Confidence intervals for the reliability characteristics via different estimation methods for the power Lindley model

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Abstract

In this article, classical and Bayes interval estimation procedures have been discussed for the reliability characteristics, namely mean time to system failure, reliability function, and hazard function for the power Lindley model and its special case. In the classical part, maximum likelihood estimation, maximum product spacing estimation are discussed to estimate the reliability characteristics. Since the computation of the exact confidence intervals for the reliability characteristics is not directly possible, then, using the large sample theory, the asymptotic confidence interval is constructed using the above-mentioned classical estimation methods. Further, the bootstrap (standard-boot, percentile-boot, students t-boot) confidence intervals are also obtained. Next, Bayes estimators are derived with a gamma prior using squared error loss function and linex loss function. The Bayes credible intervals for the same characteristics are constructed using simulated posterior samples. The obtained estimators are evaluated by the Monte Carlo simulation study in terms of mean square error, average width, and coverage probabilities. A real-life example has also been illustrated for the application purpose.

Keywords: Point estimation, Interval estimation of RC, MCMC method.
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Abbreviations

AIC	:	Akaike information criterion
ACIs	:	Asymptotic confidence intervals
BCIs	:	Bootstrap confidence intervals
BIC	:	Bayesian information criterion
CDF	:	Cumulative distribution function
CIs	:	Confidence intervals
$\mathcal{C}$	:	Coverage probability
DFR	:	Decreasing failure rate
HF	:	Hazard function
HPD	:	Highest posterior density
IFR	:	Increasing failure rate
KS	:	Kolmogrov Smirnov
LD	:	Lindley distribution
LLF	:	Linex loss function
MCMC	:	Markov Chain Monte Carlo method
MTSF	:	Mean time to system failure
MLE	:	Maximum likelihood estimation
MPSE	:	Maximum product spacing estimation
p-boot	:	Percentile bootstrap
PLD	:	Power Lindley distribution
PDF	:	Probability density distribution
RC	:	Reliability characteristics
RF	:	Reliability function
SELF	:	Squared error loss function
s-boot	:	Standard bootstrap
t-boot	:	Student's t-bootstrap
$\mathcal{W}$	:	Width of the intervals

1. Introduction

The study of the reliability characteristics, MTSF and RF, HF having great importance to study the aging pattern of any lifetime phenomenon. The aging pattern of lifetime products are varying in nature and hence modeled by suitable probability distribution. In this context, exponential distribution is the most exploited model to describe the inherent characteristics of the data. Although, its uses are restricted to the constant failure rate data. Alternatively, one parameter LD is also a good choice to analyze several survival/reliability data. The latter model received more consideration of several researchers because LD having IFR. LD was proposed by [14] as a counter example of fiducial statistics. The LD has been extensively used by several researchers to draw the inferences for the parameters using complete and censored information. For reference, the readers may be see in [1], [10], [13], [17] & [18] and the cited references therein. Let a random variable Y follow LD with parameter β, then the variable $X = Y^{1/\alpha}$ has the PLD. PLD was proposed by [9]. The PDF, CDF of PLD are, respectively, given by;

$$f(x,\alpha,\beta) = \frac{\alpha\beta^2}{(1+\beta)}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} \quad ; x \geq 0, \alpha, \beta > 0 \tag{1}$$

$$F(x,\alpha,\beta) = 1 - \left(1 + \frac{\beta}{1+\beta}x^\alpha\right)e^{-\beta x^\alpha} \tag{2}$$

where, $x \in \mathcal{R}^+$, $\alpha(> 0)$ is the shape parameter and $\beta(> 0)$ is the scale parameter. The parameter α involves the additional flexibility in terms of hazard rate as it has IFR for $\alpha > 1$ and DFR for $\alpha < 1$. PLD has been extensively used for estimation and prediction purpose and possesses all similar property as LD for $\alpha = 1$.

The theory of classical point estimation is based on the MLE because it assumes all optimum property such as consistency, sufficiency, efficiency, etc but sometimes it leads absurd result, especially for J-shaped distribution or unbounded range of distribution. Therefore, in such cases the MPSE might be better alternative. Moreover, the MLE required joint density function and MPSE required product spacing function. Whereas, the Bayes point estimation theory combines prior information and sample information supplied by likelihood function. Hence, Bayes paradigm involves the updating form of likelihood function. An important element, in Bayes estimation theory, is the loss function. The most popular one is SELF, which can be easily justified on grounds of minimum variance-unbiased estimation theory. However, the weakness of this loss function is that it is symmetric and provides an equal weight to the overestimation and underestimation of the same magnitude. But, in some real-life situation, specially in reliability analysis overestimation can lead to more severe or less severe consequences than underestimation, or vice versa. Thus, the use of asymmetric loss function is recommended. Also, use of symmetric loss function may be inappropriate as has been recognized by [4] and [22]. Thus, a number of asymmetric loss functions are available in literature, and one of the most widely used asymmetric loss function is the Linex loss function, originally proposed by [22] and popularized by [23] which has been found to be appropriate in the situation where overestimation is more serious than underestimation or vice-versa. Let, $\hat{\theta}$ be the estimate of the parameter θ and $\Delta = (\hat{\theta} - \theta)$ defines the deviation between estimated and true value of θ. The linex loss function (LLF) may be expressed as;

$$L(\Delta) \propto \left(e^{\psi\Delta} - \psi\Delta - 1\right); \qquad \psi \neq 0 \tag{3}$$

where ψ is the loss parameter which reflects the direction and degree of asymmetry. The loss parameter ψ allows different shapes of this loss function. If $\psi > 0$, then the linex loss function is quite asymmetric about zero with overestimation being more costly than underestimation and vice-versa. For ψ closes to zero, then this loss function is approximately squared error loss and therefore almost symmetric. Several authors have used this loss function in various estimation and prediction problems.

The focus of this paper is to consider the classical and Bayesian interval estimation of the MTSF, RF and HF for the PLD and its special case LD, and to develop a guideline for choosing the best estimation method that gives better estimates and CIs for RC, which would be of deep interest to applied statisticians/engineers. In classical estimation MLE and MPSE have been discussed for the estimation of the reliability characteristics. The Bayes estimators are derived under gamma informative prior using SELF and LLF. It is observed that the posterior expectation are turned in implicit form. Therefore, MCMC technique has been used to obtain the Bayes estimates based on posterior samples. Besides, ACIs using MLE and MPSE and Bayes credible/HPD are discussed. Further, different BCIs namely, standard bootstrap (s-boot), percentile bootstrap (p-boot) and student t-bootstrap (t-boot) of the reliability characteristics are proposed. To the best of our knowledge, no attempt has been made to study the aforementioned estimators, as well as CIs based on reliability characteristics for the PLD. The present work aims to fill this gap.

Evaluation of the different confidence intervals for the parameters as well RC associated with any lifetime distribution have great advantages in different fields e.g. engineering, industry, clinical trial study to predict the possible values of the lower and upper bound to achieve some standard benchmark. For example, in reliability theory several applications may be found in measuring stress level applied on a particular system. Minimum/maximum value of the stress level beyond the certain range of stress-level affects the working mode of the system/equipment. Further, the same may be seen in case of power supply in any electronic device, the minimum/maximum power supply beyond the specified limits leads the fail to functioning the electric circuit. Similarly, in industry, the experimenter may be interested to predict the quality of goods between certain limits. If the quality of lots lies in that interval then the practitioner may be interested to send/accept the lot in market otherwise reject the lot. Acceptance/rejection of the lot may lead certain level of confidence coefficient. Further, in context of the survival analysis, cancer patients are treated with drug with specified limit of doses, if a particular patient does not receive certain amount of drug/doses would causes the death of the patient. Motivated with this fact and variety of application of the confidence interval, several confidence intervals estimation for the RC have been proposed and studied for LD and PLD. Since, exact classical CIs for the considered characteristics can not be obtained because of unavailability of exact pivotal quantity, therefore, the asymptotic and bootstrap approaches have been employed to overcome the same difficulties. The similar difficulty has been encountered with the construction of Bayes interval. Therefore, the approximate Bayes interval has been constructed for RC based on generated posterior samples. The underlying RC for PLD are listed as follows;

- **MTSF:** The mean time to system failure is the simply mean of the PLD is given by;

$$\mu = \frac{(\alpha + \alpha\beta + 1)\ \Gamma\left(\frac{1}{\alpha}\right)}{\alpha^2 \beta^{\frac{1}{\alpha}} (1+\beta)} \tag{4}$$

- **RF:** It is the probability that the system performs beyond the certain time t, for PLD it is given as

$$R(t) = \left(1 + \frac{\beta}{1+\beta} t^{\alpha}\right) e^{-\beta t^{\alpha}} \tag{5}$$

- **HF:** The instantaneous failure of any system is defined by its HF

$$h(t) = \frac{\alpha\, \beta^2\, (1 + t^{\alpha})\, t^{\alpha-1}}{(1 + \beta + \beta t^{\alpha})} \tag{6}$$

The RC for the LD can be obtained by putting $\alpha = 1$ in the above expressions, respectively.

The reminder of the paper is organized as follows: Section 2, describes different methods of classical estimation. The problem of ACIs based on MLEs and MPSEs are discussed in Section of 3. The BCIs for RC are described in Section 4. Bayes estimation procedure along with Bayes computation technique have been discussed in Section 5. Section 6 presents the comparison among the classical and Bayes estimators using Monte Carlo simulations. A real-life data set has been used for illustrative purpose in Section 7. Lastly, Section 8 concludes the findings of the considered study.

