1.5733, 1.8881, 2.3203, 3.7455, 0.2501, 0.8375, 1.1733 1.7083, 1.9316, 2.3470, 3.9143, 0.3113, 0.8391, 1.2570, 1.7263, 1.9558, 2.3513, 4.8073, 0.3451, 0.8425, 1.2766, 1.7460, 2.0048, 2.4951, 5.4005, 0.4763, 0.8645, 1.2985, 1.7630, 2.0408, 2.5260, 5.4435, 0.5650, 0.8851, 1.3211, 1.7746, 2.0903, 2.9941, 5.5295, 0.5671, 0.9113, 1.3503, 1.8275, 2.1093, 3.0256, 6.5541, 0.6566, 0.9120, 1.3551, 1.8375, 2.1330, 3.2678, 9.0960, 0.6748, 0.9836, 1.4595, 1.8503, 2.2100, 3.4045). The data is skewed-to-the right with skewness =1.9794 and kurtosis =5.160
The descriptive statistics of the above data set are given in Table 6. The values in Table 7 shows that the BTE distribution leads to a better fit to the other four models.

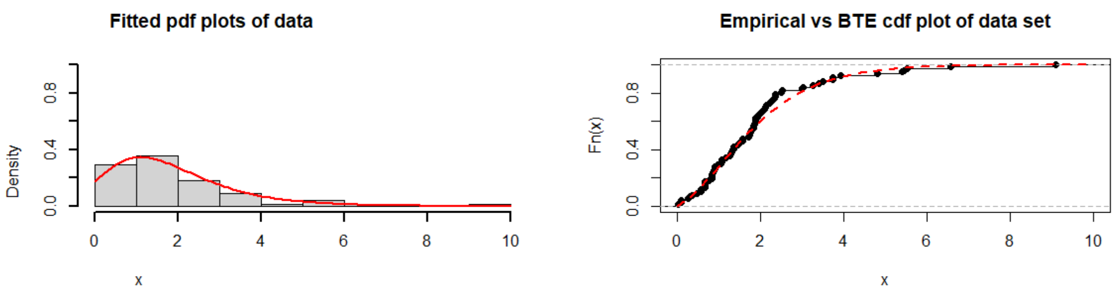
Figure 4, shows the fitted density curves, Empirical and the fitted cumulative distribution functions for the data set 2.

Table 7: *The descriptive statistics of Data set.*

Min	1st Q	Median	Mean	3rd Q	Max
0.025	0.905	1.736	1.959	2.296	9.096

Table 8: *Maximum likelihood parameter estimates and goodness of fit for various models fitted for the Data set.*

Model	parameter estimates	log L	AIC	AICC	BIC	K-S	p-value
BTE	$\hat{\beta} = 0.070$ $\hat{\theta} = 0.873$	-121.410	246.820	246.984	251.481	0.099	0.4167
EW	$\hat{\theta} = 1.101$ $\hat{\beta} = 0.609$ $\hat{c} = 1.443$	-122.166	250.332	250.665	257.324	0.0992	0.4160
W	$\hat{\beta} = 1.326$ $\hat{\theta} = 0.469$	-122.526	249.052	249.219	253.714	0.1098	0.2968
E	$\hat{\theta} = 0.510$	-127.114	256.228	256.282	258.559	0.5120	0.0266



(c) Fitted pdf plots of Data set 2

(d) Empirical and the fitted cumulative distribution functions for the data set 2

Figure 4. *Histogram with fitted pdf's (left) and Empirical cdf with fitted cdf's (right) for the data set 2.*

7. Concluding remarks

In this paper,we have proposed beta transformation in order to get a transformed distribution of some available baseline distribution. Beta transformation of $exp(\theta)$ distribution has been considered to check its application to the real problem called the Beta transformed (BTE) distribution. In the present work, we have provide expressions for the quantiles,moments,moment generating function,hazard rates,entropies and order statistics. The model parameters are estimated by maximum likelihood, Cramer-von Mises and least squares method. We have performed an extensive simulation study to compare these methods. We have compared estimators with respect to mean-squared error. The simulation results show that maximum likelihood estimators is the best performing estimator in terms of MSE. The next best performing estimator is the least square estimator followed by the Cramer-von Mises estimator.
Two real data sets are analyzed to show the importance and flexibility of this distribution.

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