2. Classical methods of estimation

2.1. Maximum likelihood estimation

In this section, we consider the classical estimation of the RC discussed in previous section. For this purpose, first we obtain the MLE of the parameters and then the MLE of the RC can be constructed by using invaraince property. Let $X_1, X_2, \cdots, X_n$ are the n iid units from the Equation (1) put on a life test. The log-likelihood function ($\ln(\alpha, \beta|x) = \mathbf{L}$) based on all n observations is

$$\mathbf{L} = n\ln(\alpha) + 2n\ln(\beta) - n\ln(1+\beta) - \beta\sum_{i=1}^{n} x_i^{\alpha} + \sum_{i=1}^{n}\ln(1+x_i^{\alpha}) + (\alpha-1)\sum_{i=1}^{n}\ln x_i \tag{7}$$

The MLEs of the parameters α, β are obtained by solving the derivatives of $\mathbf{L}$ w. r. t. α and β respectively. Let $\hat{\alpha}_m$, $\hat{\beta}_m$ be the MLEs of the parameters then the MLEs of the RC are obtained as

$$\hat{\mu}_m = \frac{\Gamma\left(\frac{1}{\hat{\alpha}_m}\right)\ (\hat{\alpha}_m + \hat{\alpha}_m\hat{\beta}_m + 1)}{\hat{\alpha}_m^2\, \hat{\beta}_m^{\frac{1}{\hat{\alpha}_m}}\ (1+\hat{\beta}_m)},\ \hat{R}(t)_m = \left(1 + \frac{\hat{\beta}_m}{1+\hat{\beta}_m}\, t^{\hat{\alpha}_m}\right) e^{-\hat{\beta}_m t},\ \hat{h}(t)_m = \frac{\hat{\alpha}_m\, \hat{\beta}_m^2\, (1+t^{\hat{\alpha}_m})t^{\hat{\alpha}_m-1}}{1+\hat{\beta}_m+\beta t^{\hat{\alpha}_m}} \tag{8}$$

2.2. Maximum product spacing estimation

The maximum product spacing method is introduced by [5], [6] as an alternative to MLE for the estimation of the unknown parameters of continuous univariate distributions. The maximum product spacing method was also derived independently by [15] as an approximation to the Kullback-Leibler measure of information. To motivate our choice, [6] proved that this method is as efficient as the MLEs and consistent under more general conditions.

Let us define the spacing function as the difference of the two consecutive CDFs

$$\mathcal{D}_i = F(x_i) - F(x_{(i-1)}) = \left(1 + \frac{\beta}{1+\beta}x_{i-1}^{\alpha}\right) e^{-\beta x_{i-1}^{\alpha}} - \left(1 + \frac{\beta}{1+\beta}x_i^{\alpha}\right) e^{-\beta x_i^{\alpha}} \tag{9}$$

such that $\sum \mathcal{D}_i = 1$,

MPSE method chooses α, β which maximizes the geometric mean of the spacing defined in equation (9)

$$\mathcal{G} = \left(\prod_{i=1}^{n+1}\mathcal{D}_i\right)^{\frac{1}{n+1}} \tag{10}$$

The equation (10) defines the alternative likelihood function using spacing. The MPS estimates can be obtained with the help of above equation by maximizing w.r.t. the parameters using iterative procedure. Once MPSE of the parameters say $\hat{\alpha}_p, \hat{\beta}_p$ are obtained, the MPSE of the reliability characteristics are obtained by simply using the invariance property.

$$\hat{\mu}_p = \frac{\Gamma\left(\frac{1}{\hat{\alpha}_p}\right)\ (\hat{\alpha}_p + \hat{\alpha}_p\hat{\beta}_p + 1)}{\hat{\alpha}_p^2\, \hat{\beta}_p^{\frac{1}{\hat{\alpha}_p}}\ (1+\hat{\beta}_p)},\ \hat{R}(t)_p = \left(1 + \frac{\hat{\beta}_p}{1+\hat{\beta}_p}\, t^{\hat{\alpha}_p}\right) e^{-\hat{\beta}_p t},\ \hat{h}(t)_p = \frac{\hat{\alpha}_p\, \hat{\beta}_p^2\, (1+t^{\hat{\alpha}_p})t^{\hat{\alpha}_p-1}}{1+\hat{\beta}_p+\beta t^{\hat{\alpha}_p}} \tag{11}$$

respectively.

3. Asymptotic confidence intervals for RC

3.1. ACIs using the usual likelihood function

In most of the two parameter lifetime distributions the construction of exact confidence intervals (CIs) usually is not an easy task due to implicit form of the MLEs. Therefore, $100(1-\tau)\%$ ACIs may be considered based on asymptotic distribution of MLEs. It is noted that $\sqrt{n}\,(\Theta - \hat{\Theta}) \sim AN(0, I(\hat{\Theta}))$, where $\Theta = (\alpha, \beta)$ and $\hat{\Theta}$ is the estimate of Θ. Hence, for this purpose the Fisher Information matrix is computed as;

$$I(\hat{\alpha}, \hat{\beta}) = \mathbf{E} \begin{pmatrix} -\frac{\partial^2 \mathbf{L}}{\partial \alpha^2} & -\frac{\partial^2 \mathbf{L}}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \mathbf{L}}{\partial \beta \partial \alpha} & -\frac{\partial^2 \mathbf{L}}{\partial \beta^2} \end{pmatrix}_{(\hat{\alpha}, \hat{\beta})} \tag{12}$$

where;

$$\frac{\partial^2 \mathbf{L}}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \beta \sum_{i=1}^{n} (\ln x_i)^2 x_i^{\alpha} + \sum_{i=1}^{n} \frac{x_i^{\alpha} (\ln x_i)^2}{1 + x_i^{\alpha}} \left(1 - \frac{x_i^{\alpha}}{1 + x_i^{\alpha}}\right) + 2 \sum_{i=1}^{n} x_i^{\alpha} \ln x_i + \sum_{i=1}^{n} x_i^{\alpha} (\ln x_i)^2 \{(\alpha - 1)\}$$

$$\frac{\partial^2 \mathbf{L}}{\partial \alpha \partial \beta} = \frac{\partial^2 \mathbf{L}}{\partial \beta \partial \alpha} = \sum_{i=1}^{n} x_i^{\alpha} \ln x_i, \qquad \frac{\partial^2 \mathbf{L}}{\partial \beta^2} = \frac{-2n}{\beta^2} + \frac{n}{(1+\beta)^2}.$$

All the above derivatives are evaluated at $(\hat{\alpha}, \hat{\beta})$. The above matrix given in equation (17) can be inverted to obtain the estimate of the asymptotic variance- covariance matrix of the MLEs and diagonal elements of $I^{-1}(\hat{\alpha}, \hat{\beta})$ provides asymptotic variance of α and β respectively. Then using large sample theory, two sided $100(1-\tau)\%$ approximate confidence interval for α, β is constructed as

$$\hat{\alpha} \pm Z_{1-\frac{\tau}{2}} \sqrt{var(\hat{\alpha})}, \quad \hat{\beta} \pm Z_{1-\frac{\tau}{2}} \sqrt{var(\hat{\beta})}.$$

Since, the MLEs of the RC's are constructed easily by applying invariance property of MLE but at the same time similar difficulties arise in construction of CIs for RCs, because no explicit distributions are available for the RCs. As we have seen from previous equations, RCs are the function of parameters α, β. Hence, the intervals for μ, $R(t)$ and $H(t)$ are constructed by applying the concept of Δ-method. The Δ-method is a general approach for computing confidence intervals for functions of maximum likelihood estimates. Let $g(\Theta)$ is any function of Θ such that it is differentiable w.r.t. the parameter(s), then

$$\sqrt{n}\,(g(\Theta) - g(\hat{\Theta})) \sim AN(0, \sigma^2_{\Theta}\, g'(\hat{\Theta})^2)$$

CI for μ:

For large sample, it may verified that,

$$\frac{\sqrt{n}\,(\mu - \hat{\mu})}{\sqrt{\sigma^2_{\hat{\mu}}}} \sim AN(0, 1)$$

where, variance of μ ($\sigma^2_{\hat{\mu}}$) is given as;

$$\sigma^2_{\hat{\mu}} = \sigma^2_{\hat{\alpha}} \left(\frac{\partial \mu}{\partial \alpha}\right)^2 + \sigma^2_{\hat{\beta}} \left(\frac{\partial \mu}{\partial \beta}\right)^2,$$

$$\frac{\partial\mu}{\partial\alpha} = \frac{\Gamma\frac{1}{\alpha}(\alpha+\alpha\beta+1)}{\alpha^2\beta^{\frac{1}{\alpha}}(1+\beta)}\left[-\frac{1}{\alpha^2}\Psi(1/\alpha)+\frac{1+\beta}{\alpha+\alpha\beta+1}-\frac{2}{\alpha}+\frac{\log\beta}{\alpha^2}\right]$$

$$\frac{\partial\mu}{\partial\beta} = \frac{\Gamma\frac{1}{\alpha}(\alpha+\alpha\beta+1)}{\alpha^2\beta^{\frac{1}{\alpha}}(1+\beta)}\left[\frac{\alpha}{\alpha+\alpha\beta+1}-\frac{1}{\alpha\beta}-\frac{1}{1+\beta}\right]$$

$\hat{\sigma}^2_\alpha$ and $\hat{\sigma}^2_\beta$ are the variances of the parameter α, β respectively. The $100(1-\tau)\%$ CIs for the μ is given by,

$$\hat{\mu}_m \pm Z_{\frac{\tau}{2}}\sqrt{\sigma^2_{\hat{\mu}}}$$

CI for R(t):
Similarly, for reliability function $R(t)$,

$$\frac{\sqrt{n}\left(R(t)-\hat{R}(t)\right)}{\sqrt{\sigma^2_{\hat{R}(t)}}} \sim AN(0,1)$$

where, variance of $\hat{R}(t)(\sigma^2_{\hat{R}(t)})$ is given as;

$$\sigma^2_{\hat{R}(t)} = \sigma^2_{\hat{\alpha}}\left(\frac{\partial R(t)}{\partial\alpha}\right)^2 + \sigma^2_{\hat{\beta}}\left(\frac{\partial R(t)}{\partial\beta}\right)^2$$

$$\frac{\partial R(t)}{\partial\alpha} = \left(1+\frac{\beta}{1+\beta}t^\alpha\right)e^{-\beta t^\alpha}\left[-\beta\,t^\alpha\,\log t+\frac{\beta\,t^\alpha\,\log t}{1+\beta+\beta\,t^\alpha}\right]$$

$$\frac{\partial R(t)}{\partial\beta} = \left(1+\frac{\beta}{1+\beta}t^\alpha\right)e^{-\beta t^\alpha}\left[-t^\alpha+\frac{t^\alpha}{(1+\beta)(1+\beta+\beta\,t^\alpha)}\right]$$

The $100(1-\tau)\%$ CIs for the $R(t)$ is given by,

$$\hat{R}(t)_m \pm Z_{\frac{\tau}{2}}\sqrt{\sigma^2_{\hat{R}(t)}}$$

CI for h(t): Similarly for hazard rate;

$$\frac{\sqrt{n}\left(h(t)-\hat{h}(t)\right)}{\sqrt{\sigma^2_{\hat{h}(t)}}} \sim AN(0,1)$$

where, variance of $\hat{h}(t)$, $\sigma^2_{\hat{h}(t)}$ is given as;

$$\sigma^2_{\hat{h}(t)} = \sigma^2_{\hat{\alpha}}\left(\frac{\partial h(t)}{\partial\alpha}\right)^2 + \sigma^2_{\hat{\beta}}\left(\frac{\partial h(t)}{\partial\beta}\right)^2$$

$$\frac{\partial h(t)}{\partial\alpha} = \frac{\alpha\,\beta^2\,(1+t^\alpha)\,t^{\alpha-1}}{(1+\beta+\beta t^\alpha)}\left[\frac{1}{\alpha}+\frac{t^\alpha\,\log t}{1+t^\alpha}+\log t-\frac{\beta\,t^\alpha\,\log t}{1+\beta+\beta\,t^\alpha}\right],\ \frac{\partial h(t)}{\partial\beta} = \frac{\alpha\,\beta^2\,(1+t^\alpha)\,t^{\alpha-1}}{(1+\beta+\beta t^\alpha)}\left[\frac{2}{\beta}+\frac{1+t^\alpha}{1+\beta+\beta\,t^\alpha}\right]$$

The $100(1-\tau)\%$ CIs for the $h(t)$ is given by,

$$\hat{h}(t)_m \pm Z_{\frac{\tau}{2}}\sqrt{\sigma^2_{\hat{h}(t)}}.$$

3.2. ACIs using spacing function

In this section, we have obtained the asymptotic confidence intervals using MPSE. As it was mentioned by [6] that the MPS method has the similar properties as MLE and is asymptotically equivalent. Estimation using MPSE has been also discussed by [19] and they showed mathematically that $\hat{\theta}_{MPS} = \hat{\theta}_{ML} + o(n^{-\frac{1}{2}})$ i.e. (the asymptotic or bootstrap inference around parameters based on MPSE may be carried out by utilizing the ML asymptotic). Utilizing the same concept, as MPSEs do not yield closed form of the estimators, hence the ACIs using MPSE for the parameters have been constructed. Let $I^{'}(\tilde{\alpha}, \tilde{\beta})$ be the observed Fishers information matrix and is defined as

$$I^{'}(\tilde{\alpha}_p, \tilde{\beta}_p) = \begin{bmatrix} -\mathcal{G}'^{''}_{\alpha\alpha} & -\mathcal{G}'^{''}_{\alpha\beta} \\ -\mathcal{G}'^{''}_{\beta\alpha} & -\mathcal{G}'^{''}_{\beta\beta} \end{bmatrix}_{(\tilde{\alpha},\ \tilde{\beta})} \tag{13}$$

The elements of the above matrix are given below

$$\begin{aligned}
\mathcal{G}'^{''}_{\alpha\alpha} = & \frac{1}{n+1}\left[\frac{F(x_1,\alpha,\beta)F^{''}_{\alpha\alpha}(x_1,\alpha,\beta) - \left(F^{'}_{\alpha}(x_1,\alpha,\beta)\right)^2}{F(x_1,\alpha,\beta)^2}\right] \\
& + \frac{1}{n+1}\left[\sum_{i=2}^{n}\frac{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}\left\{F^{''}_{\alpha\alpha}(x_i,\alpha,\beta) - F^{''}_{\alpha\alpha}(x_{i-1},\alpha,\beta)\right\}}{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}^2}\right] \\
& - \frac{1}{n+1}\left[\frac{\left\{F^{'}_{\alpha}(x_i,\alpha,\beta) - F^{'}_{\alpha}(x_{i-1},\alpha,\beta)\right\}^2}{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}^2}\right] - \frac{1}{n+1}\left[\frac{\{1 - F(x_n,\alpha,\beta)\}F^{''}_{\alpha\alpha}(x_n,\alpha,\beta) + \left\{F^{'}_{\alpha}(x_n,\alpha,\beta)\right\}^2}{\{1 - F(x_n,\alpha,\beta)\}^2}\right]
\end{aligned}$$

Similarly, the second derivative of the function $\mathcal{G}$ with respect to β is given by,

$$\begin{aligned}
\mathcal{G}'^{''}_{\beta\beta} = & \frac{1}{n+1}\left[\frac{F(x_1,\alpha,\beta)F^{''}_{\beta\beta}(x_1,\alpha,\beta) - \left(F^{'}_{\beta}(x_1,\alpha,\beta)\right)^2}{F(x_1,\alpha,\beta)^2}\right] \\
& + \frac{1}{n+1}\left[\sum_{i=2}^{n}\frac{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}\left\{F^{''}_{\beta\beta}(x_i,\alpha,\beta) - F^{''}_{\beta\beta}(x_{i-1},\alpha,\beta)\right\}}{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}^2}\right] \\
& - \frac{1}{n+1}\left[\frac{\left\{F^{'}_{\beta}(x_i,\alpha,\beta) - F^{'}_{\beta}(x_{i-1},\alpha,\beta)\right\}^2}{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}^2}\right] - \frac{1}{n+1}\left[\frac{\{1 - F(x_n,\alpha,\beta)\}F^{''}_{\beta\beta}(x_n,\alpha,\beta) + \left\{F^{'}_{\beta}(x_n,\alpha,\beta)\right\}^2}{\{1 - F(x_n,\alpha,\beta)\}^2}\right]
\end{aligned}$$

and the second derivative of the function $\mathcal{G}$ with respect to α, β is given as:

$$\begin{aligned}
\mathcal{G}'^{''}_{\alpha\beta} = \mathcal{G}'^{''}_{\beta\alpha} = & \frac{1}{m+1}\left[\frac{F(x_1,\alpha,\beta)F^{''}_{\alpha\beta}(x_1,\alpha,\beta) - F^{'}_{\alpha}(x_1,\alpha,\beta)F^{'}_{\beta}(x_1,\alpha,\beta)}{F(x_1,\alpha,\beta)^2}\right] \\
& + \frac{1}{m+1}\left[\sum_{i=2}^{n}\frac{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}\left\{F^{''}_{\alpha\beta}(x_i,\alpha,\beta) - F^{''}_{\alpha\beta}(x_{i-1},\alpha,\beta)\right\}}{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}^2}\right] \\
& - \frac{1}{m+1}\left[\frac{\left\{F^{'}_{\alpha}(x_i,\alpha,\beta) - F^{'}_{\alpha}(x_{i-1},\alpha,\beta)\right\}\left\{F^{'}_{\beta}(x_i,\alpha,\beta) - F^{'}_{\beta}(x_{i-1},\alpha,\beta)\right\}}{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}^2}\right] \\
& - \frac{1}{m+1}\left[\frac{\{1 - F(x_m,\alpha,\beta)\}F^{''}_{\alpha\beta}(x_m,\alpha,\beta) + \left\{F^{'}_{\alpha}(x_m,\alpha,\beta)\right\}\left\{F^{'}_{\beta}(x_m,\alpha,\beta)\right\}}{\{1 - F(x_m,\alpha,\beta)\}^2}\right]
\end{aligned}$$

where,

$$F'_{\alpha}(x_i, \alpha, \beta) = -\left(1 + \frac{\beta}{1+\beta} x_i^{\alpha}\right) e^{-\beta x_i^{\alpha}} \left[-\beta\, x_i^{\alpha}\, \log x_i + \frac{\beta\, x_i^{\alpha}\, \log x_i}{1+\beta+\beta\, x_i^{\alpha}}\right],$$

$$F''_{\alpha\alpha}(x_i, \alpha, \beta) = F'_{\alpha}(x_i, \alpha, \beta) \left[-\frac{F'_{\alpha}(x_i, \alpha, \beta)}{1 - F(x_i, \alpha, \beta)} + \log x_i + \frac{\beta(1+\beta)\, x_i^{\alpha}\, \log x_i}{(1+\beta)(1+\beta+\beta\, x_i^{\alpha}) + 1} - \frac{\beta\, x_i^{\alpha}\, \log x_i}{1+\beta+\beta\, x_i^{\alpha}}\right]$$

$$F'_{\beta}(x_i, \alpha, \beta) = -\left(1 + \frac{\beta}{1+\beta} x_i^{\alpha}\right) e^{-\beta x^{\alpha}} \left[-x_i^{\alpha} + \frac{x_i^{\alpha}}{(1+\beta)(1+\beta+\beta\, x_i^{\alpha})}\right]$$

$$F''_{\beta\beta}(x_i, \alpha, \beta) = F'_{\beta}(x_i, \alpha, \beta) \left[\frac{F'_{\beta}(x_i, \alpha, \beta)}{1 - F(x_i, \alpha, \beta)} - \frac{2 + x_i^{\alpha} + 2\beta + 2\beta\, x_i^{\alpha}}{(1+\beta)(1+\beta+\beta\, x_i^{\alpha}) + 1} - \frac{1}{1+\beta} - \frac{1 + x_i^{\alpha}}{1+\beta+\beta\, x_i^{\alpha}}\right]$$

$$F''_{\beta\alpha}(x_i, \alpha, \beta) = -\frac{F'_{\beta}}{1 - F(x_i, \alpha, \beta)} + \frac{2 + x_i^{\alpha} + 2\beta + 2\beta\, x_i^{\alpha}}{(1+\beta)(1+\beta+\beta\, x_i^{\alpha}) + 1} - \frac{1}{1+\beta} - \frac{1 + x_i^{\alpha}}{1+\beta+\beta\, x_i^{\alpha}}.$$

Thus, we can obtain an estimator of the information matrix as I($\hat{\alpha}, \hat{\beta}$), where $\hat{\alpha}$=$\hat{\alpha}_p$ and $\hat{\beta}$=$\hat{\beta}_p$ are the MPS estimator of the parameters and $V(\hat{\alpha})$ and $V(\hat{\beta})$ are the diagonal elements of $I^{-1}(\hat{\alpha}, \hat{\beta})$ which denotes the variance and covariance matrix. The approximate $(1-\tau)100\%$ confidence intervals for the parameters α and β is, therefore, given as, $\hat{\alpha} \pm Z_{\frac{\tau}{2}} \sqrt{V(\hat{\alpha})}$ and $\hat{\beta} \pm Z_{\frac{\tau}{2}} \sqrt{V(\hat{\beta})}$ respectively, where $Z_{\frac{\tau}{2}}$ is the upper $(\frac{\tau}{2})$ percentile of standard normal distribution. The interval estimate of RC using MPSE can be constructed in same way as discussed in previous subsection.

4. Bootstrap confidence interval

The confidence regions of parameters of a distribution have been determined using aspects of the distribution of the data. In particular, these regions have often been specified by appealing to the central limit theorem and normal approximations. The notion behind bootstrap techniques begins with the concession that the information about the source of the data is insufficient to perform the analysis to produce the necessary description of the distribution of the estimator. Thus, in this section, we considered an alternative procedure to usual method of CIs called as bootstrap method. The bootstrap method of finding confidence interval of parameters of a distribution is a most efficient sampling and re-sampling procedures without need of pivotal quantity, for more detail see, [7-8], [11]. Here, we discuss the different types of bootstrap confidence interval (BCIs), namely standard bootstrap (s-boot), percentile boot (p-boot) and students t-bootstrap (t-boot). The following steps may be used to construct the 95% BCI's.

1. Specify the value of sample size n and model parameters α, β.
2. Generate a sample $x_1, x_2, \cdots, x_n$ from equation (1)
3. Compute MLE $\hat{\alpha}$, $\hat{\beta}$ of α, β using $x_1, x_2, \cdots, x_n$.
4. Again generate bootstrap samples $x_1^*, x_2^*, \cdots, x_n^*$ from equation (1) using $\hat{\alpha}$, $\hat{\beta}$ as a population value and then compute MLE of RC $\hat{\rho} = [\hat{\mu}, \hat{R}(t), \hat{h}(t)]$.
5. Repeat step 2-3, B times and simulate $\hat{\rho}_i^*$; $i = 1, 2, \cdots, B$.

4.1. s-boot

Let $\bar{\rho}^*$ and σ_ρ^* be the sample mean and sample standard deviation of $\hat{\rho}^*, i = 1, 2, \cdots, B$.

$$\bar{\rho}^* = \frac{1}{B}\sum_{i=1}^{B}\hat{\rho}_i^* \quad \text{and} \quad \sigma_\rho^* = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\rho}_i^* - \bar{\rho}^*)^2}$$

respectively. Thus, $100(1-\tau)\%$ s-boot confidence interval for ρ is given by

$$[\hat{\rho}_L^s, \hat{\rho}_U^s] \in [\hat{\rho}^* - Z_{\tau/2}\,\sigma_\rho^* \qquad \hat{\rho}^* + Z_{\tau/2}.\sigma_\rho^*]$$

4.2. p-boot

Let $\hat{\rho}^{*(\delta)}$ be the δ-percentile of $(\hat{\rho}_{(i)}^*;\ i = 1, 2, \cdots, B)$ and $\hat{\rho}^{*(\delta)}$ is such that

$$\frac{1}{B}\sum_{i=1}^{B} I(\hat{\rho}_{(i)}^* \leq \hat{\rho}^{*(\delta)}) = \delta \quad : 0 \leq \delta \leq 1$$

where, $I(.)$ is the indicator function. Then $100(1-\tau)\%$ p-boot confidence interval is given by

$$\left(\hat{\rho}_L^p, \hat{\rho}_U^p\right) \in \left(\hat{\rho}^{*[B\frac{\tau}{2}]}, \hat{\rho}^{*[B\frac{1-\tau}{2}]}\right)$$

4.3. t-boot

The students t-bootstrap confidence interval is obtained by the following additional steps;

- Generate again bootstrap sample $x_1^{**}, x_2^{**}, x_n^{**}$ of size n from equation (1) using $\hat{\rho}^*$.
- Compute MLE of ρ say $\hat{\rho}^{**}$.
- Calculate $\sigma_\rho^{**} = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\rho}_i^{**} - \bar{\rho}^{**})^2}$ where $\bar{\rho}^{**} = \frac{1}{B}\sum_{i=1}^{B}\hat{\rho}_i^{**}$
- Compute the statistic $T = \dfrac{\hat{\rho}^{**} - \bar{\rho}^{**}}{\sigma_\rho^{**}}$. The $100(1-\rho)\%$ t-boot confidence interval for ρ is given by

$$\left(\hat{\rho}_L^p,\ \hat{\rho}_U^p\right) \in \left(\bar{\rho}^{**} - T^{\tau/2}\sigma_\rho^{**},\quad \bar{\rho}^{**} + T^{\tau/2}\,\sigma_\rho^{**}\right)$$

To study the different CIs, we consider their estimated $\mathcal{W}$ and $\mathcal{C}$. For each of the considered methods, the average width of the BCIs is computed based on the B different trials. The average width and coverage probability are given by

$$\mathcal{W} = \frac{\sum_{i=1}^{B}(U_i - L_i)}{B}, \quad \mathcal{C} = \frac{\#(L \leq \rho \leq U)}{B}$$

where L and U are the $100(1-\tau)\%$ CI based on B replicates.

5. Bayesian estimation and credible interval

In this section, the Bayes estimators of the RC have been derived under gamma priors and different loss function as mentioned in Section 1. Let $\underline{X} = (X_1, X_2, X_3, ..., X_n)$ be the random observations of size n from (1). Since, Bayes paradigm combines sample information with prior distribution and provide the updated distribution, termed as posterior distribution, hence, the posterior distribution is derived and the respective Bayes estimates are computed under SELF and LLF. For review of the parametric inference on Bayesian paradigm one may see, [16], [18], [20] etc. Let us consider the priors for α, β are;

$$\pi_1(\alpha) \propto \alpha^{a-1}e^{-b\alpha}, \quad \pi_2(\beta) \propto \beta^{c-1}e^{-d\beta}$$

Since, the considered priors are independent and flexible in nature, hence the joint prior $\pi(\alpha, \beta)$ of (α, β) is given by;

$$\pi(\alpha, \beta) \propto \alpha^{a-1}e^{-b\alpha}\beta^{c-1}e^{-d\beta} \tag{14}$$

where a, b, c and d are the hyper-parameters, assuming to be known and positive. The prior defined in the equation above accommodates the different shapes of other distributions which depends over the values of hyper-parameters. Jeffrey's non-informative prior is also a particular case of the above prior and obtained by assuming $a, b, c, d \to 0$, given by

$$\pi(\alpha, \beta) \propto \frac{1}{\alpha\,\beta}; \quad \alpha, \beta > 0$$

Although, the prior defined above is improper in nature but the resulting posterior always remains proper. The joint posterior distribution is obtained as

$$p(\alpha, \beta|\underline{x}) \propto \alpha^{n+a-1}\beta^{2n+c-1}(1+\beta)^{-n}\exp\left\{-b\alpha - d\beta - \beta\sum_{i=1}^{n} x_i^{\alpha}\right\}\prod_{i=1}^{n}\left\{(1+x_i^{\alpha})x_i^{\alpha-1}\right\} \tag{15}$$

The Bayes estimators of the RC under SELF is the posterior mean and is given by

$$\hat{\Theta}_{sf} = E_p(\Theta|x) \tag{16}$$

where

$$E_p(\Theta|\underline{x}) = K^{-1}\int\limits_{\alpha=0}^{\infty}\int\limits_{\beta=0}^{\infty}\Theta\,\alpha^{n+a-1}\beta^{2n+c-1}(1+\beta)^{-n}\exp\left\{-b\alpha - d\beta - \beta\sum_{i=1}^{n} x_i^{\alpha}\right\}\prod_{i=1}^{n}\left\{(1+x_i^{\alpha})x_i^{\alpha-1}\right\} d\alpha\, d\beta \tag{17}$$

SELF is the most popular and most widely used symmetric loss function, although sometimes in reliability inference SELF does not provide more accurate result due to over and under estimation. The details of LLF is given in Section 1. The Bayes estimates of the considered characteristics under the LLF is obtained by using the following expression.

$$\hat{\Theta}_{lf} = -\frac{1}{\psi}\log\left(E_p\left[e^{-\psi\Theta}|\underset{\sim}{x}\right]\right) \tag{18}$$

provided that the expectation $E_p\left[e^{-\psi\Theta}|\underset{\sim}{x}\right]$ exists and is finite, where $\Theta = [\alpha, \beta, \mu, R(t), h(t)]$ and

$$E_p\left[e^{-\psi\Theta}|\underset{\sim}{x}\right] = K^{-1}\int\limits_{\alpha=0}^{\infty}\int\limits_{\beta=0}^{\infty}\alpha^{n+a-1}\beta^{2n+c}(1+\beta)^{-n}\exp\left\{-b\alpha - d\beta - \beta\sum_{i=1}^{n} x_i^{\alpha} - \psi\Theta\right\}\times\prod_{i=1}^{n}\left\{(1+x_i^{\alpha})x_i^{\alpha-1}\right\} d\alpha\, d\beta. \tag{19}$$

5.1. Bayes computation via Markov Chain Monte Carlo method

From the previous section, It has been observed that the form of Bayes estimators can not be solved analytically. The evaluation of the posterior expectation will be complicated and it will be the ratio of two intractable integrals. In such situations, Markov Chain Monte Carlo (MCMC) technique can be effectively used to generate sample from full conditional posterior distributions. For more detail about MCMC method see, [12], [20], [21]. Thus concept of Gibbs under Metropolis-Hastings (M-H) sampling procedure has been utilized to generate sample from the posterior density function (22) under the assumption that parameters α and β has independent gamma priors with hyper-parameters (a, b) and (c, d) respectively. To implement Gibbs under M-H algorithm the full conditional posterior densities of α and β are given by;

$$p_1(\alpha|\beta, \underline{x}) \propto \alpha^{n+a-1} e^{-b\alpha} \exp\left\{-b\alpha - \beta \sum_{i=1}^{n} x_i^{\alpha}\right\} \prod_{i=1}^{n} \left\{(1 + x_i^{\alpha}) x_i^{\alpha-1}\right\} \tag{20}$$

$$p_2(\beta|\alpha, \underline{x}) \propto \beta^{2n+c-1} (1+\beta)^{-n} \exp\left\{-d\beta - \beta \sum_{i=1}^{n} x_i^{\alpha}\right\} \tag{21}$$

The following steps are taken to generate posterior samples from full conditional distribution.

- Start with $j = 1$ and initial values (α_0, β_0)
- Using M-H algorithm generate posterior sample for α and β from (25) and (26) respectively, where asymptotic normal distribution of full conditional densities are considered as a proposal.
- Repeat step 2, for all $j = 1, 2, 3, \cdots, N$ and obtained $(\alpha_1, \beta_1), (\alpha_2, \beta_2), ...(\alpha_N, \beta_N)$
- Generate the sequence of μ, $R(t)$ and $h(t)$ for specified t by plugin the sequences of $(\alpha_j, \beta_j); j = 1, 2, \cdots, N$, as

$$\mu_1, \mu_2, \cdots, \mu_N, \quad R_1, R_2, \cdots, R_N, \quad h_1, h_2, \cdots, h_N$$

- The Bayes estimates of the RC under SELF are given by

$$\hat{\mu}_{sf} \approx \frac{1}{N - N_0} \sum_{j=1}^{N-N_0} \mu_j, \quad \hat{R}(t)_{sf} \approx \frac{1}{N - N_0} \sum_{j=1}^{N} R_j, \quad \hat{h}(t)_{sf} \approx \frac{1}{N - N_0} \sum_{j=1}^{N} h_j$$

- The Bayes estimates under LLF are obtained as;

$$\hat{\mu}_{lf} = -\frac{1}{\psi} \log\left(\frac{1}{N - N_0} \sum_{j=1}^{N} \exp(-\psi\mu_j)\right), \ \hat{R}(t)_{lf} = -\frac{1}{\psi} \log\left(\frac{1}{N - N_0} \sum_{j=1}^{N} \exp(-\psi R_j)\right), \ \hat{h}(t)_{lf} = -\frac{1}{\psi} \log\left(\frac{1}{N - N_0} \sum_{j=1}^{N} \exp(-\psi h_j)\right),$$

  respectively. N_0 is the burn in period of Markov Chain.

5.2. HPD credible interval

After extracting the posterior samples we can easily construct the HPD credible intervals for α and β, see [3]. Therefore for this purpose order $\alpha_1, \alpha_2, ..., \alpha_N$ as $\alpha_1 < \alpha_2 < ... < \alpha_N$ and $\beta_1, \beta_2, ..., \beta_N$ as $\beta_1 < \beta_2 < ... < \beta_N$. Then $100(1-\tau)\%$ credible intervals of α and β are

$$(\alpha_1, \alpha_{[N(1-\tau)]}), ..., (\alpha_{[N\tau]}, \alpha_N) \text{and} (\beta_1, \beta_{[N(1-\tau)]}), ..., (\beta_{[N\tau]}, \beta_N)$$

Using the sequence of μ, $R(t)$ and $h(t)$ the $100(1-\tau)\%$ credible intervals for RC can be constructed by proceeding in similar way. Here $[x]$ denotes the greatest integer less than or equal to x. Then, the HPD credible interval is that interval which has the shortest length.

6. Comparison of estimators by a simulation study

In this section, we carry out a simulation study to assess the performance of the proposed point (classical & Bayesian) and interval estimates (AICs, BCIs & HPD) for PLD and in particular for Lindley distribution. To perform simulation study, a set of sample sizes $n = 10, 20, 30, 50, 100$ with different parametric combinations $(\alpha, \beta) = (0.75, 0.85), (1, 0.75), (2.5, 1.5), (2, 2.5)$ &$(3.5, 2)$ are taken. Since, the PLD reduces to LD when $\alpha = 1$, therefore the choice $(\alpha, \beta) = (1, 0.75)$ among the considered choices corresponds the result for LD. In classical setup, the MLE, MPSE of μ,

$R(t)$ and $h(t)$ have been computed for specified $t = 0.75$. The ACIs based on MLEs and MPSEs are constructed for the considered characteristics. Also, for each design, $B = 1,000$ bootstrap samples with each of size n are drawn from the original sample and BCIs are constructed based on replicated $K = 3000$ times. Next, we discussed the Bayesian estimation procedure for the estimation of the same characteristics using MCMC technique and construct HPD credible interval based on generated posterior samples. The Bayes estimates are reported under informative gamma prior and non-informative prior using SELF and LLF. For LLF, the choices of loss parameter ψ are taken as $(-2, 1.5)$. The negative/positive choices of ψ indicated the departure from symmetry. Average mean square error (MSEs) of the RC for each set up are reported in Tables 4-6. In all the simulation Tables, $(\bullet)_m$, $(\bullet)_p$ denote the estimates obtained via MLE, MPSE and $(\bullet)_{sf}$, $(\bullet)_{lf1}$, $(\bullet)_{lf1}$ denote the Bayes estimates obtained under SELF and LLF ($\psi = -2, \psi = -1.5$) respectively. Tables 4-6 describe the average estimates and MSEs for MTSF, RF and HF obtained via different classical methods of estimation and Bayes estimation method, respectively. Form this simulation study it is noted that the MSEs of the classical estimates obtained through MLE and MPSE methods are very close to each other and more or less similar to the MSEs of the Bayes estimates obtained under non-informative prior. However, the Bayes estimates under informative prior information provide better results in terms of MSE than classical estimates and Bayes estimates with non-informative prior. The MSEs of all the proposed estimates ensure the property of consistency through increasing sample size, also the MSEs of the Bayes estimators under SELF and Bayes estimates under LLF are almost same and the significance differences are very small for all the considered choices of parameters and sample sizes.

The estimated average widths $(\mathcal{W})$ and coverage probabilities $(\mathcal{C})$ of 95% ACIs based on MLE and MPSE, different BCIs and HPDIs of the RC are reported in Tables 7-9. We observe that as the sample size increases, the average widths decreases in all the cases as expected. The width of the Bayes interval is less as compared to the width of ACIs and BCIs. In comparison of ACIs and BCIs, the width of BCIs are lesser and boot-p perform better. Consequently, the smaller width affects the coverage probability. All simulations were performed using programs written in the open source statistical package R. Moreover, among the three methods of BCIs, the average width of p-boot is minimum in most of the cases and the average widths follows the order p-boot < s-boot < t-boot for all the considered variations of the sample size and model parameters. Therefore, we conclude that p-boot shows overall better performance of the BCIs for PLD.

7. Real data analysis

In this section, a real-life data set has been considered to show the applicability of proposed study. The data set is reported by [2] which represents the strength measurements in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows. The data set is given below; The average strength of single carbon fiber is 1.451 with standard deviation 0.495. The summary of the data set is presented via box plot in Figure 1, and noted that the median and mean of the data are very close to each other. However, the quartiles are equidistance from median value which indicates the symmetricity of the data. The fitting of the PLD for the above data set is checked by different model selection tools and compared with most popular two parametric probability distributions namely, Weibull distribution (WD), gamma distribution (GD), normal distribution (ND), logistic distribution (LGD) and generalized exponential distribution (GED). The considered selcetion tools are: negative of log-likelihood $-\mathbf{L}$, Akaike information criterion (AIC) [AIC=-2 $\mathbf{L}$+2k], Bayesian information criterion (BIC) [BIC=-2 $\mathbf{L} + k \log n$], Kolmogrov-Smirnov (KS) test [KS=$Sup_x|F_n(x) - \hat{F}(x)|$]. The model would be taken as best with least value of these measures. The values of the considered measures along with the $p-$value are given in Table 1, and observed that PLD has least values of $-\hat{\mathbf{L}}$, AIC, BIC, KS with higher p-value. Further, the estimated density with histogram and empirical cumulative distribution function plots for the different models are displayed in Figure 2 and Figure 3, respectively. From these Figures, it may be easily verified that the PLD might be a better choice as compared to other considered probability distributions.

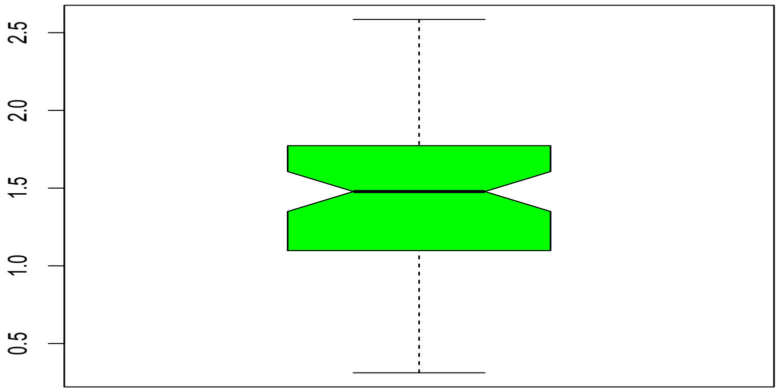

Figure 1: *Summary of the data via box plot.*

Table 1: *Values of different tools for model selection.*

Model	MLE	-LogL	AIC	BIC	KS	P-VALUE
PLD	$\hat{\alpha}$=2.6959 $\hat{\beta}$=0.4864	-48.6797	101.3594	105.8276	0.04054	0.9999
WD	$\hat{a}$=3.2487 $\hat{b}$=1.0171	-49.00054	102.0011	106.4693	0.043752	0.9994
GD	$\hat{\alpha}$=6.9968 $\hat{\beta}$=4.8209	-53.08266	110.1653	114.6335	0.0884	0.6536
ND	$\hat{\mu}$=1.4513 $\hat{\sigma}$=0.4915	-48.90256	101.8051	106.2733	0.037603	0.9992
LGD	$\hat{\mu}$=1.4533 $\hat{s}$=0.2796	-49.40943	102.8189	107.2871	0.047889	0.9974
GED	$\hat{\alpha}$=8.8283 $\hat{\lambda}$=1.8965	-56.66857	117.3371	121.8054	0.11192	0.3531

The classical estimates (MLEs, MPSE) of the RC, μ, $R(t)$ and $h(t)$ are computed for arbitrarily chosen $t = 1.5$. Since in case of real-life data set no any prior information is available, thus one may use most suited non-informative prior which may be proper or improper but it leads proper posterior. Here, we have taken the same non-informative prior where losses are SELF and LLF. The Bayes estimates are calculated under non-informative prior using MCMC method, reported in Table 2. In order to perform Bayes computation using MCMC method, well mixing of the chain has been checked via tuning of the variance of the MLE. To achieve stationarity of the Markov Chain, ($N_0 = 500$) samples (burn in period) are discarded out of 12000 generated posterior deviates. It has been verified that the generated posterior samples are well mixed and assume the stationary property. Further, different interval estimates namely ACIs, BCIs and HPD credible are constructed for the same characteristics, given in Table 3. From Table 3, it is clearly visible that the width of the interval obeys the pattern $\mathcal{W}_m \approx \mathcal{W}_p > \mathcal{W}_{s-boot} > \mathcal{W}_{p-boot} \approx \mathcal{W}_{t-boot} > \mathcal{W}_{Bayes}$.

Table 2: *Estimates of the RC for* $t = 1.5$ *of the considered data set.*

RC	$\hat{\Theta}_m$	$\hat{\Theta}_p$	$\hat{\Theta}_{sf}$	$\hat{\Theta}_{lf1}$	$\hat{\Theta}_{lf2}$
μ	1.4519	1.4488	1.4563	1.4586	1.4546
$R(t)$	0.4632	0.4752	0.4673	0.4647	0.4647
$h(t)$	1.7200	1.7236	1.7121	1.7520	1.6840

Table 3: *Width of the interval of RC for the considered data set when $t = 1.5$.*

	MLE	MPSE	s-boot	p-boot	t-boot	HPD
RC	$\mathcal{W}$	$\mathcal{W}$	$\mathcal{W}$	$\mathcal{W}$	$\mathcal{W}$	$\mathcal{W}$
μ	0.4149	0.4147	0.2318	0.2242	0.2269	0.1860
$R(t)$	0.3346	0.3426	0.0906	0.0846	0.0798	0.0532
$h(t)$	1.8786	1.8626	0.3030	0.3000	0.3221	0.1843

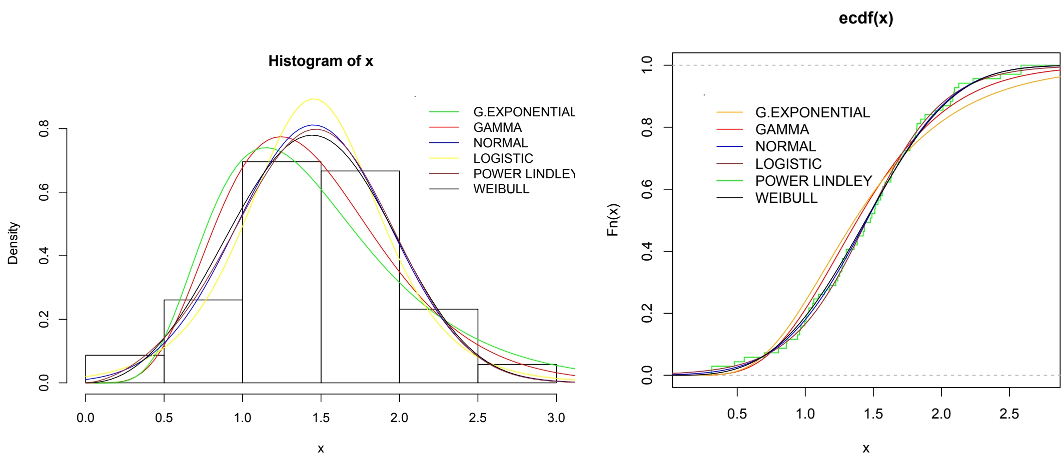

Figure 2: *Estimated density and ECDF plots based on considered data.*

Table 4: *Average MSEs of the classical and Bayes estimators of MTSF and RF.*

n	μ	Classical estimators		Bayes(informative)			Bayes (non-informative)		
		$\hat{\mu}_m$	$\hat{\mu}_p$	$\hat{\mu}_{sf}$	$\hat{\mu}_{lf1}$	$\hat{\mu}_{lf2}$	$\hat{\mu}_{sf}$	$\hat{\mu}_{lf1}$	$\hat{\mu}_{lf2}$
10		0.8280	0.8472	0.6872	0.6972	0.6197	0.6964	0.6988	0.6269
20		0.5430	0.6847	0.5356	0.4203	0.3421	0.5739	0.6125	0.3905
30	2.5445	0.3997	0.4273	0.3416	0.3611	0.2436	0.3434	0.3499	0.2500
50		0.2170	0.2123	0.1856	0.1914	0.1561	0.2016	0.3007	0.1644
100		0.0917	0.0940	0.0851	0.0903	0.0773	0.0915	0.1152	0.0840
10		0.5299	0.5285	0.4791	0.4136	0.3923	0.5175	0.4658	0.2759
20		0.1858	0.2002	0.1813	0.1282	0.1502	0.1946	0.2113	0.1551
30	2.0952	0.1053	0.1230	0.1262	0.1249	0.1094	0.1228	0.2076	0.1389
50		0.0700	0.0780	0.0723	0.0886	0.0661	0.0742	0.0921	0.0674
100		0.0304	0.0313	0.0310	0.0340	0.0298	0.0350	0.0390	0.0331
10		0.0213	0.0233	0.0175	0.0209	0.0167	0.0234	0.0214	0.0221
20		0.0101	0.0109	0.0096	0.0100	0.0094	0.0103	0.0107	0.0100
30	0.6951	0.0068	0.0066	0.0062	0.0064	0.0062	0.0070	0.0072	0.0069
50		0.0040	0.0042	0.0037	0.0037	0.0036	0.0041	0.0042	0.0041
100		0.0020	0.0020	0.0020	0.0020	0.0020	0.0022	0.0022	0.0021
10		0.0089	0.0086	0.0081	0.0083	0.0080	0.0104	0.0323	0.0103
20		0.0045	0.0044	0.0042	0.0042	0.0042	0.0049	0.0049	0.0049
30	0.7621	0.0033	0.0033	0.0030	0.0030	0.0030	0.0031	0.0031	0.0031
50		0.0022	0.0021	0.0020	0.0020	0.0020	0.0022	0.0022	0.0021
100		0.0010	0.0010	0.0009	0.0009	0.0009	0.0010	0.0011	0.0010
10		0.0091	0.0087	0.0083	0.0085	0.0082	0.0100	0.0102	0.0100
20		0.0045	0.0043	0.0043	0.0043	0.0043	0.0045	0.0046	0.0045
30	0.8841	0.0034	0.0033	0.0030	0.0030	0.0030	0.0032	0.0033	0.0032
50		0.0017	0.0018	0.0016	0.0016	0.0016	0.0020	0.0020	0.0020
100		0.0010	0.0009	0.0009	0.0009	0.0009	0.0009	0.0009	0.0009
n	$R(t)$	Classical estimators		Bayes(informative)			Bayes (non-informative)		
		$\hat{R}(t)_m$	$\hat{R}(t)_p$	$\hat{R}(t)_{sf}$	$\hat{R}(t)_{lf1}$	$\hat{R}(t)_{lf2}$	$\hat{R}(t)_{sf}$	$\hat{R}(t)_{lf1}$	$\hat{R}(t)_{lf2}$
10		0.0143	0.0143	0.0136	0.0134	0.0138	0.0144	0.0142	0.0147
20		0.0080	0.0074	0.0069	0.0069	0.0069	0.0079	0.0079	0.0080
30	0.6907	0.0051	0.0049	0.0045	0.0045	0.0046	0.0052	0.0052	0.0053
50		0.0031	0.0033	0.0029	0.0029	0.0032	0.0031	0.0031	0.0031
100		0.0015	0.0015	0.0014	0.0014	0.0014	0.0015	0.0015	0.0015
10		0.0119	0.0116	0.0114	0.0111	0.0116	0.0120	0.0117	0.0122
20		0.0066	0.0068	0.0063	0.0062	0.0064	0.0064	0.0063	0.0064
30	0.7529	0.0045	0.0047	0.0043	0.0042	0.0043	0.0044	0.0045	0.0047
50		0.0027	0.0028	0.0025	0.0025	0.0025	0.0027	0.0027	0.0027
100		0.0013	0.0013	0.0012	0.0012	0.0012	0.0013	0.0013	0.0013
10		0.0190	0.0128	0.0135	0.0137	0.0134	0.0196	0.0199	0.0194
20		0.0089	0.0088	0.0076	0.0077	0.0076	0.0082	0.0082	0.0081
30	0.3909	0.0052	0.0054	0.0051	0.0051	0.0051	0.0053	0.0056	0.0055
50		0.0033	0.0029	0.0028	0.0027	0.0028	0.0033	0.0033	0.0033
100		0.0016	0.0015	0.0016	0.0016	0.0016	0.0017	0.0017	0.0017
10		0.0182	0.0123	0.0133	0.0134	0.0133	0.0187	0.0187	0.0188
20		0.0084	0.0087	0.0075	0.0075	0.0075	0.0087	0.0087	0.0087
30	0.5000	0.0062	0.0061	0.0051	0.0051	0.0051	0.0056	0.0056	0.0056
50		0.0036	0.0038	0.0034	0.0034	0.0034	0.0031	0.0032	0.0031
100		0.0017	0.0018	0.0016	0.0016	0.0016	0.0018	0.0018	0.0018
10		0.0161	0.0131	0.0119	0.0117	0.0121	0.0165	0.0162	0.0168
20		0.0081	0.0072	0.0067	0.0066	0.0068	0.0074	0.0074	0.0075
30	0.6655	0.0058	0.0054	0.0046	0.0045	0.0046	0.0054	0.0054	0.0055
50		0.0031	0.0031	0.0027	0.0027	0.0027	0.0033	0.0033	0.0033
100		0.0016	0.0016	0.0015	0.0015	0.0015	0.0016	0.0016	0.0016

Table 5: *Average MSEs of the classical and Bayes estimators of HF.*

n	$h(t)$	Classical estimators		Bayes(informative)			Bayes (non-informative)		
		$\hat{h}(t)_m$	$\hat{h}(t)_p$	$\hat{h}(t)_{sf}$	$\hat{h}(t)_{lf1}$	$\hat{h}(t)_{lf2}$	$\hat{h}(t)_{sf}$	$\hat{h}(t)_{lf1}$	$\hat{h}(t)_{lf2}$
10		0.0387	0.0392	0.0365	0.0345	0.0321	0.0351	0.0360	0.0378
20		0.0141	0.0195	0.0115	0.0121	0.0111	0.0146	0.0156	0.0139
30	0.4148	0.0079	0.0076	0.0070	0.0073	0.0069	0.0079	0.0082	0.0077
50		0.0048	0.0047	0.0045	0.0046	0.0044	0.0046	0.0047	0.0046
100		0.0020	0.0022	0.0019	0.0019	0.0019	0.0021	0.0022	0.0021
10		0.0394	0.0324	0.0320	0.0365	0.0297	0.0327	0.0375	0.0300
20		0.0155	0.0150	0.0145	0.0153	0.0141	0.0145	0.0153	0.0140
30	0.4257	0.0097	0.0074	0.0094	0.0073	0.0094	0.0096	0.0099	0.0098
50		0.0057	0.0058	0.0052	0.0054	0.0055	0.0056	0.0057	0.0056
100		0.0035	0.0033	0.0025	0.0025	0.0024	0.0026	0.0027	0.0026
10		0.7851	0.7478	0.7298	0.7252	0.7206	0.7124	0.7503	0.7864
20		0.4959	0.4832	0.3240	0.4209	0.2428	0.3953	0.5106	0.3887
30	1.9937	0.2342	0.1563	0.2093	0.2179	0.1739	0.2124	0.2891	0.2176
50		0.1168	0.1320	0.1085	0.1293	0.0976	0.1179	0.1412	0.1055
100		0.0487	0.0432	0.0420	0.0412	0.0477	0.0531	0.0580	0.0503
10		0.5859	0.6898	0.4941	0.4781	0.4453	0.5271	0.6132	0.5805
20		0.5130	0.6181	0.3966	0.4567	0.2981	0.4765	0.5106	0.5461
30	2.4304	0.2681	0.2955	0.2371	0.2319	0.2000	0.2595	0.3539	0.2162
50		0.1771	0.1747	0.1454	0.1727	0.1320	0.1605	0.1532	0.1603
100		0.0978	0.0819	0.0649	0.0713	0.0616	0.0713	0.0773	0.0683
10		0.7301	0.4766	0.4056	0.4707	0.3060	0.6604	0.4941	0.4933
20		0.3194	0.3469	0.2161	0.2860	0.1830	0.2875	0.2577	0.2668
30	1.7233	0.1819	0.1732	0.1228	0.1428	0.1132	0.1325	0.1528	0.1230
50		0.0681	0.0705	0.0640	0.0620	0.0613	0.0773	0.0848	0.0736
100		0.0375	0.0383	0.0335	0.0352	0.0327	0.0373	0.0392	0.0363

Table 6: *Average width and coverage probability of different CIs for MTSF.*

n	μ	ACIs				BCIs						HPD interval			
		MLE		MPSE		s-boot		p-boot		t-boot		Informative Prior		non-informative Prior	
		$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$
10		5.8205	0.913	6.7708	0.971	3.5789	0.861	3.5142	0.875	4.21954	0.885	3.4797	0.873	3.5598	0.866
20		4.0643	0.959	5.3867	0.984	2.5230	0.892	2.4962	0.892	2.84413	0.913	2.3161	0.898	2.3989	0.864
30	2.5445	3.3274	0.963	4.0324	0.989	2.0414	0.915	2.0259	0.926	2.25289	0.935	1.8731	0.896	1.8774	0.889
50		2.5589	0.976	2.9189	0.992	1.6106	0.919	1.5997	0.929	1.74043	0.936	1.3967	0.879	1.4197	0.894
100		1.8102	0.988	1.9574	0.995	1.1540	0.946	1.1468	0.947	1.21691	0.954	0.9679	0.891	0.9877	0.900
10		3.6159	0.944	5.4237	0.978	2.1069	0.885	2.0834	0.886	2.33669	0.893	2.0846	0.870	2.0677	0.887
20		2.5600	0.969	3.1146	0.986	1.5304	0.917	1.5174	0.919	1.64496	0.93	1.3822	0.893	1.3843	0.874
30	2.0952	2.1025	0.980	2.4159	0.990	1.2545	0.917	1.2459	0.921	1.33214	0.933	1.1029	0.876	1.1205	0.899
50		1.6140	0.985	1.7783	0.991	0.9838	0.952	0.9780	0.949	1.03119	0.958	0.8393	0.896	0.8558	0.878
100		1.1449	0.993	1.2156	0.997	0.7020	0.948	0.6976	0.945	0.72394	0.956	0.5945	0.891	0.5993	0.892
10		0.5463	0.947	0.8013	0.994	0.5138	0.902	0.5096	0.905	0.54385	0.913	0.3918	0.863	0.4134	0.795
20		0.3889	0.951	0.4841	0.983	0.3729	0.897	0.3707	0.899	0.38834	0.907	0.2772	0.835	0.2898	0.836
30	0.6951	0.3188	0.945	0.3733	0.981	0.3100	0.917	0.3082	0.922	0.32090	0.931	0.2264	0.842	0.2315	0.822
50		0.2479	0.952	0.2754	0.972	0.2427	0.941	0.2416	0.943	0.24979	0.946	0.1759	0.845	0.1767	0.828
100		0.1752	0.948	0.1865	0.968	0.1736	0.943	0.1726	0.945	0.17659	0.945	0.1240	0.852	0.1244	0.815
10		0.3811	0.943	0.5222	0.986	0.3525	0.897	0.3504	0.895	0.35813	0.894	0.2687	0.861	0.2832	0.823
20		0.2722	0.949	0.3281	0.981	0.2632	0.917	0.2618	0.915	0.26621	0.918	0.1909	0.852	0.1994	0.831
30	0.7621	0.2227	0.952	0.2553	0.975	0.2165	0.934	0.2151	0.931	0.21781	0.934	0.1572	0.843	0.1589	0.833
50		0.1724	0.958	0.1894	0.965	0.1698	0.949	0.1691	0.947	0.17076	0.946	0.1218	0.828	0.1243	0.859
100		0.1224	0.954	0.1291	0.969	0.1212	0.941	0.1207	0.941	0.12160	0.939	0.0865	0.839	0.0872	0.814
10		0.3523	0.951	0.4558	0.989	0.3415	0.894	0.3389	0.891	0.34075	0.886	0.2531	0.810	0.2651	0.801
20		0.2517	0.941	0.2942	0.977	0.2538	0.921	0.2523	0.919	0.25420	0.916	0.1809	0.826	0.1861	0.820
30	0.8841	0.2055	0.944	0.2313	0.973	0.2096	0.926	0.2087	0.929	0.20947	0.929	0.1483	0.812	0.1499	0.813
50		0.1606	0.947	0.1735	0.957	0.1648	0.93	0.1641	0.928	0.16484	0.927	0.1147	0.838	0.1156	0.800
100		0.1141	0.936	0.1188	0.946	0.1171	0.954	0.1166	0.955	0.11693	0.956	0.0808	0.832	0.0812	0.815

Table 7: *Average length and width of the CIs for RF.*

n	$R(t)$	ACIs				BCIs						HPD interval			
		MLE		MPSE		s-boot		p-boot		t-boot		Informative Prior		non-informative Prior	
		$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$
10		0.4153	0.866	0.4235	0.925	0.4541	0.877	0.4444	0.908	0.4199	0.827	0.2728	0.743	0.2789	0.737
20		0.3043	0.898	0.3064	0.927	0.3299	0.919	0.3256	0.936	0.3202	0.896	0.2045	0.776	0.2077	0.749
30	0.6907	0.2519	0.912	0.2532	0.926	0.2705	0.922	0.2679	0.934	0.2655	0.905	0.1713	0.788	0.1730	0.762
50		0.1958	0.919	0.1975	0.928	0.2113	0.938	0.2096	0.943	0.2089	0.928	0.1347	0.755	0.1348	0.758
100		0.1398	0.924	0.1401	0.940	0.1503	0.947	0.1494	0.951	0.1492	0.943	0.0966	0.807	0.0968	0.787
10		0.3662	0.830	0.3880	0.911	0.4028	0.857	0.3914	0.895	0.3545	0.793	0.2437	0.716	0.2480	0.716
20		0.2720	0.879	0.2810	0.920	0.2995	0.906	0.2949	0.932	0.2813	0.880	0.1837	0.743	0.1871	0.756
30	0.7529	0.2252	0.885	0.2303	0.926	0.2483	0.917	0.2454	0.929	0.2378	0.903	0.1548	0.748	0.1552	0.762
50		0.1768	0.902	0.1789	0.922	0.1938	0.930	0.1922	0.942	0.1883	0.917	0.1213	0.751	0.1215	0.756
100		0.1263	0.920	0.1268	0.927	0.1385	0.943	0.1375	0.942	0.1358	0.940	0.0870	0.783	0.0876	0.783
10		0.4871	0.895	0.5628	0.969	0.5093	0.893	0.5060	0.914	0.5158	0.898	0.3202	0.843	0.3310	0.755
20		0.3480	0.931	0.3733	0.964	0.3566	0.914	0.3550	0.915	0.3604	0.908	0.2356	0.812	0.2416	0.809
30	0.3909	0.2843	0.928	0.2991	0.967	0.2897	0.934	0.2885	0.933	0.2921	0.929	0.1953	0.829	0.1989	0.813
50		0.2205	0.945	0.2279	0.967	0.2225	0.944	0.2215	0.946	0.2237	0.944	0.1531	0.829	0.1550	0.822
100		0.1561	0.943	0.1589	0.960	0.1563	0.948	0.1555	0.947	0.1570	0.953	0.1094	0.839	0.1099	0.815
10		0.5178	0.900	0.5944	0.975	0.5191	0.897	0.5148	0.913	0.5177	0.894	0.3279	0.837	0.3460	0.795
20		0.3653	0.929	0.3914	0.974	0.3657	0.910	0.3636	0.913	0.3697	0.908	0.2443	0.836	0.2520	0.818
30	0.5000	0.2976	0.941	0.3120	0.970	0.2980	0.938	0.2964	0.941	0.3019	0.937	0.2028	0.833	0.2077	0.827
50		0.2299	0.952	0.2371	0.961	0.2295	0.931	0.2283	0.931	0.2322	0.930	0.1592	0.825	0.1614	0.856
100		0.1623	0.952	0.1654	0.965	0.1614	0.930	0.1608	0.932	0.1631	0.931	0.1139	0.837	0.1146	0.809
10		0.4261	0.863	0.4923	0.952	0.4643	0.886	0.4548	0.915	0.4351	0.842	0.2857	0.781	0.3002	0.742
20		0.3045	0.884	0.3300	0.943	0.3381	0.926	0.3345	0.938	0.3316	0.909	0.2167	0.812	0.2216	0.795
30	0.6655	0.2504	0.904	0.2661	0.950	0.2762	0.933	0.2739	0.942	0.2736	0.929	0.1794	0.793	0.1823	0.777
50		0.1947	0.915	0.2025	0.931	0.2161	0.945	0.2145	0.949	0.2152	0.940	0.1411	0.809	0.1430	0.786
100		0.1379	0.908	0.1412	0.930	0.1530	0.948	0.1520	0.944	0.1528	0.943	0.1011	0.805	0.1019	0.792

Table 8: *Average width and corresponding coverage probability for different CIs of HF.*

n	$h(t)$	ACIs				BCIs						HPD interval			
		MLE		MPSE		s-boot		p-boot		t-boot		Informative Prior		non-informative Prior	
		$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$
10		0.7809	0.960	0.7237	0.972	0.9071	0.936	0.8213	0.920	1.1156	0.954	0.4329	0.811	0.4433	0.803
20		0.5370	0.977	0.5118	0.980	0.4753	0.945	0.4721	0.920	0.5716	0.959	0.2997	0.840	0.3094	0.822
30	0.6907	0.4361	0.983	0.4213	0.985	0.3573	0.948	0.3567	0.943	0.4119	0.961	0.2449	0.856	0.2487	0.852
50		0.3343	0.984	0.3276	0.991	0.2658	0.951	0.2656	0.937	0.2958	0.966	0.1903	0.846	0.1895	0.839
100		0.2358	0.987	0.2323	0.992	0.1819	0.946	0.1814	0.943	0.1958	0.964	0.1341	0.878	0.1346	0.861
10		0.8098	0.933	0.7739	0.967	0.7706	0.939	0.7213	0.929	1.0156	0.955	0.4422	0.777	0.4517	0.810
20		0.5705	0.963	0.5534	0.981	0.5011	0.938	0.4984	0.928	0.5762	0.953	0.3146	0.813	0.3206	0.834
30	0.7529	0.4635	0.972	0.4522	0.984	0.3834	0.948	0.3823	0.946	0.4241	0.965	0.2605	0.817	0.2590	0.827
50		0.3602	0.977	0.3526	0.989	0.2882	0.951	0.2871	0.948	0.3089	0.967	0.2013	0.822	0.2010	0.820
100		0.2544	0.986	0.2502	0.991	0.2003	0.950	0.1997	0.946	0.2100	0.961	0.1420	0.836	0.1433	0.851
10		3.4746	0.973	3.2738	0.919	16.0037	0.990	7.8693	0.839	13.4845	0.995	2.1102	0.862	2.4974	0.798
20		2.0997	0.948	1.9753	0.928	2.9501	0.989	2.8136	0.905	4.1606	0.997	1.4083	0.836	1.4482	0.830
30	0.3909	1.6443	0.941	1.5721	0.933	2.0297	0.971	1.9845	0.907	2.7617	0.992	1.1319	0.824	1.1537	0.833
50		1.2301	0.949	1.2003	0.934	1.3968	0.974	1.3810	0.924	1.7973	0.994	0.8618	0.836	0.8765	0.832
100		0.8548	0.949	0.8384	0.943	0.9079	0.970	0.9014	0.948	1.0936	0.991	0.6007	0.837	0.6079	0.830
10		3.8633	0.950	3.7965	0.941	9.6634	0.981	6.9379	0.879	11.1590	0.984	2.2629	0.855	2.6232	0.775
20		2.3826	0.955	2.3127	0.946	3.1208	0.986	3.0400	0.914	4.2482	0.993	1.5854	0.846	1.6504	0.821
30	0.5000	1.8798	0.952	1.8309	0.935	2.2095	0.974	2.1795	0.926	2.8706	0.992	1.2797	0.840	1.3125	0.841
50		1.4160	0.953	1.3905	0.938	1.5640	0.961	1.5500	0.928	1.9286	0.986	0.9800	0.820	0.9854	0.831
100		0.9842	0.950	0.9782	0.945	1.0323	0.965	1.0267	0.953	1.2018	0.985	0.6911	0.833	0.6921	0.826
10		2.4993	0.923	2.4870	0.951	4.0126	0.946	3.6154	0.914	5.0554	0.955	1.5872	0.794	1.6935	0.779
20		1.6623	0.933	1.6466	0.954	1.9417	0.960	1.9263	0.936	2.3671	0.977	1.1402	0.818	1.1386	0.821
30	0.6655	1.3441	0.943	1.3374	0.967	1.4932	0.951	1.4867	0.925	1.7524	0.971	0.9148	0.817	0.9246	0.823
50		1.0293	0.951	1.0244	0.956	1.0857	0.954	1.0832	0.940	1.2279	0.969	0.7074	0.842	0.7161	0.819
100		0.7218	0.944	0.7210	0.950	0.7354	0.945	0.7335	0.945	0.8005	0.968	0.5021	0.832	0.5048	0.826

8. Conclusion

In this paper, we have considered the classical and Bayesian point and interval estimation of the reliability characteristics (RC) for the PLD based on complete observations. In classical estimation, MLE, MPSE are discussed for RC. The Bayes estimators are derived with informative and non-informative priors under SELF and LLF for the same characteristics. Further, different CIs, as ACIs based on MLE and MPSE, three BCIs, namely s-boot, p-boot, t-boot and Bayes credible HPD intervals based on generated posterior samples are obtained. The theoretical comparison of the point and interval estimates obtained via different methods of estimation are not feasible. Hence, the Monte Carlo simulation study has been performed to make the extensive comparison in terms of average MSEs and average width of the respective CIs. Monte Carlo simulation results showed that p-boot CIs achieve better performance than the other BCIs and ACIs in terms of width for all the considered choices. Among the methods of estimation, Bayes estimates under informative prior are the best performing estimator in terms of the average MSEs as well as average width of CIs. Coverage probabilities do not follow any specific trend but for shorter length of the CIs, $\mathcal{C}$ decreases and reaching to the nominal values. Lastly, a practical data set has been used to illustrate the proposed methodology, and observed that it echoed the same pattern as simulation.

Conflict of interest

The authors declare that they have no conflict of interest.

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